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## Q.954 and A.954, Quickie Problem and Solution

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**Q954.** Proposed by Arthur Benjamin, Harvey Mudd College, Claremont, CA, and Michel Bataille, Rouen, France.

Show that for positive integer  $n$ ,

$$\sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} = \sum_{k=0}^n 2^k \binom{n}{k}^2.$$

**A954.** Let  $[n]$  denote the set  $\{1, 2, \dots, n\}$  and  $S$  denote the set of ordered pairs  $(A, B)$  where  $A$  is a subset of  $[n]$  and  $B$  is an  $n$ -subset of  $[2n]$  that is disjoint from  $A$ . We can select elements for  $S$  in two ways:

(1) For  $0 \leq k \leq n$ , let  $Z$  be a  $k$ -subset of  $[n]$ . Let  $A = Z^c$ , the complement of  $Z$ , which is an  $(n - k)$ -subset of  $[n]$ , and let  $B$  be an  $n$ -subset of  $\{n + 1, \dots, 2n\} \cup Z$ . This yields

$$|S| = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k}.$$

(2) For  $0 \leq k \leq n$ , choose a  $k$ -subset  $B_1$  from  $\{n + 1, \dots, 2n\}$  and a  $k$ -subset  $B_2$  of  $[n]$ . Then form  $B = B_1 \cup B_2^c$ , and choose  $A$  from among the  $2^k$  subsets of  $B_2$ . This leads to

$$|S| = \sum_{k=0}^n 2^k \binom{n}{k}^2.$$

This completes the proof.

*Note.* Another proof, using lattice paths, can be found in Robert A. Sulanke's article, Objects Counted by the Central Delannoy Numbers, *The Journal of Integer Sequences*, Vol 6, 2003. A proof by polynomials is in Michel Bataille's paper Some Identities about an Old Combinatorial Sum, *The Mathematical Gazette*, March 2003, pp. 144-8.

A slight change in the above proof leads to

$$\sum_{k=0}^n \binom{n}{k} \binom{m+k}{n} = \sum_{k=0}^n 2^k \binom{n}{k} \binom{m}{k},$$

for  $m \geq n$ , a generalization proved by Li Zhou using lattice paths in *The Mathematical Gazette*.