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**Abstract**: This document describes how one can derive the solutions to a linear constant coefficient homogeneous differential equation with repeated roots in the characteristic equation with Abel's Theorem.

## 1 Introduction

The $n^{th}$ order homogeneous linear differential equation

$$y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + p_2(t)y^{(n-2)}(t) + \cdots + p_{n-1}(t)y'(t) + p_n(t)y(t) = 0 \qquad (1.1)$$

is encountered by every differential equations student. We most often study second order equations, the case when $n = 2$, due to Newton's laws of motion and with ease one can generalize the theory to higher order.

We consider the case when the coefficient functions $p_i(t)$ are constant. Under these conditions, one makes the ansatz $y = e^{rt}$ which transforms (1.1) into an algebraic equation, which is more easily solved. In particular, if $p_i = a_i$, with the above ansatz, we have the characteristic equation

$$r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_{n-1}r + a_n = 0. \qquad (1.2)$$

By factoring out the $e^{rt}$, one finds that solving an algebraic equation yields solutions to the original differential equation. Most textbooks first handle distinct real roots and later come back to discuss when $r = r_1$ is a repeated solution to (1.2). We show an approach that relies on Abel's Theorem and the idea of reduction of order can constructively establish that the repeated root $r = r_1$ with multiplicity $m$ gives rise to $m$ linearly independent solutions of the form $y_i(t) = t^{i-1}e^{r_1 t}$ for $1 \le i \le m$.

With this approach, one can reduce (1.1) to a lower order problem which is easily solved. We first establish some preliminaries and the reduction of order method outlined in [3]. Recall that if $y_1, \ldots, y_n$ are $n$ solutions to (1.1), one can use the Wronskian to determine if the solutions are linearly independent.

**Definition 1.1.** The Wronskian of $n$ functions $y_1, \ldots, y_n$ is given by

$$W(y_1, \ldots, y_n)(t) = \begin{vmatrix} y_1(t) & y_2(t) & \ldots & y_n(t) \\ y_1'(t) & y_2'(t) & \ldots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \ldots & y_n^{(n-1)}(t) \end{vmatrix}. \tag{1.3}$$

**Theorem 1.2** (Abel's Theorem). *Assume that $p_i(t)$ are continuous on some interval $I$ for $1 \leq i \leq n$, the Wronskian of $n$ solutions $y_1, y_2, \ldots, y_n$, to (1.1) is given by*

$$W(y_1, y_2, \ldots, y_n)(t) = Ke^{-\int_{t_0}^{t} p_1(s)\, ds}.$$

*This formula is valid for any $t, t_0 \in I$, the constant $K = W(y_1, \ldots, y_n)(t_0)$ depends only on the solutions $y_1, \ldots, y_n$, and is independent of $t$. $K = 0$ if and only if the functions $y_1, \ldots y_n$ are linearly dependent.*

This theorem is often introduced but not used in any non-trivial sense in introductory differential equations courses. In [3], it is shown how one can use Abel's Theorem to quickly and cleanly derive the reduction of order technique for second order equations. It is also shown that the approach extends to higher order equations which we use in Section 2.2. Here we show another use of Abel's Theorem for introductory differential equations courses.

# 2 Repeated roots of the characteristic equation

## 2.1 Second Order

In the case of a root or order two to the characteristic equation, we often show our students that two linearly independent solutions to

$$y'' - 2r_1 y' + r_1^2 y = 0 \tag{2.1}$$

have the form $y_1 = e^{r_1 t}$ and $y_2 = te^{r_1 t}$. Most texts posit a solution of the form $y_2(t) = u(t)e^{r_1 t}$ and derive a differential equation for $u(t)$. We show that this problem can be easily solved via reduction of order with Abel's theorem.

The characteristic equation for (2.1) is

$$(r - r_1)^2 = 0.$$

At this point, it is clear that $r = r_1$ is a root of the characteristic equation which corresponds to the solution $y_1 = e^{r_1 t}$. However, since there are two linearly independent solutions to (2.1), we must find another solution. We now demonstrate how one can use Abel's Theorem to find the other solution,

$$\begin{vmatrix} e^{r_1 t} & y \\ r_1 e^{r_1 t} & y' \end{vmatrix} = W(e^{r_1 t}, y) = K \exp\Big( - \int -2r_1\, dt \Big).$$

Cleaning up a bit, and without loss of generality taking $K = 1$, we see

$$y' - r_1 y = e^{r_1 t}.$$

The integrating factor for this equation is clearly $\mu(t) = e^{-r_1 t}$, so that upon integrating we have

$$ye^{-r_1 t} = t + C,$$

and the new linearly independent solution is

$$y(t) = te^{r_1 t},$$

as claimed.

## 2.2 Higher Order

Oftentimes, we do not explain why higher order repeated roots yield solutions of the form $t^\ell e^{r_1 t}$. Here, we show that Abel's Theorem can be used to explain this as well. Our approach is inductive and follows by connecting a homogeneous equation to a lower order non-homogeneous equation. Once we have done this, we use Abel's Theorem to establish the correct form of the solution. Our approach can be thought of as a sort of converse to the method of annihilators, which connects lower order non-homogeneous equations to higher order homogeneous equations.

We wish to show that the the functions $y_i(t) = t^{i-1}e^{r_1 t}$ for $1 \leq i \leq n+1$ compose a fundamental set of solutions to

$$\begin{aligned} y^{(n+1)} + \binom{n+1}{1}(-r_1)y^{(n)} + \binom{n+1}{2}(-r_1)^2 y^{(n-1)} + \dots \\ + \binom{n+1}{n}(-r_1)^n y' + (-r_1)^{n+1} y = 0. \end{aligned} \tag{2.2}$$

We proceed in an inductive manner. We take the case of a double root, (2.1) with its solutions $e^{r_1 t}$ and $te^{r_1 t}$ as the base case. Now, for the inductive hypothesis, assume that if the characteristic equation has an $n$-fold repeated root that it gives rise to the solutions $y_i(t) = t^{i-1}e^{r_1 t}$, $1 \leq i \leq n$. In particular, we note that these are the solutions to the differential equation

$$y^{(n)} + \binom{n}{1}(-r_1)y^{(n-1)} + \dots + \binom{n}{n-1}(-r_1)^{n-1}y' + (-r_1)^n y = 0. \tag{2.3}$$

The goal is now to show that an $(n+1)$-fold repeated root will give rise to the additional solution $y_{n+1}(t) = t^n e^{r_1 t}$. As an $(n+1)$-fold root is also an $n$-fold root, we have the solutions $y_i(t) = t^{i-1}e^{r_1 t}$ for $1 \leq i \leq n$ from the inductive hypothesis. Abel's Theorem and the inductive hypothesis now tell us that the Wronskian of these functions, which are solutions to the equation (2.3), is

$$W(y_1, \dots, y_n) = Ke^{nr_1 t}. \tag{2.4}$$

We now try to find the final solution to (2.2), where there is an $(n+1)$-fold repeated root. Having $n$ linearly independent solutions (2.2), a reduction of order for $n+1^{st}$ order equation yields a final solution of the form

$$y_{n+1}(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{W_m(t)e^{(n+1)r_1 t}}{[W(t)]^2}\, dt, \tag{2.5}$$

with $W(t)$ the Wronskian of the $n$ solutions $y_j(t)$, $1 \leq j \leq n$ and $W_m(t)$ the 'Cramer's rule Wronksian,' the determinant in (1.3) with the $m^{th}$ column replaced with $(0, 0, \ldots, 0, 1)^T$. This is done by reducing the $n+1^{st}$ order homogeneous equation to an $n^{th}$ order non-homogeneous equation. Writing out the Wronskian as a determinant and applying Abel's Theorem yield the non-homogeneous problem

$$\begin{vmatrix} y_1(t) & \ldots & y_n(t) & y_{n+1}(t) \\ y_1'(t) & \ldots & y_n'(t) & y_{n+1}'(t) \\ \vdots & \ddots & \vdots & \vdots \\ y_1^{(n)}(t) & \ldots & y_n^{(n)}(t) & y_{n+1}^{(n)}(t) \end{vmatrix} = Ke^{(n+1)r_1 t}. \tag{2.6}$$

The key fact is that the functions $y_i(t)$, $1 \leq i \leq n$ form a fundamental set of solutions to the associated $n^{th}$ order homogeneous equation. One can then apply the standard method of variation of parameters to obtain (2.5).

It is clear that $W_m(t) = P_m(t)e^{(n-1)r_1 t}$ with $P_m(t)$ a polynomial as every entry in $W_m(t)$ is a polynomial multiplied by $e^{r_1 t}$, except for the column replaced by $(0, \ldots, 0, 1)^T$. Thus, we can express (2.5) as

$$y_{n+1}(t) = \sum_{m=1}^{n} t^{m-1} e^{r_1 t} \int_{t_0}^{t} P_m(t)\, dt = Q(t)e^{r_1 t},$$

for some polynomial $Q(t)$. This follows by the induction hypothesis in (2.4) to obtain $W(t) = Ke^{nr_1 t}$, so that all the exponentials in the integrand cancel. By the principle of superposition, the new solution must be of the form

$$y_{n+1}(t) = t^{\ell} e^{r_1 t}$$

for some integer $\ell \geq n$.

Assume that $\ell > n$. Then the coefficient of $t^{\ell-(n+1)}$ is

$$\frac{\ell!}{(\ell-(n+1))!} \neq 0.$$

This power of $t$ only occurs in the $y^{(n+1)}$ term of (2.2). However, this implies that $t^{\ell} e^{r_1 t}$ cannot be a solution, as the $t^{\ell-(n+1)}$ is non-zero and cannot be cancelled out for all $t$ in an interval.

As $\ell$ cannot be greater than $n$, and $\ell$ cannot be less than $n$ as this would be a solution to the homogeneous analogue of (2.6). We see that the final linearly independent solution must be

$$y_{n+1}(t) = t^n e^{r_1 t}.$$

# 3 Conclusions

The proof presented here depends on ideas from the introductory differential equations course, namely the use of Abel's Theorem. The second order case is quicker than the standard method of positing a solution of the form $y(t) = u(t)e^{r_1 t}$ and showing that $u(t) = kt + C$. It does not require the use of partial derivatives as in [1, p. 173]. The direct proof in [2, p. 304] which employs factorization of differential operators is quick and clean, similar proofs are offered in [5, p. 168] and [4, p. 93].

While this method is not quicker than simply positing the solutions should be of the form $y_i(t) = t^{i-1}e^{r_1 t}$ and checking that these solutions hold for small values of $n$, it does have a few benefits. It strongly connects homogeneous equations to non-homogeneous equations of lower order. The method demonstrated here constructively solves the problem through the use of Abel's Theorem and offers a chance for more mathematical thinking instead of a simple "plug and chug" approach.

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