1-1-2009

Review: Hankel and Toeplitz Transforms on $H^1$: Continuity, Compactness and Fredholm Properties

Stephan Ramon Garcia

Pomona College

Recommended Citation
Hankel and Toeplitz transforms on $H^1$: continuity, compactness and Fredholm properties.

(English summary)

Integral Equations Operator Theory 61 (2008), no. 4, 573–591.1420-8989

The authors discuss boundedness and compactness of Hankel and Toeplitz operators acting on the Hardy space $H^1$. Among other things, they provide a new proof (Theorem 1.5) of the fact that a Hankel operator $H_a$ having symbol $a$ is bounded on $H^1$ if and only if $a$ has bounded logarithmic mean oscillation (recall that a function $f$ belongs to $\text{BMO}_\text{log}$ if $f \in L^1$ and
\[
\|f\|_{**} = \sup_I \frac{\log \left( \frac{4\pi}{|I|} \right)}{|I|} \int_I |f(\zeta) - f_I||d\zeta| < \infty
\]

Moreover, the authors also prove (Theorem 1.6) that the operator norm of $H_a$ acting on $H^1$ is comparable to $\|P_1 a\|_{\text{BMO}_\text{log}}$. Here $P_1$ denotes the (unbounded) truncation operator
\[
P_1 f(\zeta) \sim \sum_{n=1}^{\infty} \hat{f}(n) \zeta^n.
\]

Concluding their treatment of Hankel operators on $H^1$, the authors show that $H_a$ is compact on $H^1$ if and only if $P_1 a \in \text{VMO}_\text{log}$ (Theorem 1.7). Here the subspace $\text{VMO}_\text{log}$ of $\text{BMO}_\text{log}$ is defined to be the set of all $f \in L^1$ such that
\[
\lim_{\delta \to 0^+} \sup_{|I| \leq \delta} \frac{\log \left( \frac{4\pi}{|I|} \right)}{|I|} \int_I |f(\zeta) - f_I||d\zeta| = 0.
\]

The second part of the article deals with the spectral properties of Toeplitz operators. In particular, the authors prove a theorem (Theorem 1.8) for $p = 1$ motivated by the well-known result of R. G. Douglas [Bull. Amer. Math. Soc. 74 (1968), 895–899; MR0229070 (37 #4648)] concerning the Fredholm properties of Toeplitz operators on $H^2$ with symbols $a \in C + H^\infty$. Since the criteria for boundedness and compactness are different in the cases $p = 1$ and $1 < p < \infty$, it turns out that $C$ must be replaced by $C \cap \text{VMO}_\text{log}$ and $H^\infty$ must be replaced by $H^\infty \cap \text{BMO}_\text{log}$ in order for the analogous statements to go through.

Reviewed by Stephan R. Garcia
References

12. D. A. Stegenga, *Bounded Toeplitz operators on $H^1$ and applications of the duality between $H^1$ and the functions of bounded mean oscillation*. Amer. J. of Math. 98, no. 3 (1976), 573–589. MR0420326 (54 #8340)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

© Copyright American Mathematical Society 2009, 2013