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James M. Cargal

Troy State University, Montgomery

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The Reform Calculus Debate and the Psychology of Learning Mathematics

James M. Cargal
Troy State University
PO Box 4419
Montgomery, AL 36103
e-mail: JMCargal@aol.com

It is a basic truth for the vast majority who learn mathematics, that they come to understand and appreciate the foundations of a subject only when they have some maturity in that subject.

This essay is prompted by a recent debate in the American Mathematical Monthly between Professor Tucker[1] and Professor Swann[2]. Their debate is a good example of the national, sometimes acrimonious debate on "reform calculus," with Professor Tucker on the reform side and Professor Swann on the side of tradition. I claim that the debate is misdirected, that at issue are long standing problems in mathematics education.

THE NEED FOR FORMAL PROOFS IN BEGINNING CALCULUS (THERE ISN'T ANY)

Professor Tucker says at the beginning of his article (p. 231): "I have serious doubts about the need for formal theorems and proofs in a standard calculus course." Likewise, Professor Swann says near the beginning of his article (p. 241): "I agree that proofs of the extreme value theorem and other global results from basic principles do not belong in today's beginning calculus texts in the present educational climate." This is amazing in that the debate then focuses on the foundations of calculus and like so many such discussions centers on the mean value theorem (for derivatives). Hence the exchange is basically like a knife fight between two pacifists. To be fair, Professor Swann has hedged his philosophy with the key phrase *in the present educational climate*. Later (p. 242) he adds "It surely is imperative to introduce the notion of proof in beginning multisequence calculus and keep it alive though actual proofs are few." I understand his misgivings about today's educational environment, and I think that during the basic calculus sequence the students should be exposed to δ - ϵ proofs perhaps twice in the perhaps vain hope of priming the pump: that maybe some idea may germinate or at least the gist of what a precise statement from calculus looks like will make some lasting (positive) impression.

Whereas I see periodically in print that δ - ϵ proofs are inappropriate for the basic calculus sequence, the fact is, that many teachers employ them and this is true in

junior college classrooms on up. This is symptomatic of more widespread problems in mathematics pedagogy and will bring us back to the reform calculus debate. The use of δ - ϵ proofs in the basic calculus sequence is simply a bad idea. Too few students are ready for it. Students who endure it often learn to do various δ - ϵ exercises by rote. It adds a great deal of stress to a course that has enough stress and it diverts attention from basic concepts that have to be learned. The problem is simple: δ - ϵ proofs are foundational in nature and the students are not ready for foundations.

FOUNDATIONS COME LAST

It is a basic truth for the vast majority who learn mathematics, that they come to understand and appreciate the foundations of a subject only when they have some maturity in that subject. Most learning in elementary mathematics starts with some rote. Rote learning seems to be universally condemned, yet if one starts with theory, the students usually have no idea what one is talking about. Those education experts who condemn rote learning so often wind up teaching nothing at all. It is important to recognize that rote learning in itself isn't necessarily bad; it is bad only when it is the main object of learning rather than a foundation for higher learning. If you want to teach a child arithmetic, you teach her to count largely by rote. Later she must learn addition and multiplication tables and again this is largely rote. Perhaps you would rather try to explain Peano's axioms to her? Then, of course, define addition and multiplication recursively and prove their basic properties. Then explain to her how important associativity is and then, of course, how Hindu-Arabic notation is a product of genius.

THE CRISIS OF SHALLOWNESS

Part of the crisis in calculus education stems from the lament that students who have completed the basic sequence have a rather shallow understanding of what they are doing. It is said that they do problems me-

chanically without appreciation of the substance at hand. The reformers want to use geometry and calculation to instill greater insight into the fundamentals of calculus. However, this occurs at the expense of symbolic manipulation. I don't think it makes much difference: one type of lesson is going to be replaced with another and in any case the student is going to wind up with a shallow understanding of the subject.

I would like to know what are the math areas where the students come out of an introductory course with deep understanding. Sure most students come out of the calculus sequence having learned to solve problems by rote. During their courses they ignore the proofs and often the theorem statements and they study the text examples to learn to do the homework problems and if the instructor uses similar problems

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on the tests then they may graduate. What they get out of lecture is inversely proportional to the time spent on proofs and theory. The fact is, these are the good students. Many students are unable to learn to do the examples by rote and they must go to other areas. As for those who do get past the sequence through learning by rote, give them time; some of them will acquire depth and will be good prospects for graduate school. The real problem in calculus is that too many of the students capable of rote learning and later deeper learning are filtered out with the others. They change majors not because they didn't have the potential and the work ethic to make it in calculus, but because they had a bad instructor, often a totally useless instructor at a critical time. They get the instructor who thinks his job is to present the material. And he thinks that first semester calculus is a fine time to introduce the students to full rigor. This teacher is untroubled by the fact that he flunks eighty per cent of his students. These students just don't have the right stuff. Again and again this happens and the teacher never seems to realize that the students who pass are students who have had the material before, along with one or two students who are simply brilliant. By teaching the course the way he understands the material himself, he is in effect teaching a review course, and the lecture is stimulating for the students who already know the material. I claim that a significant portion of mathematics teachers perform in a review mode most of the time they teach, and this drives away stu-

dents who might otherwise do quite well in mathematics.

ONE SEMESTER EQUALS FIFTY YEARS

Although a historical approach to teaching a subject can be over done, in learning it is safe to say that ontogeny should follow phylogeny (or should I say ontopedagogy should follow phylopedagogy?). It is not an accident that the foundations of calculus followed the discovery of calculus by two-hundred years. A good rule of thumb is: when students first reach calculus, each semester is equivalent to fifty years of history. First semester calculus is the late seventeenth century: the definition of derivative and the fundamental theorem of calculus. Second and third semester calculus are the eighteenth century: loads of applications and much problem solving. The semester following the calculus sequence is the time of Cauchy: it is time to ponder what has been learned and to start thinking more precisely about fundamentals. The semester after that (the fifth semester) brings us to the time of Weierstrass: now is the time to look at δ - ϵ proofs. Those students who simply cannot handle it should not do graduate work in mathematics but may have opportunities elsewhere. Interestingly, there are students who two years earlier thought δ - ϵ definitions were completely unintelligible and now find them rather simple, almost obvious, perhaps elegant. These students have promise in the mathematical sciences. Their promise as teachers is another matter. Because, it is rather easy for the student to convince himself that he could have understood this material two years earlier but it was not explained properly. However, it is really quite clear and he knows that when it is his time to teach calculus he will show the students δ - ϵ definitions and proofs because there is no reason that the students can't understand them right off the bat. It is quite simple really, we only need to be precise, we must have rigor and so on, and the students are stuck with another mathematics teacher from hell.

Consider the (promising) student just out of the calculus sequence. At that point calculus is a large basket of definitions, theorems, formulas, techniques, and a huge variety of problems. Although it was covered in one of his tests, he doesn't remember what the mean value theorem says. The fundamental theorem of calculus says that to evaluate an integral you find the anti-derivative and plug in the end points: that is all of it he remembers and with just that amount of pre-

cision. Following the sequence, some of his calculus is reinforced by a course in differential equations. The material starts to gel and he starts to think and work more precisely. He takes an analysis sequence which makes him think more rigorously and to finally contemplate and start to appreciate the foundations of analysis. In graduate school he gets a great deal more analysis, and also he teaches calculus recitation classes. On leaving graduate school he gets a job as a teacher and it so happens that virtually every year he teaches some calculus course or analysis course. Now the interesting thing is that virtually every time he faces the material of calculus, he learns something new. To be sure there may be no great revelations, and perhaps he is not always conscious of the new insight. But after twenty years of this he can get extremely involved in discussions on calculus and, say, the mean value theorem. More important is that as he has acquired knowledge over the last thirty years, the calculus has become simpler and simpler to him. What once was horrendously complex seems simple; as far as our professor is concerned calculus boils down to maybe

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four things: the completeness of the reals, the definition of derivative, the fundamental theorem of calculus and the mean value theorem. Having great insights, he endeavors to pass it on to his students. Each lecture is filled with insights gathered over the years, and he is handing this hard-won knowledge over to the students so that they can get it immediately without trudging through the swamps as he did. However, the damn students appreciate none of this. They nod their heads politely during lecture and concentrate on learning to do the exercises in the text. This, of course, is a crisis. We need to change the way we teach calculus because the current students who complete the sequence seem to have a shallow understanding of the topic.

SLOW DOWN, DAMN IT

If you really want to increase the understanding of your students on leaving the course, you might try the following trick: cover less material!! Consider the undergraduate abstract algebra sequence. Often it is taught out of a text like Fraleigh[3]. At the second half of the second semester the professor covers Galois theory and at the very end he proves the non-existence of a quintic formula. Out of his fifteen students,

ten have no idea what he is talking about most of the second semester. Four others are picking up fragments. The fifteenth student, the young woman who actually has comprehended most of the course, is the student who will be offered scholarships to do graduate work at Harvard, Berkeley, and Cal Tech. At least four students and probably more, could have gotten much more out of the course. That the course went too fast is because it was taught as the teacher sees the subject; the lectures were appropriate for a review course.

Surprisingly, it can be more work to teach less material. It requires effort to focus on the students and what they actually comprehend. A course that is taught in a theorem-proof format, much like the way Edmund Landau wrote books, is relatively easy to teach, and that is one reason that many teachers use that style. It is, of course, enjoyable to teach the one brilliant student; she comprehends so much more than the others do. But she is the one student who does not need much detailed attention. She might benefit more from reading a book, say the text on Galois theory by John Stillwell[4]. It is a lousy choice for a text, but it is an enlightening read, and could be inspirational to our brilliant student. One other point: covering less material does not mean giving less homework. Not only that, but the students might do more homework because they are not lost all of the time.

A RELEVANT NOTE

In 1995 the Mathematical Association of America published a book about Ralph P. Boas, Jr. who died in 1992[5]. On page 98 of that book, Professor Boas says: I once heard Wiener admit that, although he had used the ergodic theorem, he had never gone through a proof of it. Later, of course, he did prove (and improve) it. It so happens that I once expressed surprise at this to Professor Boas. Here is his reply[6]:

I do not think my story about Wiener is very surprising. One can't always be going back to first principles.

I quite agree that—at least for some people (I am one of them) calculation precedes understanding. I have probably said before that I knew how to calculate with logarithms long before I knew how they worked. The idea that proofs come first is, I think, a modern

fallacy. Certainly—even in this calculator age—a child learns that $2 \times 2 = 4$ before understanding why. The trouble with "new math" was (in part) the fallacy of thinking that understand needs to come first.

At the risk of overstating the point, let me repeat the key phrase: The idea that proofs come first is, I think, a modern fallacy.

THE REFORM CALCULUS QUESTION

There are reasons why the reform calculus is attractive. There are also good arguments for some of the more traditional approaches. I have not given any of these arguments here because they are irrelevant to my point. Whichever approach we choose will trade

off one superficial knowledge for another. Any approach, if it is at all a feasible approach, and if it is done well, will provide a foundation for the student to go on to a deeper knowledge of calculus and its applications. If the student is not going to go further, then his taking the sequence in the first place was probably a waste of his time and the teacher's, and the approach used is still irrelevant. Perhaps in twenty years, something of a consensus will emerge about what approach to teaching elementary calculus is most fruitful. In the mean time, the controversy does not deserve the present rancor. It is true that today's students are less well prepared than thirty years ago, and are less inclined to work. Nonetheless, were this not so, I would still suggest: try teaching less material, try to save the theoretical overview for later, and lighten up.

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6. Letter to J. M. Cargal from Ralph Boas, post marked 24 May 1991, dated the same day.

NUMBERS - 2

Numbers, delight and entrance
Necessary for music and dance

Patterns abound with numbers
Some are found rhyming this verse

Such pure simplicity
Hides vast complexity

Eternal truths found by addition
Show questions without solution

The most basic being
One, two, three
What art thee?