Prove It!

Kenny W. Moran

Pomona College, Claremont, CA
kennywmoran@gmail.com

"Sixty-eight equals seven plus sixty-one. Eight equals three plus five. One-hundred equals seventy-one plus twenty-nine. Eight hundred and eighty-eight equals two hundred sixty-nine plus six hundred nineteen. Don't you get it?" said Sarah as she stared wide-eyed at her father. She spoke softly but quickly, eagerly informing her father of the fruits of her thought.

"Of course I see what you're getting at; you can write all of those even numbers as the sum of two primes. And of course all of that's perfectly valid. But I could do the same thing and – well, look at it this way. If I sum the digits of ten, one hundred, one thousand, or one million, each time I will get one. You don't see me walking around claiming that's true of every number." Frank spoke in an assuring tone, but his face revealed that he was visibly disturbed.

"Of course not – that's because that would be wrong. You would be right if you told everyone that it's true for any number that has a one at the beginning and all zeroes after it," she snapped, smugly.

Frank took a deep breath before beginning his slow, nuanced explanation: "I know. That's my point, though – there's a reason I'd be able to make my claim in the limited case, and it's very simple – I can prove it. Once we had a good operational definition of digits, and that necessary condition you pointed out, and basic things like succession and addition, we'd be able to show, using a clever technique called algebra, that it's always going to work out that way. We can PROVE, dear, that IF the number is a power of ten, THEN its digits must sum to one. That's why it's true – because we can prove it. Examples aren't good enough in math, because numbers can be very tricky."

"No, Daddy, you don't get it. That's stupid. That's silly. It's not true because we can prove it – we can prove it because it's true!" exclaimed Sarah in a voice that was a tad too loud for a university classroom in the afternoon. Frank noticed her

indignance. Despite her sustained excitement, his refusal to accept her discovery on its face had her unsettled. "What I told you earlier was right, though, so it only makes sense that you would be able to prove it with this algebra thing."

Frank's skin turned flush as he was hit with a wave of guilt. He found himself in a difficult position; as often was the case, Sarah's mathematical prowess forced him to speak to her with all the callous frankness he would speak to his colleagues with. Then again, his treatment of his daughter over mathematical matters could just as easily be seen as extreme respect. Whenever Sarah found herself wandering into advanced mathematical concepts, Frank was forced to weigh these two competing concerns; failing to praise his daughter sufficiently was just as amiss, in his eyes, as failing to sufficiently stimulate and nurture her budding genius by patronizing her or giving her a watered-down notion of mathematics. "That's not what it's about, honey. It's about how we know something is true, and why we think something is true. Examples like the ones you gave me give us a good idea of what we might want to prove – but that doesn't make something true on its own. We can use specific examples of things like this to show that the big claims like the one you made, that claims about all of the natural numbers are false – for example, we know that the digits of all numbers don't sum to one, because that's not true of twenty-two. But if I want to actually make one of these claims, or if I want to convince another mathematician that they're true, then I can't do it with a few examples. Even if I go on giving examples all day, it won't be enough. When we're dealing with ALL the natural numbers, we're dealing with an infinity. And humans like us can't ever say anything about an infinity with a couple examples alone. To do that, we need some clever tricks that allow us to PROVE these things we believe about numbers."

Sarah sat down in her chair and stared pensively at the wall to her right. If he hadn't known any better, Frank would have thought she had lost interest, or that she had stopped paying attention. For the daughter of a top mathematics professor at an Ivy League institution, though, this was a standard indication that what had just been said had left her deep in thought, ponderously exploring her own mathematical world.

Unblinking, moving nothing but her mouth, Sarah began thinking aloud through a particularly intense stare: "I don't know... I still don't really get it. Why do I need to prove any of this? It's already true! I don't have any work to do! I just noticed it was true. If I was trying to explain something like why it rains or why my feet smell sometimes, then yeah, maybe I would have to 'prove' why I thought those things happened – but with math it's totally different. With math, we already know about how the numbers work. We know how to add and multiply them, and all that stuff.

And THAT'S why you can make any even number by adding primes to each other. Because of the way that numbers and evenness and adding and primeness work. I mean, I can just tell! Can't you?"

"Listen, Sarah, I understand completely why you think it's so obvious. And you're not the first one to come up with this idea. Mathematicians have been trying to prove that idea, that any even number can be expressed as a sum of two primes, for more than two hundred fifty years. And they still haven't been able to. So while I know where you're coming from, you don't really KNOW for sure that that's true."

"Yes I do. And two hundred fifty is eighty-three plus one hundred sixty-seven." pouted Sarah, frowning with indignation.

Frank sighed and put a palm to his face. He shook his head, carefully choosing his words so he could get through to the naïve yet brilliant little girl he had raised. He tried to figure out how he could get through to her that math doesn't work like that, that the truth isn't something we get to declare and dole out whenever we have a hunch – but he struggled, because, after all, didn't he himself feel the same way she did, or even more so? Sarah had only seen some primitive arithmetic, and perhaps she had had a taste of fractions and negative numbers. Frank, though, had seen too many attempts to prove the Goldbach conjecture to count. He had become intimately familiar with the Prime Number Theorem, and he knew all of the informal justifications of the conjecture of even the slightest relevance. He himself had come within sniffing distance of a complete and valid proof quite a few times in his undergraduate years, before he grew out of such idealism.

Frank had seen all the evidence that human civilization had ever produced that Goldbach should be right – and yet, he wasn't nearly as convinced as his daughter, who hadn't seen anything more than a handful of sums she had calculated, all of which were limited to what amounted to an infinitesimal slice of the infinite set her conjecture intended to tame. What could possibly be driving her conviction? Was he so brainwashed by the obsession with formalism that he couldn't see something that was plain as day? Was the entire mathematical establishment fooling itself away from the truth? Or was his daughter just a little too haughty for her own good?

"Honey, what makes you think that you really KNOW that? Why don't you just THINK it's true? Because I think it's true, too. But I don't know it, because to know something, especially in math, you need proof."

"Because I can tell! Because it's obvious! When I'm thinking about the world of numbers, I just look at a blank wall or a piece of paper, and I ask the numbers to

show me all about themselves. And then they show me! That's how I knew that all the evens could be made with two primes, because they told me! Just like they told me that there are just as many evens as there are odds, and that you'll never run out of prime sandwiches, and that you can sum and then multiply or multiply both and then sum and you'll get the same number either way . . . "

Frank did a double-take and looked at his daughter with disbelief: "What was that you said, about prime sandwiches?"

"Prime sandwiches? Oh, that's what I call it when two primes are almost next-door neighbors. See, they can't be right next to each other because the even numbers can't be prime, except for 2. But if there's a prime number, then an even one, then a prime one, all right next to each other, then that's a prime sandwich."

"And what did you say about them? What did the numbers tell you about prime sandwiches?"

"Well, they told me that you can never run out, of course! Didn't you know that? If you keep counting up, you'll never find the last prime sandwich, just like you'll never find the last prime and you'll never find the last number. There's always a bigger one!"

Frank looked at his daughter with a surreal mixture of shock, confusion, disbelief, humility, and pride. Never before had he seen such a disturbingly compelling demonstration of innocent, Platonic, informal mathematics. Sarah had no agenda to defend, she wasn't indoctrinated into one philosophical school or the next, and she wasn't even familiar enough with mathematical foundations to understand how egregious her method of reason would be seen as by any academic who hadn't given birth to her. Sarah's method of thought had circumvented centuries of painstaking work trying to demonstrate these truths, and she didn't even realize it. Many mathematicians before her had come up with the same conjectures, of course – Sarah's ideas were nothing new. But something deep inside Frank gave him the suspicion that Sarah wasn't just being colorfully metaphoric when she spoke of how numbers talked to her. Something about her uncanny skills in arithmetic, her unwavering insistence on the validity of the most famous conjectures of number theory, and the confidence with which she was willing to assert such abstract mathematical theorems told him he shouldn't discount the possibility that this was supernatural, or exceptional in some way. It may have been arrogant of him to suppose that his offspring would be gifted with inhuman intuition into the nature of mathematics, but it would have been much too irrational of him to disregard these anomalies altogether.

The same thought kept plaguing him as he reflected, though. If Sarah has some supernatural insight into the natural numbers, that implies that the natural numbers exist in a sense that he would have never thought they existed after being introduced to the formal foundations of number theory. It would imply that there really was a world of idealized numbers, and that his work was really descriptive in nature. It would imply that mathematics means more than its axioms, and mathematics derives meaning not only from its interpretations and applications. It would mean that Frank's profession itself was an exercise in characterizing the universe, in understanding the structure of reality, in exploring the mind of God.

Frank was torn between two responses to his daughter's newest pseudo-theorem. He could either go to the blackboard and start listing the Peano axioms, or he could hug her and congratulate her brilliance. He pondered the decision for what seemed like an hour, before realizing that the inescapable fact of the matter was that he just didn't KNOW which one to choose. He didn't know, and he couldn't know. But perhaps, maybe, there was some infinitesimal chance that Sarah really could KNOW something so certainly, without a single line of proof to back it up. The idea was so intellectually preposterous that he wouldn't dare embrace it in public – but there was still a fleeting suspicion in his mind that introducing Sarah to foundational mathematics, trying to convince her that those numbers talking to her weren't really real and were just abstract representations her brain built of these nine bland, soulless axioms – nine axioms that hadn't been able to produce the results that Sarah had produced just by listening to her numerical acquaintances – would be missing the point. There was too much risk in his mind that to expose her to such a radically different approach to math would be destroying something beautiful.

Frank bent down to scoop his daughter up in his arms. He kissed her on the cheek and embraced her tightly, whispering in her ear how proud he was of his little genius. Frank told her nicely to sit down in the front row and get ready to learn. Perhaps it wasn't yet the time to see how this savant dealt with stuffy academic formality. Going backwards and delving into the foundations, he concluded, could only do harm here. What Sarah needed was not soul-crushing axioms, but new potential and new methods. He grabbed some chalk and began writing on the blackboard: "3x+2=11".

"Now, let's say I have three bags of apples and two oranges too, and I have eleven fruits in total. If I want to figure out how many apples are in each bag . . ."