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Review: Nontangential Limits in Pt(μ)-spaces and the Index of Invariant Subgroups

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Nontangential limits in $P^t(\mu)$-spaces and the index of invariant subspaces. (English summary)


This deep and interesting article answers a number of fundamental questions about boundary behavior in $P^t(\mu)$ spaces and the relationship between the measure $\mu$ and the index of invariant subspaces of $P^t(\mu)$. The introduction is well written and inviting and hence we freely paraphrase portions of it below.

In order the summarize the results of this article, a few preliminary definitions are required. First, let $\mu$ denote a finite positive measure on the closed unit disk $\overline{D}$, let $1 \leq t < \infty$, and let $P^t(\mu)$ denote the closure of the analytic polynomials in $L^t(\mu)$. Multiplication by $z$ is a bounded linear operator on $P^t(\mu)$ and is denoted by $S$. An invariant subspace of $P^t(\mu)$ is a closed linear subspace $M$ of $P^t(\mu)$ which satisfies $SM \subseteq M$.

The two most familiar examples of $P^t(\mu)$ spaces are those corresponding to $\mu = \frac{1}{2\pi} m$ (normalized Lebesgue measure on $\partial D$) and $\mu = \frac{1}{\pi} A$ (normalized Lebesgue measure on $D$). These cases satisfy $\mu(\partial D) > 0$ and $\mu(\partial D) = 0$, respectively, and they illustrate the type of phenomenon that motivates the article. In particular, observe that:

- For $\mu = \frac{1}{2\pi} m$, one obtains the classical Hardy spaces $H^t$. It is well known that each $f \in H^t$ has a nontangential limit $f^*(z)$ at $m$-almost every $z \in \partial D$ and that $f^* = f$ as elements on $L^t(\mu)$. Moreover, one has a strong uniqueness criterion, for $f^* = 0$ on a set of positive $m$-measure implies that $f = 0$. By Beurling’s Theorem ($t = 2$) and its extension to other values of $t$, it follows that every nonzero invariant subspace of $H^t$ has index 1.

- For $\mu = \frac{1}{\pi} A$ one obtains the Bergman spaces $L^t_a$. Functions in $L^t_a$ need not have nontangential limits at any point of $\partial D$. Furthermore, for any $1 \leq n \leq \infty$ there exist invariant subspaces of $L^t_a$ having index $n$ [C. Apostol et al., J. Funct. Anal. 63 (1985), no. 3, 369–404; MR0808268 (87i:47004a); J. Eschmeier, Math. Ann. 298 (1994), no. 1, 167–186; MR1252824 (94k:47010); H. Hedenmalm, S. Richter and K. Seip, J. Reine Angew. Math. 477 (1996), 13–30; MR1405310 (97i:46044)].

In light of these results the authors are led to study the case where $D$ is the set of analytic bounded point evaluations for $P^t(\mu)$ and $P^t(\mu)$ contains no nontrivial characteristic functions (i.e., $P^t(\mu)$ is irreducible). It is known in this case that the restriction of $\mu$ to $\partial D$ must be of the form $h |dz|$.

If $\mu(\partial D) > 0$, then one has a boundary function $f|\partial D$ for each $f \in P^t(\mu)$. However, the precise relationship between $f|\partial D$ and the limiting behavior of the analytic function $f|D$ is not immediately clear. The following questions are suggested by the examples above:

1. Is $f|\partial D$ the boundary value function of $f$ in some suitable sense?
2. Is $f|\partial D = f^* |\partial D$-almost everywhere?
3. Is \( f \) determined by \( f|_{\partial \mathbb{D}} \)? In other words, does \( f|_{\partial \mathbb{D}} = 0 \) imply that \( f = 0 \)?

4. Is the index of every nonzero invariant subspace equal to 1?

The first major theorem of this paper answers all four of these questions in the affirmative:

Theorem A. Suppose that \( \mu \) is supported in \( \overline{\mathbb{D}} \) and is such that the set of analytic bounded point evaluations for \( \mathcal{P}^t(\mu) \) is equal to \( \mathbb{D} \) and \( \mathcal{P}^t(\mu) \) is irreducible, and that \( \mu(\partial \mathbb{D}) > 0 \). Then:

(a) If \( f \in \mathcal{P}^t(\mu) \), then the nontangential limit \( f^*(z) \) of \( f \) exists for \( \mu|_{\partial \mathbb{D}} \)-almost all \( z \), and \( f^* = f|_{\partial \mathbb{D}} \) as elements of \( \mathcal{L}^t(\mu|_{\partial \mathbb{D}}) \).

(b) Every nonzero invariant subspace of \( \mathcal{P}^t(\mu) \) has index 1.

An important consequence of this work is an affirmative answer to a conjecture of J. B. Conway and L. M. Yang [in Holomorphic spaces (Berkeley, CA, 1995), 201–209, Cambridge Univ. Press, Cambridge, 1998; MR1630651 (99e:47027)]. In particular, the present authors show that for \( 1 < t < \infty \) one has \( \dim M/zM = 1 \) for every nonzero invariant subspace \( M \) of \( \mathcal{P}^t(\mu) \) if and only if \( h \neq 0 \).

On the other hand, away from the part of \( \partial \mathbb{D} \) where \( \mu \) has mass, the boundary behavior of \( \mathcal{P}^t(\mu) \) functions can be wild. To be more specific, there is a natural notion of interpolating sequences for \( \mathcal{P}^t(\mu) \) spaces and Theorem B of this article shows (under the hypotheses of Theorem A) that for \( t \in (1, \infty) \) and \( E \subseteq \partial \mathbb{D} \) with \( \mu(E) = 0 \), there is an interpolating sequence for \( \mathcal{P}^t(\mu) \) which clusters nontangentially at \( m \)-almost every point of \( E \). In particular, this implies that there are functions in \( \mathcal{P}^t(\mu) \) which have nontangential limits at \( m \)-almost no points of \( E \). In the case \( \mu(\partial \mathbb{D}) = 0 \) and \( t \in (1, \infty) \), this proves the existence of an interpolating sequence for \( \mathcal{P}^t(\mu) \) that clusters nontangentially at \( m \)-almost every point of \( \partial \mathbb{D} \). The argument used to prove [A. Aleman, S. Richter and C. Sundberg, J. Anal. Math. 86 (2002), 139–182; MR1894480 (2003g:30058) (Proposition 7.3)] then yields invariant subspaces of \( \mathcal{P}^t(\mu) \) of index greater than 1. Thus one has a new proof of the results of [C. Apostol et al., op. cit.] and [J. Eschmeier, op. cit.] on the index of invariant subspaces of the Bergman spaces \( \mathcal{L}^t_a \).

For bounded point evaluations \( \lambda \) for \( \mathcal{P}^t(\mu) \), let \( e_\lambda \) denote the associated evaluation functional and let \( M_\lambda = \|e_\lambda\|_{\mathcal{P}^t(\mu)^*} \). One of the key elements in proving Theorem A is obtaining the inequality

\[
\limsup_{\lambda \to z} (1 - |\lambda|^2)^{1/t} M_\lambda \leq \frac{C}{h(z)^{1/t}},
\]

for \( m \)-almost all \( z \in \partial \mathbb{D} \) where \( C \) is some constant. Remarkably, for \( t > 1 \) the authors are even able to prove the following asymptotic result:

Theorem C. Under the hypotheses of Theorem A, if \( t > 1 \), then

\[
\lim_{\lambda \to z} (1 - |\lambda|^2)^{1/t} M_\lambda = \frac{1}{h(z)^{1/t}}
\]

for \( m \)-almost all \( z \in \partial \mathbb{D} \).

The proofs of these results utilize portions of J. E. Thomson’s proof of the existence of bounded point evaluations [Ann. of Math. (2) 133 (1991), no. 3, 477–507; MR1109351 (93g:47026)] along with X. Tolsa’s recent work on analytic capacity [Acta Math. 190 (2003), no. 1, 105–149; MR1982794 (2005c:30020)]. The methods used are similar to those developed independently by J. E. Brennan in his recent proof, based upon Tolsa’s work, of Thomson’s Theorem [Algebra
The authors are even able to refine Thomson’s result somewhat. For a compactly supported measure \( \nu \) in \( \mathbb{C} \), they describe the locations of bounded point evaluations for \( P_t(\mu) \) in terms of the Cauchy transform of an annihilating measure (Corollary 2.2).

Reviewed by Stephan R. Garcia

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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