Generalized Connectors

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GENERALIZED CONNECTORS*

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Abstract. An $n$-connector is an acyclic directed graph having $n$ inputs and $n$ outputs and satisfying the following condition: given any one-to-one correspondence between inputs and distinct outputs, there exists a set of vertex-disjoint paths that join each input to the corresponding output. It is known that the minimum possible number of edges in an $n$-connector lies between lower and upper bounds that are asymptotic to $3n \log_3 n$ and $6n \log_3 n$ respectively. A generalized $n$-connector satisfies the following stronger condition: given any one-to-many correspondence between inputs and disjoint sets of outputs, there exists a set of vertex-disjoint trees that join each input to the corresponding set of outputs. It is shown that the minimum number of edges in a generalized $n$-connector is asymptotic to the minimum number in an $n$-connector.

Imagine an information transmission network intended to mediate between $n$ sources of information and $n$ users of this information. At any time, any of the users may wish to be connected with any of the sources; a user can be connected with only one source at a time, but many users may wish to be connected with the same source. This paper deals with an idealized version of the problem of designing a network capable of providing any such pattern of simultaneous connections.

An $(n, m)$-graph is an acyclic directed graph with a set of $n$ distinguished vertices called inputs and a disjoint set of $m$ distinguished vertices called outputs. An $n$-graph is an $(n, n)$-graph.

An $n$-connector is an $n$-graph satisfying the following condition: given any one-to-one correspondence between inputs and distinct outputs, there exists a set of vertex-disjoint paths that join each input to the corresponding output. (A path joining an input to an output is a directed path whose origin is the input and whose destination is the output.) Let $c(n)$ denote the minimum possible number of edges in an $n$-connector; it is known that

$$3n \log_3 n \leq c(n) \leq 6n \log_3 n + O(n)$$

(see Pippenger and Valiant [4, Remark 2.2.6]).

A generalized $n$-connector is an $n$-graph satisfying the following stronger condition: given any one-to-many correspondence between inputs and disjoint sets of outputs, there exists a set of vertex-disjoint trees that join each input to the corresponding set of outputs. (A tree joining an input to a set of outputs is a directed tree whose root is the input and whose leaves are the outputs.) Let $d(n)$ denote the minimum possible number of edges in a generalized $n$-connector; that

$$d(n) \leq 10n \log_2 n + O(n)$$

for $n$ a power of 2 is implicit in the work of Ofman [1]. Thompson [5] has recently shown that

$$d(n) \leq 12n \log_3 n + O(n)$$

for $n$ a power of 3.

The object of this note is to show that

$$d(n) = c(n) + O(n),$$

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and thus that

\[ d(n) \sim c(n). \]

It is clear that

\[ d(n) \geq c(n); \]

thus it will suffice to show that

(1) \[ d(n) \leq c(n) + O(n). \]

This will be done by means of a new type of graph which will be called a generalizer. An \textit{n-generalizer} is an \textit{n}-graph that satisfies the following condition: given any correspondence between inputs and nonnegative integers that sum to at most \textit{n}, there exists a set of vertex-disjoint trees that join each input to the corresponding number of distinct outputs. Let \( g(n) \) denote the minimum possible number of edges in an \textit{n}-generalizer; it will be shown below that

(2) \[ g(n) \leq 120n + O((\log n)^2), \]

so that in particular

\[ g(n) = O(n). \]

A generalized \textit{n}-connector can be obtained from an \textit{n}-generalizer and an \textit{n}-connector by identifying the outputs of the generalizer with the inputs of the connector, as shown in Fig. 1. It is obvious that this yields a generalized \textit{n}-connector: the generalizer provides the appropriate number of copies of each input, and the connector joins these copies to the appropriate outputs. Thus

\[ d(n) \leq c(n) + g(n) \leq c(n) + O(n), \]

which completes the proof of (1).

\[ \begin{array}{c}
\text{n-GENERALIZER} \\
\{ \text{n INPUTS} \}
\end{array} \quad \begin{array}{c}
\text{n-CONNECTOR} \\
\{ \text{n OUTPUTS} \}
\end{array} \quad \text{GENERALIZED n-CONNECTOR} \]

\[ \begin{array}{c}
\text{--- INDICATES IDENTIFICATION} \\
\text{OF VERTICES (NOT EDGES)}
\end{array} \]

Fig. 1.
It remains to prove (2). To do this, two more types of graphs, called concentrators and superconcentrators, will be needed.

An \( n \)-superconcentrator is an \( n \)-graph that satisfies the following condition: given any set of inputs and any equinumerous set of outputs, there exists a set of vertex-disjoint paths that join the given inputs in a one-to-one fashion to the given outputs. Let \( s(n) \) denote the minimum possible number of edges in an \( n \)-superconcentrator; that

\[
s(n) \leq 234n
\]

was shown by Valiant [6], who first defined superconcentrators. Pippenger [3] subsequently showed that

\[
s(n) \leq 39n + O(\log n).
\]

An \((n, m)\)-concentrator is an \((n, m)\)-graph that satisfies the following condition: given any set of \( m \) or fewer inputs, there exists a set of vertex-disjoint paths that join the given inputs in a one-to-one fashion to distinct outputs. Let \( r(n, m) \) denote the minimum possible number of edges in an \((n, m)\)-concentrator; that

\[
r(n, m) \leq 29n
\]

was shown by Pinsker [2], who first defined concentrators. It will now be shown that

\[
(3) \quad r(n, \lfloor n/2 \rfloor) \leq 20n + O(\log n),
\]

where \( \lfloor \cdots \rfloor \) denotes "the greatest integer less than or equal to \( \cdots \)."

A \((n, \lfloor n/2 \rfloor)\)-concentrator can be obtained by combining \( \lfloor n/2 \rfloor \) edges with an \( \lfloor n/2 \rfloor \)-superconcentrator (where \( \lceil \cdots \rceil \) denotes "the least integer greater than or equal to \( \cdots \)"), as shown in Fig. 2. It is obvious that this yields an \((n, \lfloor n/2 \rfloor)\)-

![Diagram](image-url)
concentrator: those of the given inputs that lie among the upper \([n/2]\) inputs can be joined to distinct outputs through the edges; those that lie among the lower \([n/2]\) can be joined to other distinct outputs through the superconcentrator. Thus

\[
  r(n, [n/2]) \leq [n/2] + s([n/2]) \\
  \leq [n/2] + 39[n/2] + O(\log [n/2]) \\
  \leq 20n + O(\log n),
\]

which completes the proof of (3).

It still remains to prove (2). This will be done by means of a recursive construction: an \(n\)-generalizer can be obtained by combining an \((n, [n/2])\)-concentrator, an \([n/2]\)-generalizer, 2\([n/2]\) edges, and an \(n\)-superconcentrator, as shown in Fig. 3. This can be seen to yield an \(n\)-generalizer as follows. If an input is to be joined to \(x\)

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**Fig. 3.**
distinct outputs, one can write $x = 2y + z$, where $y$ is a nonnegative integer and $z$ is either 0 or 1. Since the $x$'s sum to at most $n$, there can be at most $\lfloor n/2 \rfloor$ inputs for which $y$ is greater than 0. Each of these inputs can therefore be joined to a distinct output of the concentrator, thence to $y$ distinct outputs of the $\lfloor n/2 \rfloor$-generalizer, and finally to $2y$ distinct outputs of the $n$-generalizer. All that remains is to join the inputs for which $z$ is 1 to other distinct outputs; this can be done through the superconcentrator. Thus

$$g(n) = g(\lfloor n/2 \rfloor) + r(n, \lfloor n/2 \rfloor) + 2\lfloor n/2 \rfloor + s(n)$$

$$\leq g(\lfloor n/2 \rfloor) + 20n + O(\log n) + 2\lfloor n/2 \rfloor + 39n + O(\log n)$$

$$\leq g(\lfloor n/2 \rfloor) + 60n + O(\log n)$$

$$\leq 120n + O((\log n)^2),$$

which completes the proof of (2).

The result of this note is satisfying from a theoretical point of view: information-theoretic considerations suggest that since

$$\log n^n = \log n! + O(n)$$

one should have

$$d(n) = c(n) + O(n)$$

as has indeed been shown to be the case. The proof technique used in this note, however, does not endow the result with any practical significance: $120n$ exceeds $6n \log_3 n$ until $n$ exceeds $3^{20} = 3,486,784,401$.

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**REFERENCES**


