Approval Voting in Box Societies

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Abstract

Under approval voting, every voter may vote for any number of candidates. To model approval voting, we let a *political spectrum* be the set of all possible political positions, and let each voter have a subset of the spectrum that they approve, called an *approval region*. The fraction of all voters who approve the most popular position is the *agreement proportion* for the society. We consider voting in societies whose political spectrum is modeled by d-dimensional space ($\mathbb{R}^d$) with approval regions defined by axis-parallel boxes. For such societies, we first consider a Turán-type problem, attempting to find the maximum agreement between pairs of voters for a society with a given level of overall agreement. We prove a lower bound on this maximum agreement and find in the literature a proof that the lower bound is optimal. By this result we find that for sufficiently large n, any n-voter box society in $\mathbb{R}^d$ where at least $\alpha\binom{n}{2}$ pairs of voters agree on some position must have a position contained in βn approval regions, where $\alpha = 1 - (1-\beta)^2/d$. We also consider an extension of this problem involving projections of approval regions to axes. Finally we consider the question of (k, m)-agreeable box societies, where a society is said to be (k, m)*-agreeable* if among every m voters, some k approve a common position. In the $m = 2k - 1$ case, we use methods from graph theory to prove that the agreement proportion is at least $1/(2d)$ for any integer $k \geq 2$.

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Chapter 1

Introduction

We consider voting in elections, where each candidate must select a set of positions on various issues to be their *platform*. The set of all possible platforms is the *political spectrum*. Many different geometric spaces can arise as political spectra, such as the real line (Berg et al., 2010) or a circle (Carlson et al., 2011).

In this thesis we model the political spectrum by $\mathbb{R}^d$. A simple interpretation of this model is that each dimension represents a separate issue upon which any voter can have any opinion. The standard example in contemporary American politics uses $d = 1$, modeling political positions as points on the real line, ranging from liberal to conservative. Some two dimensional models have also been created, such as the one seen in Figure 1.1, which splits preferences into opinions on governmental control (statism vs. anarchy) and egalitarianism (left vs. right) (Bryson and McDill, 1968). This model is closely related to the Nolan chart, named for Libertarian party founder David Nolan, and provides validation that "libertarianism seem[s] to have a place" because political ideology was complicated "beyond just left and right" (Doherty, 2007). Such splitting of issues could be carried out endlessly to better capture the diversity of potential political positions, with $\mathbb{R}^d$ representing a political spectrum where each platform corresponds to a distinct position on each of d different questions.

With a given political spectrum, an election can be modeled by considering a *society* of voters where each voter has a unique *approval region*, consisting of the set of platforms of which the voter approves. In an approval voting system, where each voter votes for or against each candidate individually, any candidate taking a position in a voter's approval region would expect to receive that voter's vote. One example of a society in $\mathbb{R}^2$

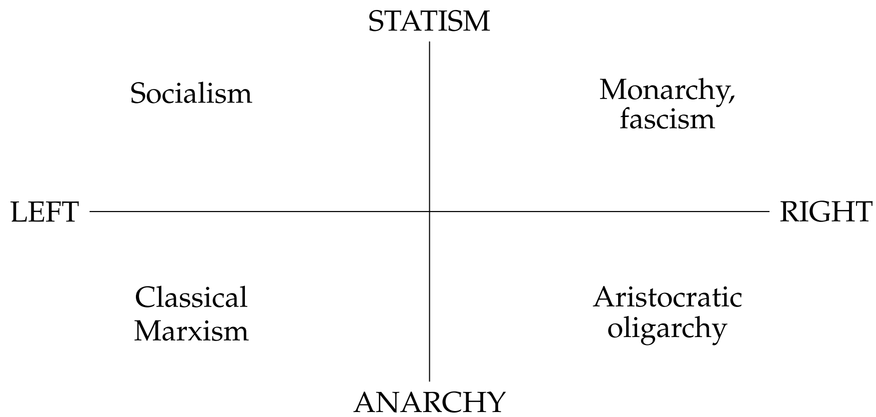

Figure 1.1 A two-dimensional political spectrum proposed by Bryson and McDill (1968).

is shown in Figure 1.2, where each voter's approval region is drawn in a different color. Note that two approval regions consist of two disconnected components.

A common question about such societies is whether there is a platform that is approved of by a particular number or percentage of voters. In other words, is there a point contained in a given number of approval regions? Many of these ideas are first explored by Berg et al. (2010). To deal with this question, we define the *agreement proportion* of a society to be the number of voters who approve of the most popular platform in the spectrum divided by the total number of voters. In other words, the agreement proportion is the maximum fraction of the population that approves a particular platform.

The usual goal when studying such models is to determine conditions that guarantee a known minimal agreement proportion, so that, for example, a candidate in a society with those conditions would know she could position herself somewhere and receive a certain percentage of voter approval. We say that such questions deal with the *agreeability* of the society. The conditions placed on societies usually deal with the shape and distribution of approval regions.

In principle, approval regions could be any subset of the spectrum, but to achieve any meaningful agreeability results, and to model rational voters, some restrictions are generally placed on approval regions. One obvious restriction is that approval regions should be convex, corresponding to

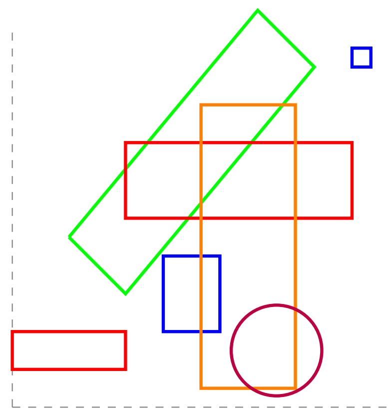

Figure 1.2 A five-voter society in $\mathbb{R}^2$.

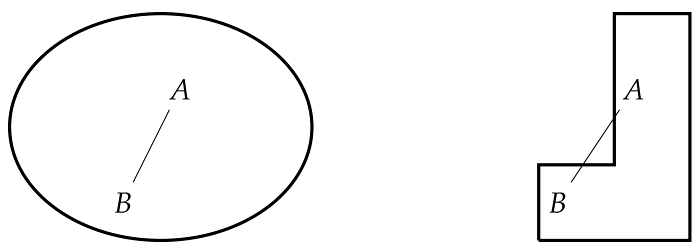

Figure 1.3 Convex and nonconvex approval regions. The lines show compromise platforms between approved platforms A and B.

the idea that a voter must approve of any platform that is a compromise between two approved platforms, as seen in Figure 1.3. Another possible restriction for the spectrum $\mathbb{R}^d$ is that approval regions must be *boxes*, here defined as the Cartesian product of d intervals in $\mathbb{R}$. Such a restriction would correspond to the idea that each axis is an independent issue upon which a voter must approve a particular set of positions, with the overall approval region being simply the set of points for which all of those individual positions are satisfied. We will briefly consider general convex societies but focus mostly on box societies.

To establish most results about agreeability, it is also necessary to formulate a hypothesis about the distribution of approval regions. That is, some conditions must be placed on the voters which will then imply that the society has some minimal agreement proportion. We will consider two different sorts of distribution hypotheses.

In Chapter 2 we connect the voting problems described above to graph theory and develop terminology for discussing graphs and societies.

In Chapter 3 we introduce one distribution condition to place on societies, based on the percentage of pairwise intersections of approval regions that are nonempty. We explore a Turán-type question that this condition generates briefly for general convex societies and more thoroughly for box societies. The result for box societies, which already existed in the literature, is in Theorem 3.3.

In Chapter 4 we consider an extension of the Turán-type question, where we consider the pairwise-agreement proportion of projections of box societies onto the coordinate axes, rather than in $\mathbb{R}^d$ itself. Our main result for this chapter is in Theorem 4.2.

In Chapter 5 we suggest a different distribution condition that looks at intersections of subsets of voters, and extend known results to new cases. The extended result is in Theorem 5.7

In Chapter 6 we consider possible future directions for research on voting in box societies in $\mathbb{R}^d$.

Chapter 2

Connecting to Graph Theory

To transform a problem about voter approval into a problem about graphs, we need to create graphs that correspond to societies. First we define a *society* to be a political spectrum together with a set of approval regions representing the voters, where each approval region is a subset of the spectrum. We let a *convex society* be a society in which all approval regions are convex, and let a *box society* be a society in which all approval regions are boxes.

Now the natural way to construct a graph to represent a society is to simply consider the intersection graph corresponding to a set of regions $\{A_i\}$ in $\mathbb{R}^d$, where each vertex v_i corresponds to a region A_i and the edge $v_i v_j$ is in the graph if and only if $A_i \cap A_j \neq \emptyset$. An example of such a graph appears in Figure 2.1.

In this chapter, we explore this connection to graph theory, considering the Helly property and its relation to convex and box societies. We also establish some terminology, useful concepts, and basic results of graph theory.

2.1 Graph Terminology

Because we are using intersection graphs, we need only consider simple graphs, with no loops, multiple edges, or directed edges. Unless otherwise specified, G will refer to such a graph, with vertex set $V(G)$ and edge set $E(G)$. We let $n = |V(G)|$ be the number of vertices and $e(G) = |E(G)|$ be the number of edges in the graph G. Given a subset W of $V(G)$, let $G[W]$ be the subgraph induced by W, with vertex set W and all edges in $E(G)$ with both endpoints in W.

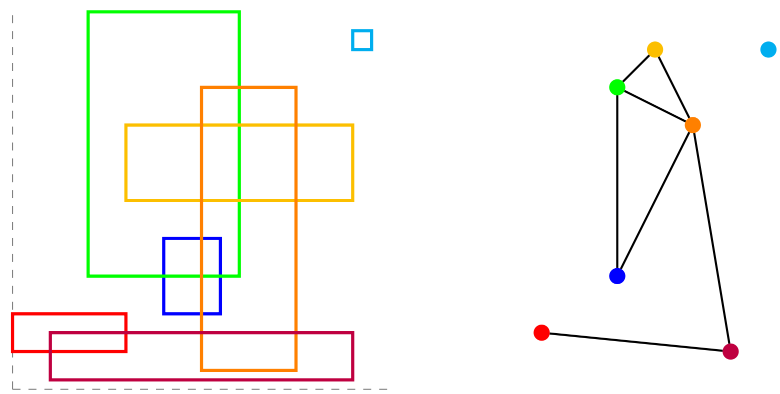

Figure 2.1 A set of approval regions in $\mathbb{R}^2$ and corresponding intersection graph.

For any vertex $v \in V(G)$, the *degree* of v, denoted $\deg(v)$, is the number of edges incident to v. A *clique* in G is a subset of vertices that induces a complete subgraph. We denote the size of the largest clique in G by $\omega(G)$, the *clique number* of G.

2.2 The Helly Property and Convex Approval Regions

To analyze mutual intersections of approval regions, we want to find a feature of the intersection graph that corresponds to such intersections. Unfortunately, as seen in Figure 2.2, two sets of convex approval regions may have the same intersection graph despite one having a mutual intersection of more regions. This corresponds to the idea that collections of convex sets in $\mathbb{R}^d$ have Helly number $d + 1$. We define a family of sets $\mathcal{B}$ to have *Helly number* h if for a finite subfamily $\mathcal{F} \subseteq \mathcal{B}$ in which every h or fewer sets of $\mathcal{F}$ have a common point, then $\bigcap \mathcal{F} \neq \emptyset$ (Matoušek, 2002).

Thus Helly's theorem only refers to pairwise intersections for $d = 1$, and we see that for $d > 1$, sets of pairwise intersecting regions do not necessarily have a point in all sets. Since intersection graphs only record pairwise intersections, they are not useful in analyzing general convex approval regions. Because we are primarily interested in pairwise intersections, we will say that a family of sets has the *Helly property* if it has Helly number 2.

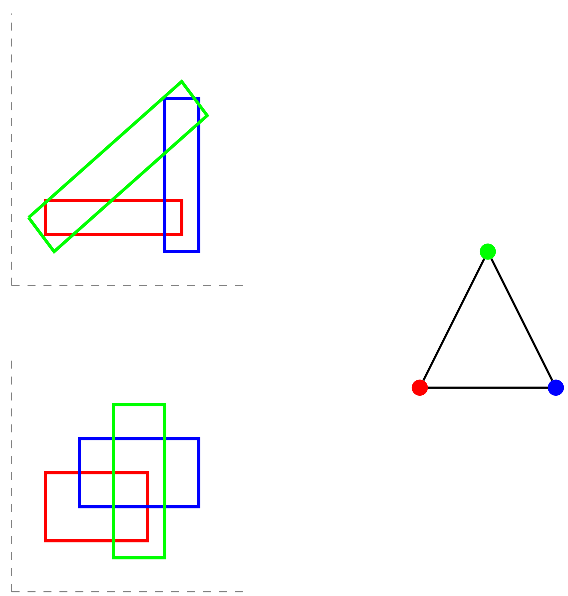

Figure 2.2 Two different sets of convex approval regions with the same intersection graph. Note that one set has an intersection of three regions while the other does not.

2.3 The Helly Property and Box Societies

We therefore consider box societies. To see the advantage provided by boxes, we first note that

Lemma 2.1. *A point* $x = (x_1, \ldots, x_d)$ *is in* $B = \bigcap_{i=1}^{k} B_i$ *for boxes*

$$B_i = (a_{i1}, b_{i1}) \times \cdots \times (a_{id}, b_{id})$$

if and only if

$$a_{ij} < x_i < b_{ij}$$

for all $1 \leq j \leq d$ *and* $1 \leq i \leq k$*.*

Proof. Clearly if $a_{ij} < x_i < b_{ij}$ for all $1 \leq j \leq d$ and $1 \leq i \leq k$, then $x \in B$. Furthermore, if there exist i' and j' such that $1 \leq j' \leq d$ and $1 \leq i' \leq k$ such that $x_{i'} \leq a_{i'j'}$ or $x_{i'} \geq b_{i'j'}$, then $x \notin B_{i'}$, and so $x \notin B$. □

Lemma 2.1 allows us to prove

Theorem 2.1. *Boxes in* $\mathbb{R}^d$ *have the Helly property. That is, any set of boxes for which every pair intersects must contain a point in all boxes.*

Proof. We first recall that intervals on the real line have the Helly property, so any set of intervals for which every pair intersects contains a point in all intervals. Now consider a set X of pairwise intersecting boxes in $\mathbb{R}^d$ and let X_i be the set of projections from those intervals to the x_i-axis for $1 \leq i \leq d$. By Lemma 2.1, X_i is a set of pairwise intersecting intervals, which therefore has a point x_i in all intervals. By the other direction of Lemma 2.1, the point $x = (x_1, \ldots, x_d)$ is in the intersection of all the boxes of X. □

Theorem 2.1 implies that any set of k pairwise intersecting vertices (a clique of size k) in the intersection graph for a box society corresponds to a nonempty mutual intersection of k approval regions. Therefore results about clique size of intersection graphs can be directly translated to results about mutual intersections of approval regions.

2.4 Boxicity

One other concept is important for connecting box societies to intersection graphs, allowing any simple graph to be interpreted as a box society in $\mathbb{R}^d$ for some $d > 1$. This idea, developed by Roberts (1969), is the *boxicity*

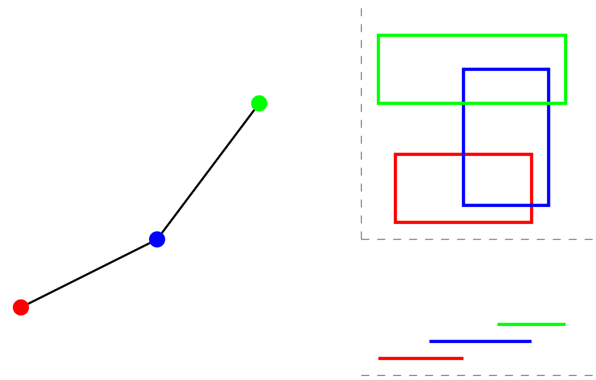

Figure 2.3 A graph with corresponding collections of boxes in $\mathbb{R}^2$ and $\mathbb{R}^1$.

of a graph G, which is the minimum dimension necessary to construct a collection of boxes with intersection graph G.

For example, the graph G in Figure 2.3 is shown with a collection of boxes in $\mathbb{R}^2$ and a collection of intervals in $\mathbb{R}$, both of which have G as their intersection graph. Thus G has boxicity at most 1. By convention, the only graph with n vertices of boxicity 0 is the complete graph K_n, which can be represented by n copies of a single point (a box in $\mathbb{R}^0$).

Roberts (1969) shows that every graph has finite boxicity, so we let $\text{box}(G)$ denote the boxicity of the graph G.

Chapter 3

A Turán-type Problem

In this chapter, we build on Carlson et al. (2011) and Abbott and Katchalski (1979), considering pairwise intersections of approval regions, trying to determine a fraction of such intersections that will guarantee a position approved by some number of the voters. We define the *pairwise-agreement proportion* of a society to be the number of pairs of voters with intersecting approval regions divided by the total number of pairs of voters.

The question leads us to a consideration of Turán's Theorem, a result of extremal graph theory. Using insight from that theorem, we develop a graph of fixed boxicity that attempts to maximize the number of edges while maintaining a given maximum clique size. We then find that this graph already exists in the literature and is proven to be maximal, providing a relationship between agreement proportion and pairwise-agreement proportion.

3.1 The Question

Given the concept of pairwise-agreement proportion, our question can be formulated as

Question 1. *What is the minimal value $\alpha \in [0,1]$ for a given $\beta \in (0,1]$ such that any n-voter society in $\mathbb{R}^d$ in which at least $\alpha\binom{n}{2}$ of the pairwise intersections of approval regions are nonempty must have a platform contained in βn approval regions, for sufficiently large n?*

We note briefly that this question is trivial for general convex approval regions in $\mathbb{R}^d$, using a construction suggested by the top set of three regions in Figure 2.2 that demonstrates an arbitrary collection of n convex voter approval regions need have no point in more than two regions, even if all regions pairwise intersect.

In loose terms, we can construct a society in $\mathbb{R}^2$ where each approval region is a rotated box that is sufficiently large in one dimension and sufficiently small in the other that each region approximates a line. Simply rotating each region to a different angle and translating them appropriately to avoid triple intersections, we can construct a society with $\binom{n}{2}$ nonempty pairwise intersections for which every platform is in at most 2 approval regions. Such a construction can clearly be extended to $\mathbb{R}^d$ simply by adding size in extra dimensions.

From this construction we see that we cannot achieve a Turán-type result stronger than the trivial statement that a single pairwise intersection indicates there is a point contained in two sets, so we turn our attention to box societies.

By Chapter 2, we see that Question 1 for box societies can be reconfigured as

Question 2. *What is the minimal value $\alpha \in [0,1]$ for a given $\beta \in (0,1]$ such that any n-vertex graph G with boxicity at most d and at least $\alpha\binom{n}{2}$ edges must contain a clique of size βn, for sufficiently large n?*

Such an approach is analogous to Turán's theorem, a result of graph theory that states that any graph with a certain number of edges must have a complete graph of a given size as a subgraph. Abbott and Katchalski (1979) consider the question for interval graphs, which corresponds to the case $d = 1$ in Question 1. Notice that in $\mathbb{R}$ convex sets and boxes are both just intervals. Thus their answer to Question 1,

$$\alpha = 1 - (1-\beta)^2, \tag{3.1}$$

applies to any linear society with convex approval regions. Carlson et al. (2011) prove a similar result for a circular political spectrum. The analogy between Question 1 and Turán's theorem suggests that a full understanding of Turán's Theorem and its proof may be helpful for answering Question 1.

3.2 Turán's Theorem

This theorem deals with maximizing the number of edges in a graph with n vertices without creating a clique of size $r+1$. Recall that a *k-partite graph* is a graph whose vertices can be partitioned into k independent sets, where an independent set is a set of vertices with no edges. Define the *complete k-partite graph* $K_{n_1,\ldots,n_k}$ to be the graph that can be partitioned into k independent sets of size $n_1, \ldots, n_k$, respectively, such that all vertices in different independent sets are adjoined by an edge. We now define the Turán graph $T_{n,r}$ to be the complete r-partite graph with n vertices whose partite sets differ in size by at most 1. If we define $a = \lfloor n/r \rfloor$ and $b = n - ra$, $T_{n,r}$ has b partite sets of size $a + 1$ and $r - b$ partite sets of size a (West, 2001).

With these definitions established, Turán's theorem states that

Theorem 3.1. *Among n vertex graphs with no clique of size $r + 1$, $T_{n,r}$ has the most edges. In other words, any n vertex graph with more edges than $T_{n,r}$ must contain a clique of size $r + 1$.*

Turán proved this result in 1941. The proof given here comes from West (2001).

Proof. First notice that no r-partite graph can contain an $(r+1)$-clique, since all members of a clique must be put into different partite sets of any partition into independent sets. Further notice that among r-partite graphs, complete r-partite graphs maximize the number of edges, as any r-partite graph that is not complete can have edges added between partite sets while remaining r-partite.

We now show that among r-partite graphs with n vertices, $T_{n,r}$ has the most edges. By the above we need only consider complete r-partite graphs, so we consider a complete r-partite graph with two partite sets of sizes i and j such that $i > j + 1$. Then we can move a vertex v from the set X of size i to the set Y of size j, which requires deleting the j edges between v and the members of Y and adding $i - 1$ edges between v and the members of X. The other edges remain the same. Therefore there is a net addition of $i - 1 - j > 0$ edges, and the partite sets have moved closer to differing in size by at most 1. Therefore the number of edges is maximized by $T_{n,r}$.

It only remains to be shown that for any graph G with no clique of size $r + 1$ there is an r-partite graph H with the same vertex set and at least as many edges. We proceed by induction on r.

For $r = 1$, any graph with no clique of size $r + 1 = 2$ has no edges, so G and H are the same graph. Now suppose the result holds for $r - 1$, and consider G with no $(r + 1)$-clique. Let x be a vertex of G with degree k, where k is the maximum degree of any vertex in G. Then let G' be the subgraph induced by all the vertices adjacent to x. Since an r-clique in G' would have all its members adjacent to x in G and create an $(r + 1)$-clique in G, G' has no r-clique. By the induction hypothesis, there exists some

$(r-1)$-partite graph H' on the same vertex set as G' with at least as many edges as G'.

Now form H by joining the vertices of H' to the set of vertices S in G but not in G'. That is, add an edge between every vertex of H' and every vertex of S, preserving the edges in H' but not any edges between members of S. Since H' is $(r-1)$-partite and S is an independent set, H is r-partite. We will show that H has at least as many edges as H.

Notice that by construction $e(H) = e(H') + k(n-k)$ and $e(H') \geq e(G')$. Notice that

$$e(G) \leq e(G') + \sum_{v \in S} d_G(v),$$

where $d_G(v)$ is the degree of v in G, since the edges in G but not in G' are all counted at least once by the sum. Because k is the maximum degree of G, we have $d_G(v) \leq k$ for $v \in S$ and $|S| = n-k$, so we have

$$e(G) \leq e(G') + (n-k)k \leq e(H') + k(n-k) = e(H),$$

as desired. □

Notice that if $T_{n,\beta n-1}$ has boxicity at most d, Turán's theorem states that it maximizes the edges being counted in Question 2, so it is important to understand the boxicity of the Turán graph. Luckily the boxicity of complete k-partite graphs is well known, with Roberts (1969: Theorem 7) proving

Lemma 3.1. *The boxicity of the complete k-partite graph $K_{n_1,\ldots,n_k}$ is equal to the number of the n_i that are strictly greater than* 1.

Proof. Let d be the number of n_i that are strictly greater than 1.

Since we are here concerned only with graphs with known upper bounds on their boxicity, we will only prove that the boxicity of $K_{n_1,\ldots,n_k}$ it at most d. We refer to Roberts (1969) for the proof that d is the precise boxicity.

To prove $\text{box}(K_{n_1,\ldots,n_k}) \leq d$ it suffices to construct a set of boxes in $\mathbb{R}^d$ with intersection graph $K_{n_1,\ldots,n_k}$. To do this, we first notice that any partite set of size $n_i = 1$ consists of a vertex adjacent to all other vertices, and so can be represented by a copy of $\mathbb{R}^d$, or alternately a box large enough to encompass all other boxes. Therefore any number of partite sets of size 1 can be accounted for, leaving only the d sets of size greater than 1.

Since we are creating boxes in $\mathbb{R}^d$, we can simply assign one direction to each of these d partite sets and separate the members of the set in the assigned direction. Each box can be made arbitrarily large in all other directions so as to intersect all boxes not part of its partite set. An example of this construction is shown in Figure 3.1.

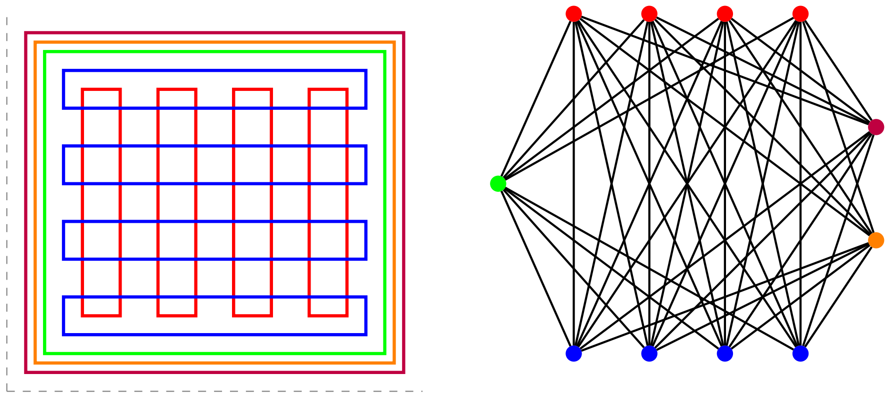

Figure 3.1 A family of boxes in $\mathbb{R}^2$ with intersection graph $K_{4,4,1,1,1}$.

□

Since $T_{n,r}$ is a complete r-partite graph with partite set sizes all near $\frac{n}{r}$, this result implies that for large n and $r > d$ the boxicity of the Turán graph will be greater than d, and so the maximal graph of boxicity at most d will have fewer edges than $T_{n,r}$.

We now use these insights from graph theory to find a lower bound on the maximal pairwise-agreement proportion for a given agreement proportion.

3.3 Lower Bound on Maximal Pairwise-Agreement Proportion

In this section we establish for Question 2 the lower bound

$$\alpha \geq 1 - \frac{(1-\beta)^2}{d} \tag{3.2}$$

for the maximal pairwise-agreement proportion α for a given agreement proportion β. To do this, we construct a graph G with n vertices, boxicity at most d, such that it does not contain a clique of size $r+1$, trying to maximize the number of edges m. This can be converted into the large-n case by setting $r+1 = \beta n$ and $m \leq \alpha\binom{n}{2}$ and then considering the limit of $m/\binom{n}{2}$ as n goes to infinity.

3.3.1 Constructing a Turán-like Graph

Recalling Turán's theorem (Theorem 3.1) and Lemma 3.1, we consider complete r-partite graphs as likely candidates to maximize the number of edges among boxicity d graphs with no clique of size $r+1$. Such graphs are especially convenient to study because their boxicity is well known.

To determine how the vertices should be distributed among the r partite sets, we recall from Lemma 3.1 that at most d of the sets can have size greater than 1. If $r \leq d$, any complete r-partite graph has boxicity at most d, so the number of edges is maximized by the Turán graph $T_{n,r}$. For $r > d$, we let $r-d$ of the partite sets have size 1. This leaves $n-(r-d)$ vertices to be distributed among d partite sets. Turán's theorem implies that the number of edges among those vertices is maximized by allowing the sizes of the sets to vary by at most 1. This gives us a complete construction of a graph, which can be achieved most simply by joining the Turán graph $T_{n-(r-d),d}$ with the complete graph K_{r-d}.

By the *join* of two graphs G and H we mean the graph achieved by including all the edges of G and H, and adding an edge from each vertex of G to every vertex of H. We define

Definition 3.1. *For given n, r, and d, let $T_{n,r,d}$ be defined as the Turán graph $T_{n,r}$ for $r \leq d$ and the join of $T_{n-(r-d),d}$ and K_{r-d} for $r > d$. We call this graph the* Turán-like graph of boxicity *d.*

Notice that $T_{n,r,d}$ is uniquely defined because the Turán graph itself is.

We can now prove

Lemma 3.2. *Among complete r-partite graphs with n vertices of boxicity at most d, the Turán-like graph $T_{n,r,d}$ maximizes the number of edges.*

Proof. By Lemma 3.1, any complete r-partite graph of boxicity at most d has at most d partite sets of size greater than 1. If $d \geq r$, the Turán-like graph is the Turán graph, and is therefore maximal by Theorem 3.1.

Suppose $d < r$. Then in any complete r-partite graph with boxicity at most d, there are b universal vertices (vertices adjacent to every other point in the graph) for some $0 \leq b \leq r-d$. The remaining $n-b$ vertices are distributed into d partite sets, a process in which the number of edges is maximized by the Turán graph $T(n-b,r)$. Suppose $b < r-d$, and consider a vertex v in a partite set of size $k > 1$. Notice that turning v into a universal vertex only involves adding $k-1 > 0$ edges, so increasing b increases the number of edges in the graph. Thus the number of edges is maximized for $b = r-d$, which gives precisely the definition of the Turán-like graph. □

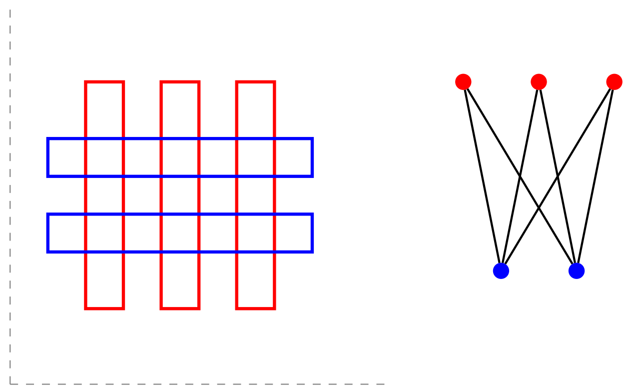

Figure 3.2 The graph $T_{5,2,2}$ and a corresponding society.

Thus for n vertices, maximum boxicity d, and maximum clique size r, we consider the Turán-like graph $T_{n,r,d}$. Before counting the edges in this graph, we consider a few examples.

3.3.2 Examples and Alternate Constructions

To get a better sense for the structure of $T_{n,r,d}$, and how to interpret the societies it represents, we consider a few examples in $\mathbb{R}^2$. We can, in all cases, use the construction from the proof of Lemma 3.1 to get a society whose intersection graph is $T_{n,r,d}$. Figure 3.2 shows $T_{5,2,2}$ along with a corresponding society. Notice that $T_{5,2} = K_{3,2}$ has boxicity 2, so $T_{5,2,2} = T_{5,2}$. Figure 3.3 shows $T_{8,3,2}$, which has one universal vertex.

At this point we consider the uniqueness of the pairwise-agreement proportion achieved by $T_{n,r,d}$. In the case $r \leq d$, then $T_{n,r,d}$ is simply the Turán graph $T_{n,r}$, which is the unique n-vertex graph with no $(r+1)$-clique that maximizes the number of edges, but no such result limits the $r > d$ case. It then becomes of interest to try to find an example of a family of boxes which matches the upper bound given by $T_{n,r,d}$ with a different intersection graph.

We find that we can construct an example for $n = 8$, $r = 3$, $d = 2$. Figure 3.3 shows $T_{8,3,2}$, which has 19 edges. Figure 3.4 shows another eight-voter society with a 19-edge intersection graph. This graph is clearly not isomorphic to $T_{8,3,2}$, as it does not have a vertex of degree 7.

One other thing that may be interesting to consider is the case $d = 1$, and what the Turán-like graph looks like in that case. Figure 3.5 shows the

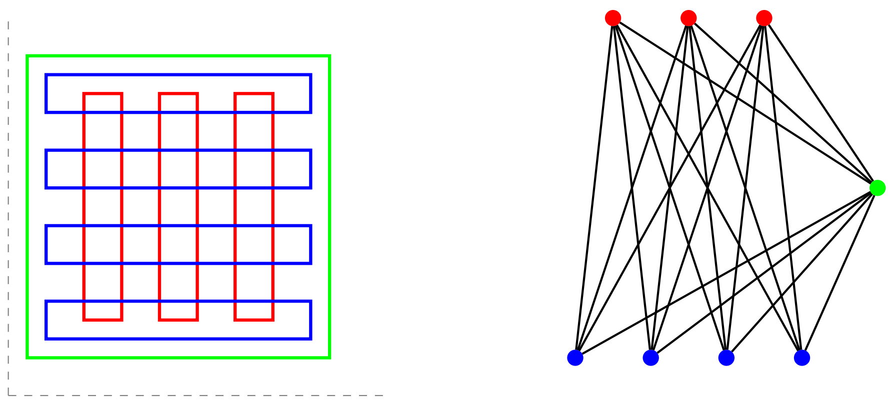

Figure 3.3 The graph $T_{8,3,2}$ and a corresponding society.

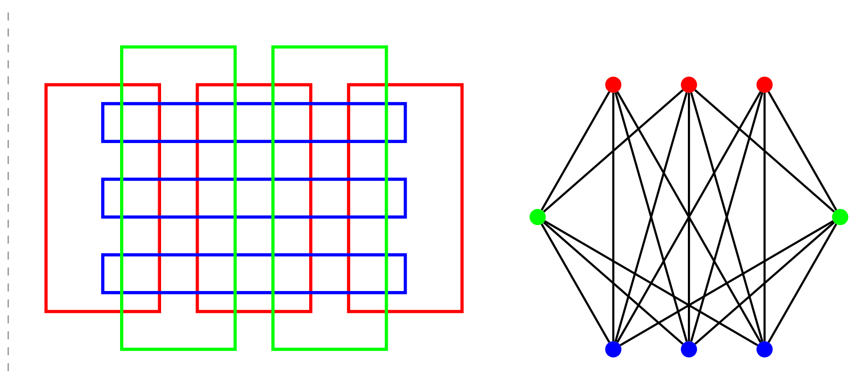

Figure 3.4 An alternate construction to achieve the pairwise agreement of $T_{8,3,2}$.

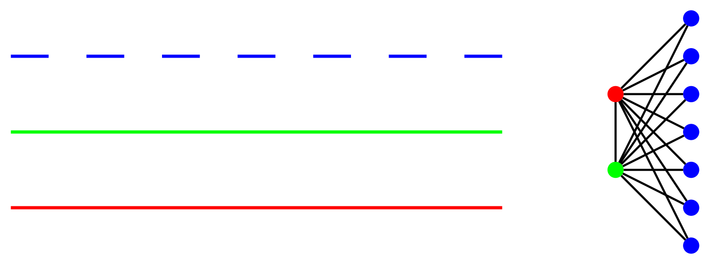

Figure 3.5 The graph $T_{9,3,1}$ and a corresponding society.

graph $T_{9,3,1}$ and representative intervals (the intervals are separated so as to be visible). Notice that this graph is just $K_{n-(r-d),1,\ldots,1}$, where there are $r-d$ 1's, which can also be thought of as the complete graph K_{r-d} joined with an independent set of size $n-(r-d)$.

3.3.3 Counting the Edges of the Turán-like Graph

We now use the Turán-like graph to prove

Theorem 3.2. *Given agreement proportion $\beta \in (0,1]$, the minimal pairwise-agreement proportion $\alpha \in [0,1]$ such that any n-vertex graph G with boxicity at most d and at least $\alpha\binom{n}{2}$ edges must contain a clique of size βn satisfies*

$$\alpha \geq 1 - \frac{(1-\beta)^2}{d},$$

for sufficiently large n.

Notice that this bound is the best possible in the case $d = 1$ by Abbott and Katchalski (1979).

Proof. We use that the Turán-like graph $T_{n,\beta n-1,d}$ to construct a family of n-vertex graphs with agreement proportion

$$1 - \frac{(1-\beta)^2}{d}$$

in the limit $n \to \infty$. By the construction of $T_{n,r,d}$, such graphs have boxicity at most d and do not contain a clique of size βn. This construction establishes the lower bound on α.

We will use the degree-sum formula to compute the number of edges m of the graph $T_{n,r,d}$. Note that each vertex in a partite set of size x has degree $n-x$, so

$$2m = (r-d)(n-1) + \sum_{i=1}^{d} n_i(n-n_i), \tag{3.3}$$

where n_i is the size of the ith partite set of $T_{n,r,d}$.

Since we have defined $T_{n,r,d}$ so that its first d partite sets correspond to the Turán graph $T_{n-(r-d),d}$, we notice

$$\sum_{i=1}^{d} n_i(n-n_i) = 2e(T_{n-r+d,d}).$$

Plugging this back in to Equation 3.3 gives

$$2m = (r-d)(n-1) + 2e(T_{n-r+d,d}). \tag{3.4}$$

We define

$$a = \left\lfloor \frac{n-r+d}{d} \right\rfloor \tag{3.5}$$

and

$$b = (n-r+d) - da \tag{3.6}$$

so $T_{n,r,d}$ has b partite sets of size $a+1$ and $d-b$ partite sets of size a along with its $r-d$ partite sets of size 1. We therefore have

$$\begin{aligned} 2m &= (r-d)(n-1) + b(a+1)(n-(a+1)) + (d-b)a(n-a) \\ &= (r-d)(n-1) + adn + bn - a^2d - 2ab - b. \end{aligned} \tag{3.7}$$

Substituting Equation 3.6 into Equation 3.7 gives

$$2m = n^2 + a^2d - 2an + 2ar - n - ad, \tag{3.8}$$

and using Equation 3.5 gives

$$2m = n^2 + \left\lfloor \frac{n-r+d}{d} \right\rfloor^2 d + \left\lfloor \frac{n-r+d}{d} \right\rfloor (2r-2n-d) - n. \tag{3.9}$$

At this point we set $r = \beta n - 1$ to find

$$\begin{aligned} m = \frac{1}{2}\Bigg(& n(n-1) + \left\lfloor \frac{n(1-\beta)+1+d}{d} \right\rfloor^2 d \\ & - \left\lfloor \frac{n(1-\beta)+1+d}{d} \right\rfloor (2n(1-\beta)+2+d) \Bigg). \end{aligned} \tag{3.10}$$

In the limit as $n \to \infty$, the n^2 terms dominate in both m and

$$\binom{n}{2} = \frac{n(n-1)}{2},$$

so we have

$$\begin{aligned}\alpha &\geq \lim_{n\to\infty} \frac{m}{\binom{n}{2}} \\ &= \lim_{n\to\infty} \left(\frac{2}{n^2}\right)\left(\frac{1}{2}\right)\left(n^2 + d\left(\frac{n(1-\beta)}{d}\right)^2 - \left(\frac{2n^2(1-\beta)^2}{d}\right)\right) \\ &= 1 - \frac{(1-\beta)^2}{d}, \qquad (3.11)\end{aligned}$$

as desired. □

3.4 Attempts to Prove an Upper Bound on Pairwise-Agreement Proportion

Since we found in Theorem 3.2 that the Turán-like graph $T_{n,r,d}$ provides a pairwise-agreement proportion that matches the maximal value proved by Abbott and Katchalski (1979) for $d = 1$, we attempt to prove that it is also maximal for the $d > 1$ case.

Because the methods of Abbott and Katchalski (1979) involve structural features of interval graphs and no analogs for boxicity at most d graphs are known, we attempt to modify a proof of Turán's theorem to show that $T_{n,r,d}$ has the maximum number of edges among n-vertex graphs of boxicity d with no clique of size $r+1$.

Lemma 3.2 proves that $T_{n,r,d}$ maximizes the number of edges among complete r-partite graphs, so following the proof of Turán's theorem presented in Section 3.2 it only remains to show that the maximum is achieved by a complete r-partite graph.

That step can be broken down into two parts: Proving that for any r-partite graph with boxicity at most d, there is a complete r-partite graph with boxicity at most d and at least as many edges and proving that for any graph with boxicity at most d and no clique of size $r+1$ there is an r-partite graph with boxicity at most d and at least as many edges.

Maintaining the boxicity requirement proves to be problematic for both parts, and straightforward adaptations of the arguments presented in Section 3.2 are insufficient.

3.5 The Existing Answer

While searching for a way to prove the maximality of the Turán-like graph, we found that Eckhoff (1988) proves

Theorem 3.3 (Eckhoff)**.** *Any collection of n boxes in* $\mathbb{R}^d$ *in which at least* $\alpha\binom{n}{2}$ *of the pairwise intersections are nonempty must have a point contained in* βn *boxes, where*

$$\alpha = 1 - \frac{(1-\beta)^2}{d},$$

for sufficiently large n.

This theorem was originally conjectured by Kalai (1984). Both Kalai and Eckhoff (1988) use the family of boxes discussed in Section 3.3, which we discovered independently. Eckhoff also mentions the intersection graph of these families, which we have defined as $T_{n,r,d}$. He uses geometric methods to prove Theorem 3.3, and extends the result to the claim that families of boxes corresponding to $T_{n,r,d}$ maximize the number of cliques of size k for all $2 \leq k \leq r$. Eckhoff (1991) also characterizes the families of boxes that achieve the maximal pairwise-agreement proportion.

After discovering Theorem 3.3, we began to consider extensions and variants of the question.

Chapter 4

Projections to Axes

The voting context suggests a variant of the Turán-type problem posed by Questions 1 and 2, where instead of considering pairwise intersections of approval regions, we consider projections of the approval regions onto each of the coordinate axes in $\mathbb{R}^d$. This corresponds to the idea of looking for agreement between pairs of voters on the individual questions or issues represented by the axes. In other words, if we know the pairwise agreement on each issue, what does it tell us about the agreement for the society as a whole?

In this chapter we construct a family of societies that maximizes the pairwise agreement of projections for a given agreement proportion.

4.1 Considering Agreement on Projections

Given a family of boxes in $\mathbb{R}^d$, it is simple to construct d families of intervals on the real line, where each family corresponds to one direction. By Lemma 2.1, two boxes intersect in $\mathbb{R}^d$ if and only if their corresponding intervals intersect in each of the d families of intervals. An example of a box society in $\mathbb{R}^2$ and its corresponding linear societies is shown in Figure 4.1.

To simplify notation, when referring to projections of a society S in $\mathbb{R}^d$ to the coordinate axes, we define $\alpha_i(S)$ for $1 \leq i \leq d$ to be the fraction of the possible pairwise intersections that are nonempty in the projection to the x_i-axis. Notice that, unlike the value α in Question 1, these values refer to specific societies, rather than families of societies. When the society in question is obvious from context, we will omit the argument S.

When dealing with $\mathbb{R}^2$, we will consider α_1 to refer to projection to the x-axis and α_2 to refer to projection to the y-axis. As an example, notice that

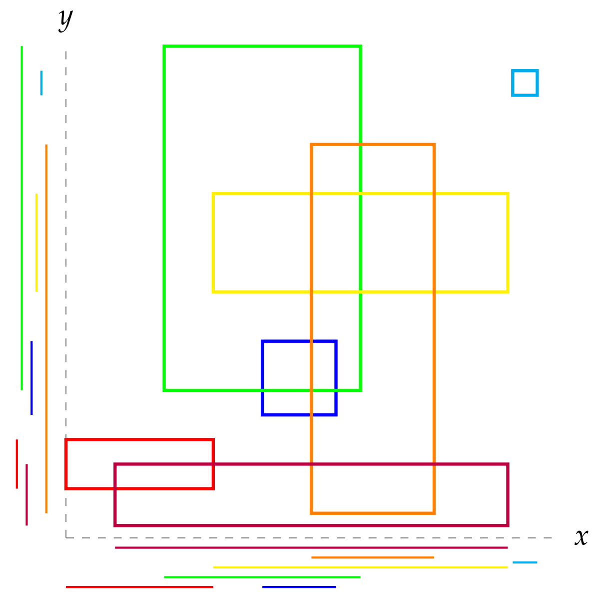

Figure 4.1 A box society in $\mathbb{R}^2$ with corresponding projections.

in Figure 4.1, $\alpha_1 = 2/3$ and $\alpha_2 = 9/15$.

An initial question that we might ask about this situation is what conditions we should place on agreement in the projections to even guarantee that any boxes overlap in the overall society. To answer this question, we construct societies in $\mathbb{R}^2$ with no nonempty intersections, trying to maximize the intersection in the projections. One natural construction to consider is shown in Figure 4.2. The idea of this construction is to split the approval regions into two sets and assign one as vertical sets and the other as horizontal. The horizontal sets are then spaced in the y direction, with identical x endpoints, and the vertical sets are placed above the horizontal ones in y, then spaced from each other in x.

As can be seen, all horizontal boxes intersect each other in the x projection and all vertical boxes intersect each other in the y projection. Furthermore, each vertical box intersects all the horizontal boxes in the x projection, but the horizontal boxes have no intersections in the y projection. If we count the total number of nonempty intersections in each projection (in Figure 4.2), we find that there are six in the y-projection and 22 in the x-projection. Since there are a total of eight boxes, each projection could have up to 28 intersections, so we have $\alpha_1 = 11/14$ and $\alpha_2 = 3/14$. Immediately

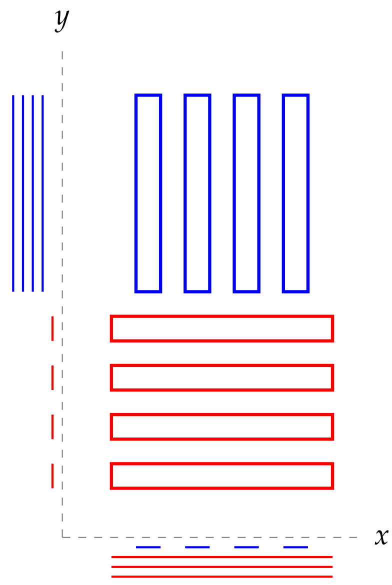

Figure 4.2 A box society with no agreement but high pairwise agreement in corresponding projections.

we notice that $\alpha_1 + \alpha_2 = 1$. Furthermore, we notice that this sum would be the same regardless of how many boxes we included in this construction, or how they were distributed, because of the relationships

$$\alpha_1 \binom{n}{2} = \binom{n}{2} - \binom{n_v}{2}$$

and

$$\alpha_2 \binom{n}{2} = \binom{n_v}{2},$$

where n_v is the number of boxes chosen to be vertical. Then $\alpha_1 + \alpha_2 = 1$ for any version of this construction in $\mathbb{R}^2$. This, along with the lack of obvious ways to increase the sum without introducing intersections between the boxes in $\mathbb{R}^2$, suggests that 1 may be an upper bound for the sum. We therefore consider

Question 3. *For box societies in $\mathbb{R}^d$ with no agreement, what is the maximum value of the sum $\sum \alpha_i$ of the pairwise-agreement proportions of the projections?*

We attempt to address this question in the next section.

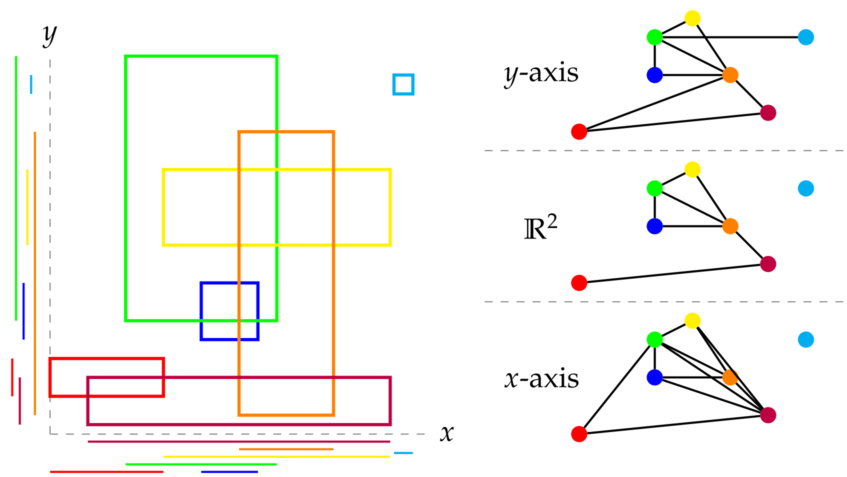

Figure 4.3 A box society with projections and intersection graphs.

4.2 Intersection Graphs of Projections

Considering the projections of a box society as linear societies, we can understand their intersection graphs exactly as we did in Chapter 2. Of course, since the societies in question have intervals in $\mathbb{R}$ as approval regions, the resulting graphs will have boxicity at most 1, making them interval graphs.

As we have already observed, two boxes intersect in $\mathbb{R}^d$ if and only if the corresponding intervals intersect in each of the d projections, so an edge exists on the intersection graph for a society if and only if it exists on the intersection graphs for each projection. In other words, the intersection graph for a society is the intersection of the projection graphs, where we consider the intersection of graphs G_i on the same vertex set to be the graph G on that vertex set such that two vertices are adjacent in G if and only if they are adjacent in each of the G_i. An example of a society in $\mathbb{R}^2$ and its projections, with corresponding graphs, is shown in Figure 4.3.

Conceiving of projections as interval graphs also evokes an alternate definition of boxicity, mentioned by Roberts (1969), who points out that a graph is representable as the intersection graph of boxes in $\mathbb{R}^d$ if and only if it is the intersection of d interval graphs. The boxicity of a graph G can then be defined as the smallest number of interval graphs G_i necessary to represent G as their intersection. This definition provides a significant advantage to the graph approach to projections, as studying $d > 2$ now only requires considering more interval graphs, rather than constructing families of boxes in $\mathbb{R}^d$.

To answer Question 3 for $d = 2$, notice that $\binom{n}{2}(\alpha_1 + \alpha_2)$ is the total number of edges among the two interval graphs, so $\alpha_1 + \alpha_2 = 1$ corresponds to there being exactly $\binom{n}{2}$ edges in the two graphs. By the pigeonhole principle, if there are more than $\binom{n}{2}$ edges in the two graphs, some edge must exist in both graphs. That is, there must be some pair of vertices that are adjacent in both graphs. Such a pair must also be in the intersection of the two graphs, indicating an intersection of boxes in $\mathbb{R}^2$. Thus we have proved the conjecture that $\alpha_1 + \alpha_2 = 1$ is maximal for the $d = 2$ case, and suggests that we can prove the following theorem.

Theorem 4.1. *Any box society in $\mathbb{R}^d$ whose projection to the x_i axis has pairwise agreement α_i such that*

$$\sum_{i=1}^{d} \alpha_i > d - 1$$

must have at least one nonempty intersection between approval regions.

Proof. Let G be the intersection graph for such a society, let G_i be the intersection graph for the projection to the x_i axis, and let n be the number of voters in the society. Then the number of edges in G_i is $e(G_i) = \alpha_i\binom{n}{2}$, so

$$\sum_{i=1}^{d} \alpha_i > d - 1$$

implies

$$\sum_{i=1}^{d} e(G_i) > (d-1)\binom{n}{2}.$$

These edges must be distributed between the d graphs G_i, each of which has only $\binom{n}{2}$ pairs of vertices. Thus there must be some pair of vertices that are adjacent in all G_i. That pair of vertices is also adjacent in G, corresponding to a nonempty intersection of approval regions. □

Notice that this bound is sharp, as a society in which all approval regions are copies of $\mathbb{R}^{d-1}$ translated in one direction has $\sum \alpha_i = d - 1$ with no intersections in $\mathbb{R}^d$.

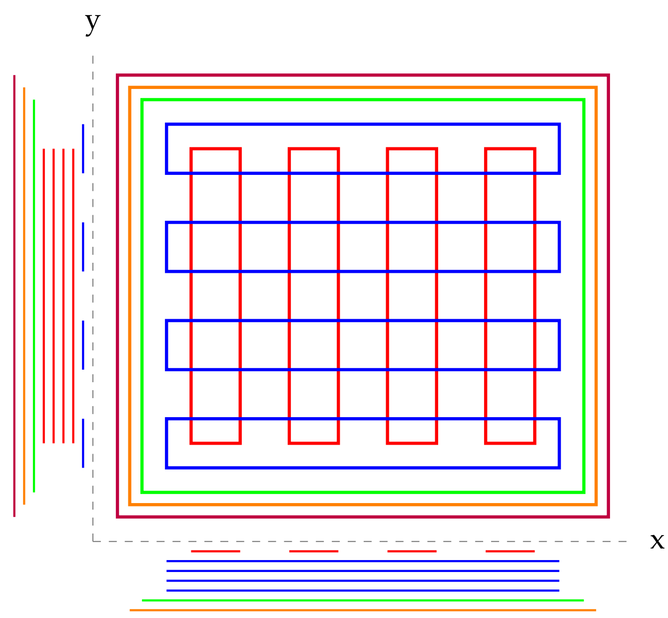

Figure 4.4 A society with intersection graph $T_{11,5,2}$ and its projections.

4.3 A Turán-like Result for Projections

We now return to the original purpose of the chapter, which is to answer an analog of Question 1 where we count the pairwise-agreement proportion in the projections to axes rather than in the society as a whole. The natural place to start for such a question is the Turán-like graph defined in Chapter 3, since by Theorem 3.3 this graph maximizes the number of overall edges.

We consider a typical example of $T_{n,r,d}$, and attempt to characterize its projections. Figure 4.4 shows a society whose intersection graph is $T_{11,5,2}$. Notice that the projections both have an intersection graph which is the join of K_8 and an independent set of size 4. Further note that the independent sets of size 4 correspond to different vertices in each projection graph. We also observe that the intervals of the projection are strongly evocative of Figure 3.5, which is not unexpected considering that figure showed a Turán-like graph for $d = 1$.

Given the standard society we use to correspond to $T_{n,r,d}$, we expect that the projection graphs will always have a similar structure, and a consideration of the graph $T_{n,r,d}$ as an intersection of d interval graph confirms this. If we let the ith largest partite sets of $T_{n,r,d}$ have size n_i for $1 \leq i \leq d$ and define H_i to be K_{n-n_i} joined with an independent set S_i of size n_i and identify the vertices of the S_i such that $S_i \cap S_j = \emptyset$ for $i \neq j$, then the intersection H of all the H_i will be $T_{n,r,d}$. Clearly every edge not between two vertices in the same S_i will exist in H, and no edges between two such vertices will, so the resulting graph will be $K_{n_1,\ldots,n_d,1,\ldots,1}$, where there are $n - \sum n_i = n - (n - (r - d)) = r - d$ partite sets of size 1. This is of course $T_{n,r,d}$.

With this construction of $T_{n,r,d}$ by the intersection of G_i, we prove

Theorem 4.2. *Among graphs with n vertices and no clique of size $r + 1$ that are the intersection of d interval graphs $G_1, \ldots, G_d$, the Turán-like graph $T_{n,r,d}$ constructed by the intersection of the graphs H_i above, maximizes the total number of edges in $G_1, \ldots, G_d$.*

Proof. Because $T_{n,r,d}$ can be constructed by intersecting the interval graphs H_i defined above, and by Theorem 3.3 $T_{n,r,d}$ has the maximum number of edges among graphs of boxicity at most d with n vertices and no clique of size $r + 1$, we need only prove that any collection of d interval graphs G_i with more edges than the collection H_i must have an intersection with more edges than $T_{n,r,d}$.

Let $x = \binom{n}{2} - e(T_{n,r,d})$ be the number of nonadjacent vertex pairs in $T_{n,r,d}$ (i.e., the number of nonedges). Notice that by the construction of the H_i, the total number of edges missing from the H_i is also x, since every nonedge of $T_{n,r,d}$ is a nonedge in precisely one of the H_i (corresponding to whichever partite set the nonadjacent vertices are in). Further notice that given a collection of interval graphs G_i, a pair of vertices is nonadjacent in their intersection only if it is nonadjacent in at least one G_i. Therefore any collection of graphs with at most $x - 1$ nonedges among the G_i can have at most $x - 1$ nonedges in the intersection, so no collection G_i can have more total edges than H_i without having more edges in the intersection than $T_{n,r,d}$. □

Chapter 5

(k, m)-Agreeability

For any society of voters, say that some subset of voters *agree* on a position if it is in all of their approval regions. Berg et al. (2010) introduce the concept of (k, m)-*agreeability*, where a society is (k, m)-agreeable if it has at least m voters, and among every m voters some subset of k voters agree on some position. Similarly, a graph with at least m vertices is said to be (k, m)-agreeable if every induced subgraph with m vertices contains a clique of size k.

In this chapter, we will consider the minimal agreement proportion for (k, m)-agreeable societies. For box societies in $\mathbb{R}^d$ we adopt the notation of Abrahams et al. (2010) to denote this minimal agreement proportion $\beta(k, m, d)$. Notice that this notation is analogous to that of Question 1, with the condition on the fraction of nonempty pairwise intersections replaced by the new (k, m)-agreeability condition.

After reviewing previous work on this question, we extend in Theorem 5.7 a result of Abrahams et al. (2010) to prove for $k \geq 2$ the lower bound

$$\beta(k, 2k-1, d) \geq (2d)^{-1}. \tag{5.1}$$

Note that due to the definition of (k, m)-agreeability, throughout the chapter we will assume $k \geq 2$, and $m \geq k$. Since the only boxicity zero graphs are the complete graphs K_n (which always have agreement proportion 1), we also assume $d \geq 1$.

5.1 Previous Work

Berg et al. (2010) extensively study the $d = 1$ case; that is, linear (k, m)-

agreeable societies, proving that for $2 \leq k \leq m$,

$$\beta(k, m, 1) \geq \frac{k-1}{m-1}. \tag{5.2}$$

The same paper also deals briefly with societies in $\mathbb{R}^d$ where approval regions are arbitrary convex subsets, using the Fractional Helly Theorem from Kalai (1984) to prove for $m > d$ the lower bound

$$\beta(k, m, d) \geq 1 - \left(1 - \frac{\binom{k}{d+1}}{\binom{m}{d+1}}\right)^{\frac{1}{d+1}}, \tag{5.3}$$

where $\binom{a}{b}$ is the binomial coefficient "a choose b". Since this bound applies for all convex approval regions, not just boxes, and is weaker than Equation 5.2 in the $d = 1$ case, Berg et al. (2010) suspect it to not be the best possible.

Berg et al. (2010) also discuss box societies in $\mathbb{R}^d$, and find a lower bound on agreement proportion for the case $k \leq m \leq 2k-2$. This case has significant agreement, an intuitively logical result because any subset of size m must have more than half of its voters in agreement. The formal result can be stated as

Theorem 5.1 (Berg et al. (2010))**.** *For integers $m, k \geq 2$ such that $k \leq m \leq 2k-2$, any (k, m)-agreeable box society in $\mathbb{R}^d$ with $n \geq m$ voters has a position approved by $n - m + k$ voters.*

We can interpret this result as the lower bound

$$\beta(k, m, d) \geq 1 - \frac{m-k}{n} \tag{5.4}$$

for $k \leq m \leq 2k-2$.

For these values of m, this result is best possible, as demonstrated by considering a graph consisting of a clique of size $n - m + k$ and $m - k$ isolated vertices. Such a graph clearly has boxicity 1, as shown in Figure 5.1 for $k = 4$, $m = 6$, and so the bound applies for all $d \geq 1$.

The result does not, however, directly address societies for which $m \geq 2k-1$, leaving significant room for investigation of other values.

Abrahams et al. (2010) deal with one such case, considering $(2, 3)$-agreeable box societies in $\mathbb{R}^d$, proving

Theorem 5.2 (Abrahams et al. (2010))**.** *For any $d \geq 1$, any $(2, 3)$-agreeable d-box society has an agreement proportion of at least $(2d)^{-1}$.*

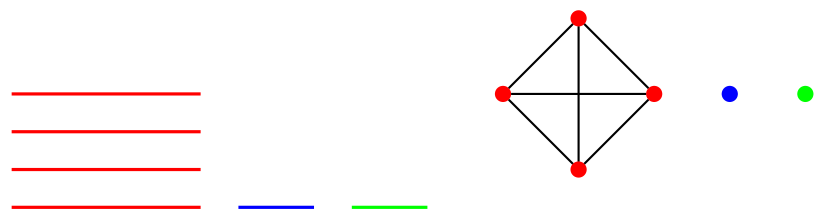

Figure 5.1 A $(4, 6)$-agreeable graph on six vertices with agreement proportion $1 - 2/6 = 2/3$, with a corresponding linear society.

This result relies on a lower bound on the boxicity of graphs in terms of their minimum degree, proven by Adiga et al. (2008). We will adapt the proof of Theorem 5.2 to extend the result to all $(k, 2k-1)$-agreeable box societies.

5.2 Notation

For $k \geq 2$, let $\mathcal{G}_k$ denote the set of all $(k, 2k-1)$-agreeable graphs. For $d \geq 1$, let $\mathcal{G}_{k,d}$ represent the subset of $\mathcal{G}_k$ consisting of graphs with boxicity at most d, and for $r \geq k$, let $\mathcal{G}_{k,d}(r)$ be the subset of $\mathcal{G}_{k,d}$ consisting of graphs with maximum clique size at most r. Similarly, let $\mathcal{G}_k(r)$ be the subset of $\mathcal{G}_k$ consisting of graphs with maximum clique size at most r.

Note that by our definition of (k, m)-agreeability, all graphs in the above sets must contain at least $2k-1$ vertices. Abrahams et al. (2010) do not impose this restriction on $(2, 3)$-agreeability, but we do in order to avoid the case of a society consisting of $2k-2$ voters with no agreement, which has agreement proportion $(2k-2)^{-1}$, a value that is in many cases smaller than the lower bound we will prove. For example, the graph consisting of four isolated vertices is "$(3, 5)$-agreeable" without the size restriction, but its agreement proportion of $1/4$ is less than the lower bound of $1/2$ stated in either Equation 5.2 or Equation 5.1.

We now define $\eta_k(r, d)$ to be the maximum number of vertices of any graph in $\mathcal{G}_{k,d}(r)$, and $\eta_k(r)$ as the maximum number of vertices of any graph in $\mathcal{G}_k(r)$. Note that $\eta_k(r, d)$ and $\eta_k(r)$ as defined may be infinite, but we will prove in Theorem 5.3 that $\eta_k(r)$ is finite for all $r \geq k \geq 2$. The finiteness of $\eta_k(r, d)$ for $d \geq 1$ follows from the inequalities

$$2k - 1 \leq \eta_k(r, 1) \leq \eta_k(r, 2) \leq \cdots \leq \eta_k(r), \tag{5.5}$$

which follow from the inclusion relationships

$$\mathcal{G}_{k,1}(r) \subseteq \mathcal{G}_{k,2}(r) \subseteq \cdots \subseteq \mathcal{G}_k(r).$$

Similarly, the inclusion relationships

$$\mathcal{G}_k(k) \subseteq \mathcal{G}_k(k+1) \subseteq \mathcal{G}_k(k+2) \subseteq \cdots$$

imply

$$2k - 1 \leq \eta_k(k) \leq \eta_k(k+1) \leq \eta_k(k+2) \leq \cdots .$$

In light of this, we define $\eta_k(k-1) = 2k-2$, noting that any $(k, 2k-1)$-agreeable graph must contain a clique of size k, and must have more than $2k-2$ vertices.

We will show $\eta_k(r) \leq k(r+1) + \frac{1}{2}(r^2 - r - 4)$ (Theorem 5.3), so $\mathcal{G}_{k,d}(r)$ is finite for each $r \geq k$ and $d \geq 1$. Once this finiteness is proven, we can define $\rho_k(r,d)$ to be the minimal agreement proportion for a $(k, 2k-1)$-agreeable graph with at least $2k-1$ vertices, boxicity at most $d \geq 1$ and clique size at most $r \geq k$. This can be equivalently stated as

$$\rho_k(r,d) = \min\{\omega(G)/|V(G)| \mid G \in \mathcal{G}_{k,d}(r)\}, \tag{5.6}$$

where $\omega(G)$ is the maximum clique size and $|V(G)|$ is the number of vertices of any graph G. We note that by the finiteness of $G_{k,d}(r)$, at least one graph in $G_{k,d}(r)$ has agreement proportion $\rho_k(r,d)$.

5.3 Bounding Vertex Degrees

To prove that $\mathcal{G}_{k,d}(r)$ is finite, we find upper and lower bounds on the degrees of vertices in $(k, 2k-1)$-agreeable graphs in terms of their number of vertices and clique number. For this discussion, we mostly do not worry about the boxicity of the graphs under consideration.

For the lower bound, we rely on the fact that $(k, 2k-1)$-agreeable graphs are “close” to the strong agreement case covered by Theorem 5.1. That is to say, we can usually find subgraphs of $(k, 2k-1)$-agreeable graphs that are $(k, 2k-2)$-agreeable and therefore covered by Theorem 5.1, since the subsets we consider to establish the agreeability condition are only one larger in the $(k, 2k-1)$ case than the $(k, 2k-2)$ case.

We formalize these ideas to prove the following theorem.

Lemma 5.1. *For $k \geq 2$, let $G \in \mathcal{G}_k$ have $n \geq 2k-1$ vertices. Then for any vertex v of G, we have*

$$\deg(v) \geq n - \omega(G) - (k-1),$$

where $\omega(G)$ is the size of the largest clique of G.

Proof. Notice that v is connected to $\deg(v)$ vertices, and let H be the subgraph induced by the $n - \deg(v) - 1$ vertices of G that are neither v nor adjacent to v.

First suppose $n - \deg(v) - 1 < 2k - 2$. This implies

$$\deg(v) > n - 2k + 1 = n - k - (k-1).$$

Because G is $(k, 2k-1)$-agreeable, $\omega(G) \geq k$, so

$$\deg(v) > n - \omega(G) - (k-1).$$

Now suppose $n - \deg(v) - 1 \geq 2k - 2$. Then H must be $(k, 2k-2)$-agreeable, because otherwise we can take $2k-2$ vertices of H along with v to create a set of $2k-1$ vertices of G that do not contain a clique of size k. Therefore by Theorem 5.1, H must contain a clique of size

$$(n - \deg(v) - 1) - (2k-2) + k = n - \deg(v) - (k-1).$$

This implies

$$\omega(G) \geq \omega(H) \geq n - \deg(v) - (k-1),$$

and therefore

$$\deg(v) \geq n - \omega(G) - (k-1),$$

as desired. □

We note that this bound is the best possible, as can be seen by considering a graph consisting of a clique of size k and $k-1$ isolated vertices. In this case the graph has $n = 2k-1$ vertices, so the lower bound on degree is $2k - 1 - k - (k-1) = 0$, which is the degree of each isolated vertex. The $k = 3$ case of this graph is shown in Figure 5.2.

We now prove an upper bound on the degree of vertices.

Lemma 5.2. *For $k \geq 2$ and $r \geq k$, let $G \in \mathcal{G}_{k,d}$. Then for any vertex v of G,*

$$\deg(v) \leq \eta_k(r-1, d) \leq \eta_k(r-1).$$

This proof is essentially identical to the proof of Abrahams et al. (2010: Lemma 4.3).

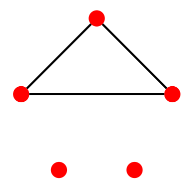

Figure 5.2 A $(3, 5)$-agreeable graph demonstrating the sharpness of Lemma 5.1.

Proof. Given vertex v, let H be the subgraph induced by the neighbors of v. If H has fewer than $2k-1$ vertices, then

$$\deg(v) < 2k - 1 \leq \eta_k(r-1, d)$$

by Equation 5.5.

Otherwise H is $(k, 2k-1)$-agreeable, since it is an induced subgraph of G with at least $2k-1$ vertices, and G is $(k, 2k-1)$-agreeable. Similarly, H has boxicity at most d.

If it were the case that $\deg(v) > \eta_k(r-1, d)$, then H would have a clique of size r (by the definition of $\eta_k(r-1, d)$), which together with v would form a clique of size $r+1$ in G, violating the assumption $\omega(G) \leq r$. This implies

$$\deg(v) \leq \eta_k(r-1, d) \leq \eta_k(r-1),$$

with the latter inequality following from Equation 5.5. □

These upper and lower bounds on degrees of vertices allow us to prove an upper bound on $\eta_k(r)$.

Theorem 5.3. *For $r \geq k$,*

$$\eta_k(r) \leq k(r+1) + \frac{1}{2}(r^2 - r - 4).$$

Again this proof follows that of Abrahams et al. (2010) for a similar result about $(2, 3)$-agreeable graphs.

Proof. Let $G \in \mathcal{G}_k(r)$ satisfy $\omega(G) = r$ and have $\eta_k(r)$ vertices. Then by Lemmas 5.1 and 5.2, for any vertex v of a G,

$$\eta_k(r) - r - (k-1) \leq \deg(v) \leq \eta_k(r-1),$$

and thus

$$\eta_k(r) - r - (k-1) - \eta_k(r-1) \leq 0.$$

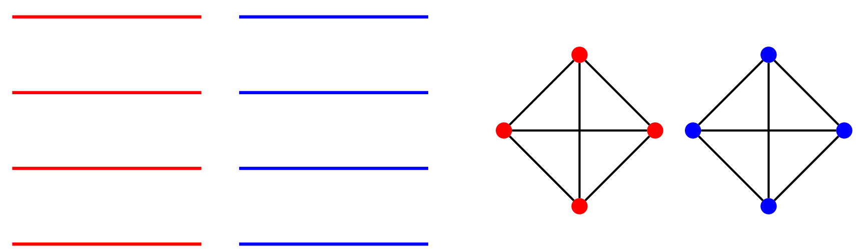

Figure 5.3 The graph G with $k = 4$, and a corresponding linear society.

This is a recurrence we can solve, given $\eta_k(k-1) = 2k-2$.

Solving the recurrence gives

$$\eta_k(r) \leq k(r+1) + \frac{1}{2}(r^2 - r - 4),$$

as desired. □

By the definition of $\eta_k(r)$, this result implies that $\mathcal{G}_k(r)$ is finite, and therefore so is $\mathcal{G}_{k,d}(r) \subseteq \mathcal{G}_k(r)$. This justifies our definition of $\rho_k(r,d)$.

5.4 Bounding Minimal Agreement Proportion

Before proving a lower bound on minimal agreement proportion, we prove a simple upper bound, which will allow us to consider only graphs that potentially minimize agreement proportion, ignoring those with too much agreement. For inspiration, we recall the graph constructed to demonstrate the sharpness of Lemma 5.1 (an example of which appears in Figure 5.2), and note that it has agreement proportion $k/(2k-1) > 1/2$.

With this in mind, we prove

Lemma 5.3. *For all $k \geq 2$, $d \geq 1$, and $r \geq k$, the minimal agreement proportion $\rho_k(r,d)$ is at most $1/2$.*

Proof. We note $\mathcal{G}_{k,1}(k) \subseteq \mathcal{G}_{k,d}(r)$ for all $d \geq 1$ and $r \geq k$, so it will suffice to construct a graph with boxicity 1 and clique number k whose agreement proportion is $1/2$.

We construct the graph G consisting of two disjoint cliques of size k. Clearly this graph has boxicity 1 and clique number k. It is shown, along with a representation as intervals, in Figure 5.3 for $k = 4$.

Notice that G has $2k$ vertices, so it has agreement proportion $1/2$. Further notice that any set of $2k - 1$ vertices simply excludes one vertex, and thus must include the entire clique not containing that vertex, so G is $(k, 2k - 1)$-agreeable. Thus G is the desired graph, proving

$$\rho_k(r, d) \leq \frac{1}{2}.$$

□

This result clearly shows that the graph we constructed to show the sharpness of Lemma 5.1 does not minimize agreement proportion. In fact it allows us to prove a stronger version of the lemma, which will prove to be the key result for proving Equation 5.1.

Theorem 5.4. *For $k \geq 2$, let $G \in \mathcal{G}_{k,d}(r)$ have $n \geq 2k - 1$ vertices and agreement proportion $\rho_k(r, d)$. Then for any vertex v of G, we have*

$$\deg(v) \geq n - \omega(G) - 1,$$

where $\omega(G) \leq r$ is the size of the largest clique of G.

Proof. Let $G \in \mathcal{G}_{k,d}(r)$ have agreement proportion $\rho_k(r, d)$, and let v be some vertex of G.

First suppose $\deg(v) > k - 2$, and construct a set of vertices X by first including all vertices not adjacent to v (excluding v itself), and then choosing some $k - 2$ neighbors of v to add to X. By this construction we have

$$|X| = (n - \deg(v) - 1) + (k - 2) = n - \deg(v) + k - 3.$$

If $|X| < 2k - 2$, then

$$n - \deg(v) + k - 3 < 2k - 2,$$

which implies

$$\deg(v) > n - k - 1 \geq n - \omega(G) - 1,$$

where the second inequality follows from $\omega(G) \geq k$. This is the desired result.

If, on the other hand, $|X| \geq 2k - 2$, let H_1 be the subgraph induced by X, and notice that H_1 must be $(k, 2k - 2)$-agreeable. This is because any set of $2k - 2$ vertices from H_1 can be paired with v to create a set of $2k - 1$ vertices of G, and such a set must contain a clique of size k. The construction of X ensures that such a clique cannot contain v, since at most $k - 2$ other vertices of the set are adjacent to v, and so the clique must be contained within H_1.

By Theorem 5.1, H_1 must contain a clique of size

$$(n - \deg(v) + k - 3) - (2k - 2) + k = n - \deg(v) - 1.$$

Since H_1 is an induced subgraph of G, this implies

$$\omega(G) \geq n - \deg(v) - 1,$$

which implies

$$\deg(v) \geq n - \omega(G) - 1,$$

which again is the desired result.

We are left to consider the case $\deg(v) \leq k - 2$. Let H_2 be the subgraph induced by all the vertices of G except for v, and note that H_2 has size $n - 1$. Further notice that any set of $2k - 2$ vertices of H_2 contains at most $k - 2$ vertices adjacent to v (in G), so H_2 must be $(k, 2k - 2)$-agreeable. This again follows from the idea that any set containing v and $2k - 2$ vertices of H_2 must contain a clique of size k in G, but such a clique cannot include v since it has at most $k - 2$ neighbors.

By Theorem 5.1, H_1 must contain a clique of size

$$(n - 1) - (2k - 2) + k = n - k + 1.$$

Since H_1 is an induced subgraph of G, this implies

$$\omega(G) \geq n - k + 1,$$

so G has agreement proportion

$$\rho_k(r, d) = \frac{\omega(G)}{n} \geq \frac{n - k + 1}{n} = 1 - \frac{k - 1}{n} > \frac{1}{2}, \tag{5.7}$$

where the last inequality follows from $n \geq 2k - 1$. Notice, however, that by Lemma 5.3, $\rho_k(r, d) \leq 1/2$, so Equation 5.7 implies

$$\frac{1}{2} \geq \rho_k(r, d) > \frac{1}{2},$$

which is a contradiction. Thus there are no vertices v such that $\deg(v) \leq k - 2$ in G, and we have considered all cases. □

We note that the bound on vertex degree in this theorem is exactly the same for agreement proportion-minimizing $(k, 2k-1)$-agreeable graphs as the one proved by Abrahams et al. (2010) for *all* $(2,3)$-agreeable graphs. This is good, since we are trying to prove the same lower bound on minimal agreement proportion.

Following the method of Abrahams et al. (2010), at this point we want to bring boxicity back in by way of a key lower bound on the boxicity of a graph in terms of its minimum degree, proven by Adiga et al. (2008).

For a graph G on n vertices, we call a vertex *universal* if its degree is $n-1$.

Theorem 5.5 (Adiga et al. (2008))**.** *Let G be a graph with no universal vertices and minimum degree δ. Then the boxicity of G has the lower bound*

$$box(G) \geq \frac{n}{2(n-\delta-1)}.$$

To apply this theorem, we need to show that the graphs in consideration do not have universal vertices.

Theorem 5.6. *For $k \geq 2$, let G be a $(k, 2k-1)$-agreeable graph on $n \geq 2k-1$ vertices with boxicity at most $d \geq 1$, clique size at most $r \geq k$, and agreement proportion equal to $\rho_k(r,d)$. Then G has no universal vertices.*

This theorem and proof are adapted from Abrahams et al. (2010: Lemma 5.2).

Proof. Let $G \in \mathcal{G}_{k,d}(r)$ have n vertices with $u \geq 0$ universal vertices and agreement proportion $\rho_k(r,d)$. We will prove $u = 0$.

First suppose

$$n - u < 2k - 1. \tag{5.8}$$

In this case, there are a lot of universal vertices, so we will show that the agreement proportion of G is large.

Let $m = 2k-1-(n-u)$, so that any set of $2k-1$ vertices of G containing all vertices of H will contain exactly m universal vertices, where m is positive by Equation 5.8. Since such a set must contain a clique of size k, this implies

$$\omega(H) \geq k - m = n - u - k + 1.$$

From here we can see that G has agreement proportion

$$\rho_k(r,d) = \frac{\omega(G)}{n} = \frac{\omega(H)+u}{n} \geq \frac{(n-u-k+1)+u}{n} = 1 - \frac{k-1}{n} > \frac{1}{2},$$

where the final inequality follows from the hypothesis $n \geq 2k-1 > 2k-2$. By Lemma 5.3, this implies

$$\frac{1}{2} \geq \rho_k(r,d) > \frac{1}{2},$$

which is a contradiction.

Therefore it must be the case that $n-u \geq 2k-1$, and notice that this implies $u < n$, since $2k-1 > 0$. In this case, there are fewer universal vertices, so we just rely on the idea that adding universal vertices to a graph can only increase its agreement proportion to show that the agreement proportion of G is not minimal if $u > 0$.

Let H be the graph induced by the nonuniversal vertices of G, and note that H has size $n-u \geq 2k-1$, so H is $(k, 2k-1)$-agreeable. Further note that $\mathrm{box}(H) \leq \mathrm{box}(G) \leq d$, as boxicity can only decrease when taking induced subgraphs. For any vertex $v \in V(H)$, we have

$$\deg_H(v) = \deg_G(v) - u < n - 1 - u = |V(H)| - 1,$$

so H has no universal vertices. Finally, notice that any maximal clique in G must contain all of the universal vertices of G, so

$$\omega(H) = \omega(G) - u.$$

This implies $H \in \mathcal{G}_{k,d}(r)$, and allows us to calculate the agreement proportion for H as

$$\frac{\omega(H)}{|V(H)|} = \frac{\omega(G)-u}{n-u} \leq \frac{\omega(G)}{n} = \rho_k(r,d). \tag{5.9}$$

But $H \in \mathcal{G}_{k,d}(r)$ implies H has agreement proportion at least $\rho_k(r,d)$, so it must be the case that equality holds, and so

$$\frac{\omega(G)-u}{n-u} = \frac{\omega(G)}{n},$$

which implies $u = 0$, as desired. □

With this theorem in hand, we can prove

Theorem 5.7. *For $k \geq 2$ and $d \geq 1$, any $(k, 2k-1)$-agreeable n-vertex graph with boxicity at most d has agreement proportion at least $(2d)^{-1}$.*

Note that this result is an extension of Theorem 5.2, which is just the $k = 2$ case. The proof method is identical to that used by Abrahams et al. (2010).

Proof. Let G be a $(k, 2k-1)$-agreeable n-vertex graph with boxicity at most d and maximum clique size r such that G has agreement proportion $\rho_k(r, d)$. By Theorem 5.6, G has no universal vertices, and so by Theorem 5.5,

$$d \geq \text{box}(G) \geq \frac{n}{2(n-\delta-1)},$$

where δ is the minimum degree of G. We rewrite this inequality as

$$n - \delta - 1 \geq \frac{n}{2d}.$$

By Theorem 5.4,

$$\omega(G) \geq n - \delta - 1.$$

These inequalities can be combined to show

$$\rho_k(r, d) = \frac{\omega(G)}{n} \geq \frac{n-\delta-1}{n} \geq \frac{1}{2d}.$$

Since by definition every $(k, 2k-1)$-agreeable graph with boxicity at most d has agreement proportion at least $\rho_k(r, d)$ for some r, this implies the desired result. □

5.5 Discussion

The most immediate reaction to Theorem 5.7 is to note that it provides exactly the same bound as Abrahams et al. (2010), for a wider range of societies. We also note that it matches the Berg et al. (2010) linear result (Equation 5.2) when $d = 1$.

It is important to note that this $d = 1$ case provides our only current examples of equality holding in Theorem 5.7. Lemma 5.3 and its proof imply that equality holds, with the graph consisting of two cliques of size k serving as a convenient example of a $(k, 2k-1)$-agreeable linear society with agreement proportion $1/2$.

For $d = 2$, Berg et al. (2010) and Abrahams et al. (2010) both refer to examples of $(2, 3)$-agreeable graphs with agreement proportion $3/8$, such as the one in Figure 5.4. Since Theorem 5.7 provides a lower bound of $1/4$ for this case, the lack of other examples suggests that it may be possible to improve the bound, even in the $k = 2$, $d = 2$ case. We have found no examples for $k > 2$ that show the bound to be sharp.

Since Theorem 5.7 depends on the bound of Theorem 5.5, one way to improve the bound might be to find a new version of the boxicity bound

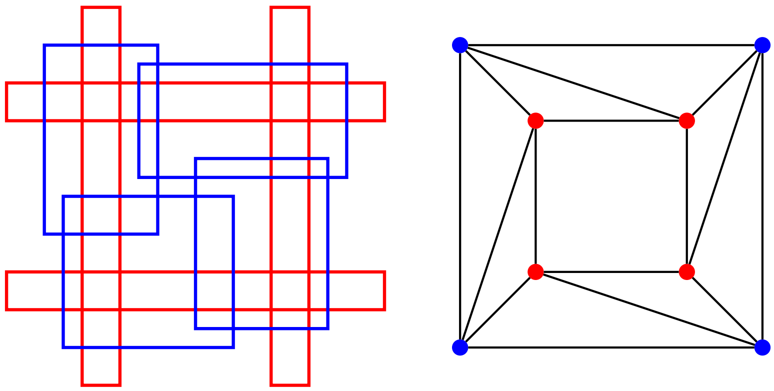

Figure 5.4 A $(2,3)$-agreeable graph with agreement proportion $3/8$, and a corresponding 2-box society (Abrahams et al., 2010).

Figure 5.5 The graph P_4, for which equality holds in Theorem 5.5.

found by Adiga et al. (2008). This path is especially tempting since boxicity only really enters Theorem 5.7 through that bound, but it is not immediately clear what form an altered Theorem 5.5 might take.

Adiga et al. (2008) make no explicit comment on the sharpness of the bound

$$\text{box}(G) \geq \frac{n}{2(n-\delta-1)}, \tag{5.10}$$

but it is possible to choose values and construct graphs for which equality holds. For example, P_4, the path on four vertices, has $\delta = 1$ and $n = 4$, and boxicity 1, and $1 = 4/(2(2))$.

Whether such examples remove all possibility for improving the bound is not immediately clear.

It may also be possible to derive new bounds on $\beta(k,m,d)$ from some of the various other lower bounds on boxicity provided by Adiga et al. (2008), but most refer to specific types of graphs. Thus applications of those results would be limited by restriction to certain sorts of societies. Without more extensive characterizations of agreement-proportion minimizing graphs (beyond the facts proved in Theorems 5.4 and 5.6), such limitations will be severe.

Despite our reservations that Theorem 5.7 may not be the best possible, we can confirm that it is better than Equation 5.3, the lower bound for agreement in convex societies in $\mathbb{R}^d$ derived from the fractional Helly Theorem. This is not unexpected, since requiring approval regions to be boxes forces more intersections than general convex sets.

For the $m = 2k - 1$ case under consideration, Equation 5.3 becomes

$$\beta(k, 2k-1, d) \geq 1 - \left(1 - \frac{k!((2k-1)-(d+1))!}{(2k-1)!(k-(d+1))!}\right)^{\frac{1}{d+1}}. \tag{5.11}$$

For fixed d, the limit of the right hand side as k goes to infinity is

$$1 - (1 - 2^{-d-1})^{1/(d+1)},$$

and this limit is greater than the actual value for all k, so the bound can be written as

$$\beta(k, 2k-1, d) \geq 1 - (1 - 2^{-d-1})^{1/(d+1)}.$$

Since $(2d)^{-1}$ is greater than $1 - (1 - 2^{-d-1})^{1/(d+1)}$ for all $d \geq 1$, the new bound is stronger than Equation 5.3.

To just get a sense for this, we can pick values of k and d (recalling that Equation 5.3 applies only for $d < 2k - 1$).

For $d = 2$, we get $1/4 = 0.25$ from Theorem 5.7, and the Equation 5.3 value ranges from 0 to 0.043 as k goes from 2 to 10. The limit is $1 - \frac{7^{1/3}}{2} \approx 0.044$.

For $d = 10$, we get $1/20 = 0.05$ from Theorem 5.7, and the Equation 5.3 value ranges from 0 to 9.1×10^{-6} as k goes from 6 to 20. The limit is approximately 4.4×10^{-5}.

These examples fit with our expectation that Theorem 5.7, which is specific to box societies, is a much stronger result than Equation 5.3.

Chapter 6

Future Work

The work of Chapter 3 leaves little in the way of open questions, as it provides a best possible lower bound on the agreement proportion of an arbitrary society with a given pairwise-agreement proportion, so the most tempting direction for future work on approval voting in box societies seems to be in the sorts of (k, m)-agreeability questions posed in Chapter 5.

Here we present a natural conjecture that arises from that chapter, along with some thoughts on a potential proof.

6.1 A Conjecture

As we have already noted, the lower bound on agreement proportion for $(k, 2k-1)$-agreeable d-box societies proved in Theorem 5.7 matches the lower bound of $(k-1)/(m-1)$ found by Berg et al. (2010) for all $2 \leq k \leq m$ in the linear ($d = 1$) case. This suggests a possible lower bound for all (k, m)-agreeable d-box societies.

Conjecture 6.1. *For $d \geq 1$ and $2 \leq k \leq m$,*

$$\beta(k, m, d) \geq \frac{k-1}{(m-1)d}.$$

In considering how to potentially prove Conjecture 6.1, we note that it would be useful to prove

$$\omega(G) \geq 2(n - \delta - 1)\left(\frac{k-1}{m-1}\right) \tag{6.1}$$

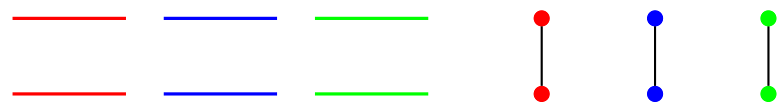

Figure 6.1 A six-vertex (2,4)-agreeable graph G with agreement proportion $1/3$, and a corresponding linear society.

for (k, m)-agreeable graphs G on n vertices that minimize the agreement proportion, where δ is the minimum degree of G. This, along with an analog of Theorem 5.6 for (k, m)-agreeable graphs, would allow us to repeat the proof of Theorem 5.7 to prove Conjecture 6.1.

However, we can show that Equation 6.1 does not hold for general (k, m)-agreeable graphs minimizing agreement proportion. To see this, we consider $k = 2$, $m = 4$, and note that these would convert Equation 6.1 into

$$\delta \geq n - \frac{3}{2}\omega(G) - 1. \tag{6.2}$$

We construct a graph G consisting of three pairs of adjacent vertices with no other edges, shown in Figure 6.1.

We first note that this graph is $(2, 4)$-agreeable because any collection of four vertices must contain at least two adjacent vertices. We can easily calculate that its agreement proportion is $1/3$, and observe that it has boxicity 1. Thus we can apply the Berg et al. (2010) result to see that $(2 - 1)/(4 - 1) = 1/3$ and thus this graph has minimal agreement proportion among $(2, 4)$-agreeable interval graphs.

Therefore if Equation 6.1 holds for all (k, m)-agreeable graphs on n vertices that minimize the agreement proportion, it must be that Equation 6.2 holds for this particular graph G.

However, for this G we have $n = 6$ and $\omega(G) = 2$, so the right hand side of Equation 6.2 becomes $6 - 3 - 1 = 2$, while the left hand side is 1 (since all of the vertices have degree 1). Thus Equation 6.1 does not hold for general k, m, and d.

It may be the case that Equation 6.1 does hold for some other specific cases, but the above example demonstrates that there is no completely general extension of the argument of Chapter 5 to prove Conjecture 6.1.

It does, however, remain possible, and perhaps likely, that Conjecture 6.1 or even a stronger version could be proved by some alternate method, or some special cases of the general framework used by Abrahams et al. (2010) and Chapter 5.

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