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What Do *We* Mean by Mathematical Proof? [1]

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Abstract

Mathematical proof lies at the foundations of mathematics, but there are several notions of what mathematical proof is, or might be. In fact, the idea of mathematical proof continues to evolve. In this article, I review the body of literature that argues that there are at least two widely held meanings of proof, and that the standards of proof are negotiated and agreed upon by the members of mathematical communities. The formal view of proof is contrasted with the view of proofs as arguments intended to convince a reader. These views are examined in the context of the various roles of proof. The conceptions of proof held by students, and communities of students, are discussed, as well as the pedagogy of introductory proof-writing classes.

What is a mathematical proof? This question, and variations on it, have been debated for some time, and many answers have been proposed. One variation of this question is the title of this article: "What do *we* mean by mathematical proof?" Here *we* may stand for the international community of mathematicians, a classroom of students, the human race as a whole, or any number of other mathematical communities. When the question is phrased this way, it becomes clear that any answer to this question must, in one way or another, take into account the fact that mathematics and mathematical proof are endeavors undertaken by people, either individually or communally.[2]

This article will discuss two answers to this question that are held by mathematicians and mathematics educators, and how those answers affect

[1]Portions of this article previously appeared in the author's doctoral dissertation, [7].

[2]By this statement, I do not mean to take sides in the debate over whether mathematics is "discovered" or "created"; in either case, it is *people* who discover or create mathematics.

the notions of proof held by communities of students. First, we will contrast the "formal" definition of proof with the "practical" meaning described by Reuben Hersh and others. Second, we will discuss these meanings in the context of the different roles played by mathematical proofs. Third, beliefs about proof that are held by students and communities of students are outlined. Finally, we will discuss some of the ways proof is introduced in the classroom, with an eye toward the effect of the meanings of proof on pedagogy.

1. Formal and Practical Notions of Proof

Proof is fundamental to mathematics; we do not know whether a mathematical proposition is true or false until we have proved (or disproved) it. Therefore, the question of "What do we mean by mathematical proof?" is not an idle one. The answer to this question given by the worldwide community of mathematicians, educators, philosophers, and others interested in mathematics continues to be negotiated and to evolve. In fact, there are at least two prevalent meanings of proof in this community. These meanings will be discussed below.

Let us take, as a starting point, the description of proof given by Gian-Carlo Rota: "Everybody knows what a mathematical proof is. A proof of a mathematical theorem is a sequence of steps which leads to the desired conclusion. The rules to be followed by such a sequence of steps were made explicit when logic was formalized early in this century, and they have not changed since" [47, p.183]. As Rota's opening sentence indicates (see also [2]), this is what most mathematicians believe their proofs accomplish. This notion could be called the *formal* meaning of proof. However, as Rota [47] points out later in the same article, and as will be discussed below, this description is *unrealistic* in describing the proofs that most mathematicians actually write.

This formal notion of proof, or something close to it, had been held by some mathematicians at the beginning of the 20th century [22]. Three schools of thought sought to ground the foundations of mathematics at this time; of these, perhaps the most well-known is the formalist school of David Hilbert. The formalists sought to show that areas of mathematics are free of contradictions by writing their statements in a formal language, and proving them using formal rules of inference [50]. For the formalists, the *meaning* of a mathematical proposition was irrelevant, proofs were exclusively based on syntactic constructs and manipulations. The original goal of the formalist school ultimately failed in 1931, when the publication of Gödel's incompleteness theorems showed that no formal system that includes even simple arithmetic can be both complete and free of contradiction.[3]

To the Hilbert school of formalists, the role of proof was to show that a particular part of mathematics was free of contradiction, and to thereby validate the theorems of that branch of mathematics. Indeed, today the most visible role of proof is to verify the truth of published theorems.[4] A present-day version of formalism can be seen in a current project to use computers and specialized languages to complete formal proofs of major mathematical theorems [19, 28]. Hales [19] puts such formal proofs in the context of a larger project to automate the formal proving of theorems.

This formal notion of proof has some important limitations. In all but the most trivial cases, a *purely* formal proof is, or would be if it were written down, far too long to be of any interest or value. One report notes that a purely formal proof of the Pythagorean theorem, beginning from Hilbert's axioms of geometry, was nearly 80 pages long ([46] cited in [59]) An even more extreme example is cited by Hales [19]: to fully expand the definition of the number "1", in terms of the primitives of Bourbaki's Theory of Sets, would require over four trillion symbols! In principle, the proof of any theorem can be written in such a formal way, but such formalizations of proofs are almost always too lengthy to be of value to human readers. Therefore, this purely formal notion of proof does not reflect the *practice* of mathematicians [20, 47, 57]. The automated proofs described by Hales [19] might, indeed, be able to prove previously unproven theorems, but Auslander [2] questions whether we would be satisfied with such a proof, that could not be read or understood by a human mathematician.

These *purely* formal proofs represent an extreme version of the formal meaning of proof. More often, the notion of formal proofs allows for proofs to be condensed using previously proven theorems or lemmas. Nonetheless, the formal meaning of proof is the understanding that proofs can be read, understood, and checked entirely within the context of an axiomatic system and formal rules of logic [53].

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[3]Nonetheless, the formalist school made many important contributions to modern mathematics. A much more complete, and yet accessible, discussion of the formalist school of thought, and the other two major schools of thought at that time, can be found in Snapper [50].

[4]As discussed below, this is really only *one* of the roles of mathematical proof.

Hersh [32] and others have argued that there is another distinct meaning of "mathematical proof," in addition to the formal meaning of proof described above. This second meaning of proof, which Hersh calls "practical mathematical proof," is informal and imprecise. As described by Hersh, "Practical mathematical proof is what we do to make each other believe our theorems" [32, p.49].[5] This is the meaning that many (but not all) mathematicians use in their practice of mathematics, in which the meaning of the mathematics is not only present but essential to the proof.

The practical meaning implies that proof has a subjective side; the goal of a proof is to convince the mathematical community of the truth of a theorem. That is, mathematics is a *human* endeavor, since proofs are written, read, understood, verified, and used by humans. This point is made by Davis and Hersh [10] in an imaginary dialogue between their Ideal Mathematician and a philosophy student who asks for a definition of proof. The Ideal Mathematician, when pressed, offers the following [10, p.39-40]:

> **I.M.:** Well, this whole thing was cleared up by the logician Tarski, I guess, and some others, maybe Russell or Peano. Anyhow, what you do is, you write down the axioms of your theory in a formal language,... Then you show that you can transform the hypothesis step by step, using the rules of logic, till you get a conclusion. That's a proof.
>
> **Student:** Really? That's amazing!... I've never seen that done before.
>
> **I.M.:** Oh, of course no one really *does* it. It would take forever! You just show that you could do it, that's sufficient.
>
> **Student:** But even that doesn't sound like what was done in my courses and textbooks. ... Then what really *is* a proof?
>
> **I.M.:** Well, it's an argument that convinces someone who knows the subject.
>
> **Student:** Someone who knows the subject? Then the definition of proof is subjective; it depends on particular persons...

[5] Hanna [20] describes these two meanings as "formal" and "acceptable" proofs, respectively, while Devlin [13] characterizes these meanings as "right-wing" and "left-wing" definitions of proof.

> **I.M.:** No, no. There's nothing subjective about it! Everybody knows what a proof is...

In their exchange, the Ideal Mathematician readily admits that the formal definition of proof does not adequately describe the proofs we actually use, but is unable to provide a satisfactory way of defining proof as *objective*.

In this dialogue, the question of how we recognize proofs is addressed. If it is impractical to write proofs as *purely formal* proofs, then we instead consider a proof to be valid when it is accepted by other mathematicians. As pointed out by Davis and Hersh above, and by others [13, 20, 27, e.g.], when a mathematician reads a proof to determine its validity, he or she makes that determination based on whether or not he or she finds the proof to be convincing.[6] That is, the mathematician makes a judgement based on *subjective* criteria. The Clay Mathematics Institute, which offers a one million dollar prize for a proof of any one of seven mathematical conjectures, stipulates that any proof must be published and accepted by the community of mathematicians for two years before a prize will be awarded. Because the validity of a proof depends on acceptance by mathematicians, that validity is inherently subjective.

This subjectivity should not be taken to imply that Hersh's practical meaning of proof abandons notions of *rigor*. Detlefsen [12] argues that formalization and rigor are independent of each other. Under what he refers to as the "common view," "Rigor is a necessary feature of proof and formalizability is a necessary condition of rigor" [12, p.16]. However, he argues that mathematical proofs are not presented in a way that makes their formalizations either apparent or routine, and in fact, such formalizations are generally *not* routine at all. However, proofs *are* presented in such a way that makes their *rigor* clear to the reader. Thus, he argues, formalization and rigor are independent of each other. While Detlefsen does not expound on what, exactly, he means by rigor, it appears from his argument that rigor is a set of standards agreed upon by the community of mathematicians, and that that set of standards does not include strict formalization of proofs. As

[6] This is not universally accepted. Weber [62] has argued that the power to convince an individual or community does not necessarily constitute a proof, nor does an argument that fails to be completely convincing necessarily fail to be a proof. Furthermore, as Bell [4, p.24] states, "Conviction is normally reached by quite other means than following a logical proof."

Davis and Hersh [10] point out, it is likely to be impossible to precisely define this set of standards. The rigor of a proof is made clear to the reader using communally defined standards for the *format* of written proofs. Proofs are generally written in such a way as to make the rigor used by the writer apparent; for instance, by showing that all relevant cases have been considered. Just as it is likely to be impossible to define the standards of rigor set by the community of mathematicians, it is also likely to be impossible to define the standards of the format of written proofs.

Just as mathematicians do not hold a uniform opinion of the meaning of proof, mathematicians' practice of creating proofs is varied. Weber [61] notes that successful mathematical reasoning can take place in at least two qualitatively different ways, which he and Alcock [63] refer to as *syntactic proof production* and *semantic proof production*. In the former, an individual produces a proof by focusing on manipulating correctly stated definitions and facts in a logically valid way; in the latter, the individual will also use "instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws" [63, p.210]. Some mathematicians work predominantly by proving syntactically, others by proving semantically. Weber [61] cites Pinto and Tall [43], who note that Poincaré distinguished between mathematicians who were "guided by intuition" and those who were "preoccupied with logic".

These two ways of reasoning mirror the "formal" and "practical" meanings of proof described above. Weber describes syntactic and semantic proof production in this way: "A syntactic proof production occurs when ... the prover focuses on the definition of the statement to be proven and the rules for deducing logical consequences from this definition without reference to other informal representations of the concept. Alternatively, a semantic proof production occurs when much of the prover's work takes place by considering other representations of mathematical concepts ... such as graphs, diagrams, kinesthetic gestures, or prototypical examples" [61, p.201]. Just as there are at least two possible meanings of what we mean by "proof," there are at least two ways to go about the task of proving.

Some researchers have made a distinction between "argumentation" and "proof." Douek [14] cites Duval [16] as arguing that these are very different, despite having similar linguistic forms. However, Doeuk argues that argumentation and proof have many aspects in common, and that argumentation is often useful to the process of proving. Pedemonte [42] considers proof to be a specialized form of argumentation, and further argues that there is evidence of *cognitive unity* [17] between the argumentation used in the construction of a conjecture and the construction of a proof of that same conjecture. Argumentation, however, may not mean the same thing to everyone: one mathematician's argumentation may be quite semantic, while another's may be more syntactic.

A distinction should also be made between *proving* and *proof*. Proving is a process, which may include arguments and trains of thought which ultimately lead nowhere. The *proof*, which is the result of this process, will not include such dead ends.[7] The process of proving, again, may be very different for different provers.

2. The Roles of Proof

The difficulty of defining mathematical proof is compounded by the fact that a proof may serve several different roles, depending on the author, the audience, and the style of the proof. These roles are identified by de Villiers [11], building on the work of others [3, 4, 20, 31], as verification, explanation, systematization, discovery, intellectual challenge, and communication [27].

The role of *verification* may be most familiar to research mathematicians; a theorem is not a theorem until it has been verified to be such by the construction of a proof. The role of verification is also presented in mathematics classrooms, but the importance of this role can be missed by students. Indeed, novice proof writers often complain that it is pointless to prove theorems that "everybody knows," or that have already been proven in the past.

The role of *explanation* is often typified by proofs in the classroom. In many classroom contexts, the goal of a proof is not to show *that* a theorem is true, but to explain *why* a theorem is true. Of course, this role is not restricted to the classroom. Mathematicians often value a new, more elementary proof of a well-established theorem for its explanatory power.

There are several examples of the use of proof for *systematization*, Euclid's *Elements* being the most famous. The *Elements* collected together many theorems first proven by earlier Greek mathematicians, and organized them in such a way that the theorems follow from definitions, axioms, and

[7] Harel [24] describes proving as an example of a *mental act*, and defines the result of a mental act as a *way of understanding*. Therefore, Harel describes proof as an example of a way of understanding.

postulates. The *Elements* also show that these definitions, axioms and postulates are sufficient to develop all of what we now call Euclidean geometry. By systematizing areas of mathematics, mathematicians are able to dispense with an overabundance of definitions and axioms, and show that a small set of these capture all of the necessary ideas.

The role of *discovery* may be somewhat unusual. Historically, the theorems of some areas of mathematics, such as non-Euclidean geometry, were arrived at through purely deductive means [11]. Non-Euclidean geometry is a particularly interesting example: by modifying only the fifth postulate, the so-called "parallel postulate," of Euclidean geometry, mathematicians have been able to derive, purely through the use of deductive inference, theorems that describe the geometry of figures drawn on curved surfaces rather than the plane. The discovery role of proof can also be seen in the *cognitive unity*, discussed above, between the arguments used to form a conjecture and the proof of that same conjecture [17, 42].

Mathematicians find the *intellectual challenge* of proofs appealing, as completing a proof can be very satisfying. de Villiers [11] paraphrases George Mallory's famous quotation about his reason for climbing Mount Everest: We prove our results because they are there. Mathematicians and mathematics educators hope that their students enjoy the intellectual challenge of writing proofs, and often try to encourage that enjoyment. The act of proving should not be a chore for our students, or nothing but a means to an end.

Finally, the role of proof as *communication* is emphasized by the fact that proofs are written and read by human beings, and therefore proofs act as a means of communicating mathematical results between parties. Moreover, proofs can illustrate a new approach or technique, which might be just what another mathematician needs to complete his or her own proof of a different theorem.

The roles identified by de Villiers do not constitute an exhaustive list of all the possible roles of proof. Auslander [2] adds the role of proof as *justification of definitions*. For example, he argues that the proof of the Intermediate Value Theorem (see below) justifies the definitions of the real number system. Weber [58] also notes this role in the context of the mathematics classroom, as well as proofs that illustrate a technique, such as mathematical induction or proof by contradiction.

These roles can, and often do, overlap. Mathematicians may write proofs with any of these roles in mind, and the way in which the proof is written depends on the intended role or roles, and on the intended audience. For example, a proof written to verify a theorem, as in a research article, is likely to be very different than a proof of the same theorem written to explain the essential ideas to students. Furthermore, the proofs written for the role of *systematization* are often quite formal, but not necessarily entirely so. Proofs written for the purpose of explanation are likely to be less formal.

Two of de Villiers' roles of proof are worth special consideration. To the research mathematician on the frontiers of mathematics, the primary role of proof is that of *verification*. In the mathematics classroom, however, the primary role of proof is that of *explanation* [32]. Mathematics educators sometimes draw a distinction between "proofs that convince" and "proofs that explain" [20, 31, 63].[8] (Of course, there are proofs that are both convincing and explanatory.) A proof that explains might include explanatory paragraphs that are not strictly necessary to the proof, may leave out details that obscure the main ideas, or might be a completely different proof altogether than a proof that convinces. If the goal of a proof is to convince the reader, the "best" proof might be the one that is the most concise or the most general, but if the goal of a proof is to explain, then the "best" proof is the one that offers the most insight.

For example, the Intermediate Value Theorem is typically justified in first-year calculus courses using a graphical argument.[9] This argument is meant to explain why the theorem must be true, but does not use the precise definition of a continuous function. The Intermediate Value Theorem may also be proved/explained by a thought experiment: If a hang-glider starts her flight by launching from a point 2,000 feet above sea level, and lands at a point at sea level, then at some point during her flight, she must be exactly 1,000 feet above sea level. These arguments are "proofs that explain," and may or may not also be convincing. They are certainly not the most general; the graphical argument found in the appendix relies on a particular, if generic, graph of a particular function[10]; the thought experiment of the hang glider is similarly restricted. Such arguments may contain implicit assumptions not noticed by the reader. For example, the graphical argument

[8]It is to this context that Weber [58] adds two other uses of proof in the mathematics classroom: proofs that justify the use of a definition or axiomatic structure, and proofs that illustrate a technique.

[9]This graphical argument can be found in Stewart [51], many other calculus textbooks, and in the appendix.

[10]Such "generic" proofs will be discussed in more detail below.

for the Intermediate Value Theorem applies only to functions on the real numbers; the theorem is not true for functions on the rational numbers, but this distinction may not be noticed by the reader [54].

A more formal proof of the Intermediate Value Theorem, based on the definition of a continuous function, is more general, but may not offer much insight. A deductive proof of the Intermediate Value theorem requires quite a bit of mathematical training, and students do not typically encounter such a proof until an advanced calculus course. Nevertheless, the Intermediate Value Theorem is used to develop other important results in calculus; the deductive proof is not necessary for these ideas to be developed if the truth of the Intermediate Value Theorem is accepted. Historically, modern deductive proofs of this theorem were not developed until the early nineteenth century, by Bolzano and Cauchy [18, 48], but the theorem was certainly used much earlier, on similar visual evidence as that presented in the graphical argument found in the appendix. Brown [5] argues that the achievement of Bolzano's "purely analytical proof" of the Intermediate Value Theorem was that it proved a theorem *independently* known to be true though convincing visual arguments.

A purely formal proof, as noted above, cannot be very complex without becoming so lengthy as to be incomprehensible to a human reader. Such a formal proof is rarely able to be explanatory, and may only be convincing to the degree that it can be read and understood by the reader or checked by a computer.

3. Students' Conceptions of Proof

As students progress from grade school, to high school, to undergraduate study, and beyond, they become part of several different mathematical communities. Throughout their educational careers, students are exposed to, and participate in, mathematical reasoning, justification, and proof [41]. The mathematical communities that students participate in have very different standards of what constitutes a "proof." For instance, the standard of proof in grade school is very different from that of a high school geometry class, which in turn is different than that of undergraduate mathematics classes. Therefore, students can acquire a wide variety of beliefs about the (subjective) criteria that make proofs valid and what constitutes a mathematical proof.

3.1. Proof Schemes

Harel and Sowder [26, 27] have created a detailed taxonomy of the cognitive characteristics of the act of proving held by students, which they call *proof schemes*.[11] This framework of proof schemes has been influential in mathematics education research and has been used, at least to some extent, in several studies (see, for instance, [25, 33, 45, 66]). Harel and Sowder define *proof scheme* with the following: "A person's (or community's) proof scheme consists of what constitutes ascertaining and persuading for that person (or community)" [27, p.809]. Harel and Sowder use the terms "ascertaining" and "persuading" to describe complementary processes of convincing oneself and convincing others. These processes make up the act of proving a statement. Harel and Sowder's taxonomy consists of seven major types of proof scheme, organized into three broader categories: *external conviction*, *empirical*, and *deductive*.

External conviction proof schemes are possessed by students who are convinced a theorem is true by external factors. Types of proof scheme in this category are *ritual*, *non-referential symbolic*, and *authoritative*. Ritual proof schemes are held by students convinced by the form or appearance of a proof. For example, students who have only seen two-column proofs may be convinced a proof is valid only because it follows a two-column format. Non-referential symbolic proof schemes are held by students who are convinced by symbolic manipulation, but the symbols and manipulations have no potential system of referents. Harel and Sowder [27] give the incorrect algebraic reduction of $(a + b)/(c + b) = (a + \cancel{b})/(c + \cancel{b}) = a/c$ as evidence of a non-referential symbolic proof scheme. The last subcategory of external conviction proof schemes is authoritative proof schemes, which describe the proof schemes of students who are convinced by an external authority, such as an instructor or textbook, that a theorem is true. There is a place for this proof scheme (for instance, many mathematicians who are not expert in elliptic curves are convinced that Fermat's Last Theorem has been proven based on the verification of their peers, not because they have read and understood Andrew Wiles' proof themselves), however, this cannot be used by a student to write his or her own proofs.

Empirical proof schemes describe the proof schemes of students who are

[11]Harel [24] describes proof schemes as *ways of thinking*, which he defines as cognitive characteristics of a mental act (in this case, the mental act of proving).

convinced by evidence from examples or perceptions, rather than logical reasoning. This category includes *inductive* and *perceptual* proof schemes. Inductive proof schemes are evidenced by a student who is convinced by seeing a few examples (or even just a single example).[12] Perceptual proof schemes are held by students who are convinced of a theorem's truth by a rudimentary mental image or perception. Perceptual proof schemes might contain some elements of a deductive proof, but the mental image possessed by the student is not developed enough to involve transformations of the image or the consequences of such transformations.

Students (and mathematicians) who hold *deductive* proof schemes prove and validate theorems by means of logical deductions which are general enough to account for all cases. This category includes *transformational* proof schemes and *axiomatic* proof schemes. When a transformational proof scheme is held, students base their proofs on a fully developed mental image capable of transformations and reasoning about an object in such a way that all the necessary relationships are explored. Transformational proof schemes require a deep understanding of the concepts at hand in order to work with these advanced mental images. It is important to note that mental image, in this context, can be defined as a mental construct which "supports thought experiments and support reasoning by way of quantitative relationships" [56, p.230], and can be thought of as a *concept image* [55] of the objects under consideration. As before, this image does not have to be visual in nature. Transformational proofs contain all the necessary logical inferences that are present for a valid proof of the theorem, but do not organize them into an axiomatic scheme. When a student understands that mathematical theorems are based, at least in principle, on axioms and can make deductions based on an axiomatic system, they are said to have an axiomatic proof scheme.[13] Some further examples of proofs that evoke different proof schemes can be found in the appendix.

Harel and Sowder consider axiomatic proof schemes to be an extension of transformational proof schemes, in the sense that students must first possess

[12] Inductive proof schemes should not be confused with the technique of mathematical induction.

[13] It is important to note that even when individuals possess deductive proof schemes, they may still make errors which invalidate the proofs they write. These errors may spring from misconceptions, faults in logic, or simple carelessness. A list of some common student errors is given in Selden and Selden [49].

transformational proof schemes before developing axiomatic proof schemes, and that students in possession of axiomatic proof schemes can also appreciate the underlying formal constructs [26]. Harel and Sowder differentiate further subcategories of transformational and axiomatic proof schemes, but one in particular is noteworthy for this discussion: the *referential symbolic proof scheme*. This proof scheme is described in the following way: "In the referential symbolic proof scheme, to prove or refute an assertion or to solve a problem, one learns to represent the situation algebraically and performs symbol manipulations on the resulting expressions, with the intention to derive information relevant to the problem at hand" [27, p.811]. If taken to the extreme, the referential symbolic proof scheme is very close to the *formal* meaning of proof.

An individual or community may hold to more than one category of proof scheme, depending on the subject matter and context at hand. For example, a mathematician might use an axiomatic proof scheme while proving something for himself or herself, a transformational proof scheme when proving a theorem to a student, and an authoritative proof scheme when accepting a theorem outside his or her field based on the word of a colleague.

Harel and Sowder's use of the term "proof schemes" underscores their subjective interpretation of the word *proof* in this context: Proof is what establishes truth for a given individual or community. This interpretation is related to that of the above discussion of the subjective nature of proof: The practical meaning of proof implies that what constitutes the act of proving for mathematicians (and their students) is agreed upon by the community itself. That said, the proof schemes of the mathematical community have changed greatly over time, and continue to evolve [8]. Roughly speaking, the mathematical community requires deductive proof schemes for the production of proofs, but as Weber [60] has found, mathematicians do not exclusively rely on deductive methods when determining the validity of a proof.

As discussed above, in other communities, different proof schemes may be used. In an elementary school classroom, the weight of several examples might be enough to "prove" that the sum of two even numbers is always even, for example. When these same students reach high school geometry classes, they may try to use similar empirical evidence to justify theorems [9], but such a proof scheme is not generally acceptable in these communities. Furthermore, as Recio and Godino [45] point out, the (informal) arguments used in daily life to convince self and others of a proposition may be quite different than those produced by deductive reasoning, and students may have difficulty distinguishing the two. For example, empirical evidence may be convincing in many "everyday" cases, but may not be convincing in the context of a mathematical community.

Therefore, proof schemes can be thought of as an example of a *sociomathematical norm* [15]. This term, coined by Yackel and Cobb [64], refers to the mathematical practices and standards developed (explicitly or implicitly) by a mathematical community. Clearly, the sociomathematical norms (including proof schemes) of different communities may vary widely. In the context of a particular classroom, sociomathematical norms are influenced by the textbook used, the feedback and comments from the instructor, and the opinions and beliefs of peers, and other, more subtle, factors. However, these influences often contradict each other. Furthermore, the sociomathematical norms (including standards of proof) are rarely explicitly discussed in courses which require proof [15]. Rather, the sociomathematical norms are usually derived implicitly from instructor feedback or comments in class. For example, Harel and Rabin [25] identify teaching practices (such as guiding students toward the teacher's preferred solution approach, a low level of interaction between students, and a lack of deductive justifications used in class) that are likely to foster *authoritative* proof schemes in students.

The beliefs about proof held by teachers can have a profound effect on the beliefs held by students. Knuth [37] found that some high school teachers may not have a clear concept of the generality of a proof, or use empirical evidence when evaluating the validity of a proof. It is possible that these beliefs may influence the proof schemes of these teachers' students.

4. Pedagogy and Proof

When discussing the pedagogy of teaching the ideas and techniques of mathematical proof, it is useful to reconsider our original question, "what do we mean by mathematical proof?" One goal of our teaching could be to help our students, at least at the undergraduate level, to begin to think like mathematicians.[14] However, it is not obvious what it means to "think like a mathematician."[15] As discussed above, mathematicians may have a

[14]As Weber [61] points out, it is unlikely that most students will be able to develop the deep intuition and conceptual understanding of mathematical domains relied on by mathematicians in the course of a single semester.

[15]Entire books have been written on the subject; see for instance, [6].

formal meaning, a practical meaning, or some combination of the two in mind when they discuss proof. Furthermore, mathematicians may hold to different meanings of proof depending on the intended role of the proof. Finally, both syntactic and semantic reasoning may be used to successfully prove theorems, and mathematicians, teachers, and students often find that they prefer syntactic reasoning over semantic, or vice-versa. It is reasonable to assume that those who prefer syntactic reasoning may be drawn to a formal meaning of proof, while those disposed to semantic reasoning may prefer the practical meaning of proof. As both types of reasoning are viable, it is important for mathematics instructors to be aware of, allow for, and even to encourage both types of reasoning, regardless of the instructor's own preferences and personal meaning of proof.

As the mathematics curriculum in the United States currently stands, students are generally insulated from the practical meaning of proof in mathematics courses at the secondary school level. Students are traditionally introduced to the term "mathematical proof" in high school geometry classes, where they are taught that proofs are highly organized, usually using two columns, in which each line must be deduced from the lines above using some "rule of geometry" [30]. In short, high school students are introduced to the formal meaning of mathematical proof.[16]

As Herbst [30] points out, this way of including mathematical proof at the high school level is advantageous in that it allows students to acquire a generalized notion of proof and brings stability to the geometry curriculum. However, at the same time, this method of formalization divorces the formal act of proving from the construction of knowledge, and as a result, high school students are implicitly asked to focus only on the former at the expense of the latter. Thus, students can come to believe that proving is a purely formal activity, and may only become engaged in proving when explicitly asked to do so, and even then, only when certain prerequisites are given [29]. Students may or may not be concerned with the questions that mathematicians are concerned with, such as whether the proof is valid or convincing [29]. While high-school geometry students may be encouraged to reason using diagrams and other less formal ideas by some instructors, other instructors may discourage these ideas, at least in the discussion of mathe-

[16]This way of introducing geometric reasoning and proof is not universal. In China, for example, teaching practices reflect the influence of Confucianism [36].

matical proof. This discouragement is reflected in the common mantra, "A picture is not a proof" [34, 35].

One stated reason for including mathematical proof in high school geometry classes is to help students understand logical reasoning, with the goal of developing the ability to apply such reasoning in broader contexts. However, it is unclear that this goal is achieved if students only associate proof with formal, two-column proofs.

After completing a high-school geometry class, students are rarely explicitly asked to write mathematical proofs again unless they study mathematics as undergraduate students. Even at the undergraduate level, students are often taught that proofs are written by applying logical rules to successive lines of a proof. In other words, students are often taught to prove syntactically. Furthermore, the proofs that are shown to students as examples are usually highly polished. The processes used by mathematicians are often rough and informal, but students typically see proofs in their final forms, and rarely witness the process of creating a "rough draft." As a result, students often do not know where to begin when writing their own proofs [40].

It is often simpler to focus on syntactic proof production when introducing proofs to students. The skills and knowledge required for semantic proof production are far more complex than for syntactic proof production. However, the scope of theorems that can be proved syntactically may be more limited than that of semantic proof production, and syntactic proofs can sometimes leave the proof writer feeling like he or she does not have a complete understanding of the concept [63]. On the other hand, Harel and Sowder argue that students who can reason intuitively but not formally are restricted in the scope of theorems they can prove [26], and intuitive ideas can certainly be incorrect.[17] Thus, it would seem that both formal and intuitive ways of thinking are valuable. Indeed, as Weber and Alcock point out [61, 63], and as discussed above, successful reasoning can be carried out both by relying on the logic and formal structures of syntactic reasoning, and by relying on the informal representations of mathematical objects of semantic

[17]Harel and Sowder give the example of a finite geometry as a context in which intuition alone is inadequate for proving theorems [27] Certainly, this is true for students beginning to work in an unfamiliar area of mathematics that bears little resemblance to our "usual" contexts. However, it is likely that mathematicians who work in such areas of mathematics so, in fact, develop such intuitions; for example, Thurston [57] describes his development of intuitions about hyperbolic three-manifolds.

reasoning.

Several proposals to improve students ability to write proofs seek to do so by helping students connect the *meaning* of mathematical concepts to the proofs they write. Thus, these proposals intend to help students to develop a semantic way of proving, intended to convince the reader, which can complement students' previous experience with syntactic reasoning.

Raman [44] has proposed that it is necessary for students to be aware of both the public and private aspects of a proof, and the *key ideas* that bridge the two. Private aspects include the mental structures that are used to formulate ideas, while public aspects are the structures needed to complete the proof and make it rigorous. A "key idea" is a heuristic idea that is developed enough that it can be made into a proof (or part of a proof) by applying appropriate rigor (by which Raman means "sufficient rigor for a particular mathematical community" [p.320]). By focusing on the key ideas of a theorem, Raman suggests that students can use those key ideas to help structure a proof.

Some mathematicians and mathematics educators have proposed that alternative forms of proof may be more useful to beginning proof writers. Among these are generic proofs [52], in which theorems are proved by generalizing from examples. A generic proof reasons from a particular example that acts as a "prototype" for a class of examples, and should be interpreted as representing a general situation. The graphical argument for the Intermediate Value Theorem, described above, is an example of a generic proof: The graph provided is a *particular* graph, but it represents *any* graph that is continuous from a to b [54]. The values of a, b, $f(a)$, $f(b)$, etc., are not important, and these values are not stated. In the version given in the appendix of this article, a "generic" graph of a function has been drawn, but the details of this graph are not important. It is not important, for example, that the value of $f(a)$ be less than that of $f(b)$, or that the function be differentiable everywhere. Harel [23] considers these generic proofs to fall under the heading of transformational proof schemes.

Another proposed alternative is structural proof [38]. Structural proofs are intended to be proofs that both convince and explain. The term "structural" describes a process of organizing the proof into conceptual levels, in which one level is built upon another. The first level provides a global view of the proof, while 'deeper' levels fill in the details. A structural proof includes, along with its "formal" argument, informal practices such as the inclusion of a short overview of the proof, and frequent explanatory passages in the proof itself. Once a beginning student has mastered these informal proofs, then the process of making proofs more rigorous can begin. A detailed exposition of generic and structural proofs is given in Alibert and Thomas [1].

In the same paper, Alibert and Thomas argue for the use of scientific debate in the classroom. That is, students are encouraged to challenge the arguments of their peers in class, providing students with first-hand experience as to what arguments are necessary to convince others of the truth of a proposition. This experience, in turn, helps students to see proof as an integral part of mathematics rather than a frustrating, unnecessary exercise.

A related method of classroom instruction is the so-called *modified Moore method*, sometimes referred to as Texas-style instruction, inquiry-based learning, or discovery learning. In this method of instruction, based on the teaching style of R. L. Moore, an instructor will introduce definitions and perhaps a few examples, but all theorems in the material will be proved by the students. Students will present proofs of theorems in class, while other students critique them. Only after a proof has been accepted by the entire class (including the instructor) will the class move on to the next theorem. If the students are unable to prove a theorem, the instructor may provide a simpler theorem, or a few very general hints, but the main points of the proof come from the students.[18] Students learning under the modified Moore method are exposed to the social aspects of proof, especially the notion that proofs must be accepted as such by a mathematical community. A recent study of prospective secondary teachers' perceptions of mathematical proof found that students in a modified Moore method classroom "held more humanistic perspectives of mathematical proof than those in traditional sections" [65, p.302]. According to this study, students taking classes using the modified Moore method were more likely to define proof as a convincing and logical argument, rather than as a formal argument, were more willing to accept different forms of proof as valid, saw a greater variety of roles of proof, and emphasized the explanatory role of proof more than their counterparts in traditional classes.

New technology has impacted mathematics teaching on all levels, and the teaching of proof is no exception. In high school level classes, the teaching of geometry and geometric proofs have been enhanced by *dynamic geometry*

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[18] A more complete description of the Moore method, especially as it is used in undergraduate courses, is given by Mahavier [39].

environments, or DGEs. Popular DGEs include Geometer's Sketchpad and Cabri Geometry. These software packages allow students to explore geometric objects and environments, test conjectures, and 'discover' new properties of geometric objects [21]. These DGEs may help to create cognitive unity [17] between the conjectures students create and the proofs of those theorems. Thus, they may help students to connect informal arguments made during the formulation of conjectures to more formal arguments made in proving those conjectures.

5. Concluding Remarks

Because mathematical proof is a human endeavor, and because purely formal proofs of interesting theorems are too lengthy to be of value to human readers, the question of "what, exactly, is a proof?" becomes a question for communities of mathematicians, educators, philosophers, and students to debate. Different communities may agree on different answers to this question; the worldwide community of mathematicians may never entirely agree on a single answer to this question. Certainly, for mathematics students and mathematicians, deductive reasoning, definitions, and axioms are all crucial to proving, but so are the meanings of mathematical objects and ideas. The proofs written by the members of a community are intended to convince other members of the same community, according to these implicitly agreed upon standards. That these standards cannot be explicitly stated does not make them any less real or important.

Complicating the matter is the fact that mathematical proof may be used in different, but related, roles. The intended role of a proof often affects how a proof is written, and we may be more comfortable with relaxing the level of formality of a proof in certain contexts. This might be most clearly seen in the distinction between "proofs that convince" and "proofs that explain." In the context of research mathematics, a proof that convinces can be quite formal; such a proof is convincing to the degree that we trust that the formal definitions and axioms adequately frame the problem and that the author has correctly followed rules of deduction. In the context of the classroom, a proof that explains is often preferred. Such a proof may not necessarily rely on definitions, axioms, and formal reasoning, but will attempt to communicate meaning and understanding to the students.

Because we are unable to precisely define what a proof is, we should not be surprised that our students often fail to successfully meet our (implicit and evolving) standards of proof, both in recognizing proofs and in creating their own proofs. Some students fail to meet our standards of *rigor*. For example, some students may consider empirical arguments to be "good enough," if they possess an inductive proof scheme. Other students may exhibit a non-referential symbolic proof scheme or ritual proof scheme by focusing too closely on formalization, and miss the meaning of the formalized objects they are working with. One challenge for educators is to help students to understand the value of formal, axiomatic proofs, while at the same time helping students to connect meaning to these formal constructions. Since both syntactic and semantic proof production are viable ways of writing proofs for mathematicians, it stands to reason that some students will be more successful if encouraged to reason syntactically, while others may reason semantically. Mathematics classes that introduce proof have tended to focus on syntactic reasoning. However, some recent efforts are underway to help students understand the meaning of the syntax and, in so doing, help students to reason semantically as well as syntactically.

Hersh has written, "In the mathematics classroom, the motto is: 'Proof is a tool in service of teacher and class, not a shackle to restrain them.' In teaching future mathematicians, 'Proof is a tool in service of research, not a shackle on the mathematician's imagination'" [32, p.60]. Ultimately, if we pretend in the mathematics classroom that mathematical proof only has a *formal* meaning, and can only be created through formal, syntactic reasoning, we treat proof as if it is a shackle. By doing so, we deceive ourselves, and worse, we deceive our students.

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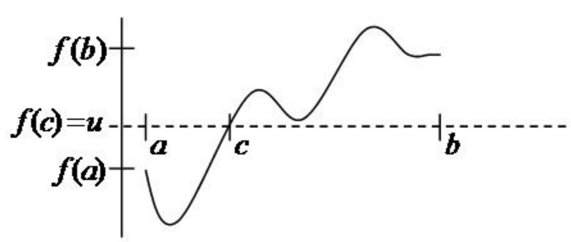
Appendix A. A graphical proof of the Intermediate Value Theorem

The Intermediate Value Theorem states that if a function f is continuous on a closed interval $[a, b]$, and u is any number between $f(a)$ and $f(b)$ inclusive, then there exists at least one number c in the closed interval $[a, b]$ such that $f(c) = u$. The graph below explains why this theorem must be true.



Clearly, in order for the function to reach $f(b)$, it must pass through a point with a value of u. The only way to avoid doing so would be to "jump" over the value u, but a function that contains such "jumps" is not continuous.

Appendix B. Some examples of proofs using different proof schemes

Below are five proofs of a theorem of set theory: if $A \subseteq B$, then $A \cup B = B$. These are meant to serve as examples of proofs that might be written, or accepted as "convincing," by individuals with different proof schemes.

Symbolic non-referential:

> If $A \subseteq B$, then $B \supseteq A$.
> Thus, $A \cup B = B \cap A$.
> Since $A \cap B \subseteq B$, $B \subseteq B \cup A$.
> Therefore, $B = A \cup B$, and $A \cup B = B$.$\square$

This (nonsensical) argument is "symbolic non-referential," as the mathematical symbols used cannot have any mathematical meaning.

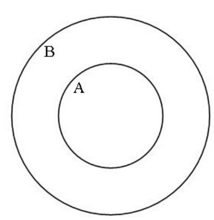
Inductive:

> Suppose that $A = \{a, b, c\}$, and $B = \{a, b, c, d, e\}$.
> Then $A \subseteq B$.
> Also, $A \cup B = \{a, b, c, d, e\} = B$.
> Therefore, if $A \subseteq B$, then $A \cup B = B$.$\square$

This argument is inductive; it determines the conclusion based on a single example.

Perceptual:

> In this diagram we can see that $A \subseteq B$:



> From this diagram, we can also see that $A \cup B = B$.$\square$

This argument is perceptual, as it argues from a particular diagram. This diagram is not flexible enough to be "transformational," but a student making this argument may be able to develop this argument further.

Transformational:

> If $A \subseteq B$, then A is completely contained in B.
> Thus, A does not add anything to B when A and B are "unioned" together.
> Thus, $A \cup B$ is contained in B.
> Also, B is contained in $A \cup B$, so $A \cup B = B$.$\square$

This argument is transformational, as it relies on a mental image capable of incorporating the necessary ideas.

Axiomatic:

> Let x be in $A \cup B$.
> If $x \in A$, then, since $A \subseteq B$, $x \in B$.
> Therefore $A \cup B \subseteq B$.
> Also, if $x \in B$, then $x \in A \cup B$ by definition.
> Therefore $B \subseteq A \cup B$.
> Since $A \cup B \subseteq B$ and $B \subseteq A \cup B$, $A \cup B = B$.$\square$

This argument is axiomatic, in that it relies (at least in part) on the definitions and axioms of the relevant mathematical objects.