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COMPLEX SYMMETRIC PARTIAL ISOMETRIES

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ABSTRACT. An operator $T \in B(\mathcal{H})$ is complex symmetric if there exists a conjugate-linear, isometric involution $C : \mathcal{H} \to \mathcal{H}$ so that $T = CT^*C$. We provide a concrete description of all complex symmetric partial isometries. In particular, we prove that any partial isometry on a Hilbert space of dimension $\leq 4$ is complex symmetric.

1. Introduction

The aim of this note is to complete the classification of complex symmetric partial isometries which was started in [10]. In particular, we give a concrete necessary and sufficient condition for a partial isometry to be a complex symmetric operator.

Before proceeding any further, let us first recall a few definitions. In the following, $\mathcal{H}$ denotes a separable, complex Hilbert space and $B(\mathcal{H})$ denotes the collection of all bounded linear operators on $\mathcal{H}$.

Definition. A conjugation is a conjugate-linear operator $C : \mathcal{H} \to \mathcal{H}$, which is both involutive (i.e., $C^2 = I$) and isometric (i.e., $\langle Cx, Cy \rangle = \langle y, x \rangle$).

Definition. We say that $T \in B(\mathcal{H})$ is $C$-symmetric if $T = CT^*C$. We say that $T$ is complex symmetric if there exists a conjugation $C$ with respect to which $T$ is $C$-symmetric.

It is straightforward to show that if $\dim \ker T \neq \dim \ker T^*$, then $T$ is not a complex symmetric operator. For instance, the unilateral shift is perhaps the most ubiquitous example of a partial isometry which is not complex symmetric (see [7, Prop. 1], [9, Ex. 2.14], [6, Cor. 7]). On the other hand, we have the following theorem from [10]:

Theorem 1. Let $T \in B(\mathcal{H})$ be a partial isometry.

(i) If $\dim \ker T = \dim \ker T^* = 1$, then $T$ is a complex symmetric operator,

(ii) If $\dim \ker T \neq \dim \ker T^*$, then $T$ is not a complex symmetric operator.

(iii) If $2 \leq \dim \ker T = \dim \ker T^* \leq \infty$, then either possibility can (and does) occur.

Although these results are the sharpest possible statements that can be made given only the data ($\dim \ker T, \dim \ker T^*$), they are in some sense unsatisfactory. For instance, it is known that partial isometries on $\mathcal{H}$ that are not complex symmetric exist if $\dim \mathcal{H} \geq 5$ and that every partial isometry on $\mathcal{H}$ is complex symmetric if $\dim \mathcal{H} \leq 3$, the authors were unable to answer the corresponding question if

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dim $\mathcal{H} = 4$. To be more specific, the techniques used in [10] were insufficient to resolve the question in the case where $\dim \mathcal{H} = 4$ and $\dim \ker T = 2$. Significant numerical evidence in favor of the assertion that all partial isometries on a four-dimensional Hilbert space are complex symmetric has recently been produced by J. Tener [13].

Suppose that $T$ is a partial isometry on $\mathcal{H}$ and let

$$\mathcal{H}_1 = (\ker T) \perp = \mathrm{ran} T^*$$

(1)
denote the initial space of $T$ and $\mathcal{H}_2 = (\mathcal{H}_1) \perp = \ker T$ denote its orthogonal complement (see [12, Pr. 127] or [2, Ch. VIII, Sect. 3] for terminology). With respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, we have

$$T = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$$

(2)

where $A : \mathcal{H}_1 \to \mathcal{H}_1$ and $B : \mathcal{H}_1 \to \mathcal{H}_2$. Furthermore, the fact that $T^*T$ is the orthogonal projection onto $\mathcal{H}_1$ yields the identity

$$A^*A + B^*B = I,$$

(3)

where $I$ denotes the identity operator on $\mathcal{H}_1$. Finally, observe that the operator $A \in B(\mathcal{H}_1)$ is simply the compression of the partial isometry $T$ to its initial space.

The main result of this note is the following concrete description of complex symmetric partial isometries:

**Theorem 2.** Let $T \in B(\mathcal{H})$ be a partial isometry. If $A$ denotes the compression of $T$ to its initial space, then $T$ is a complex symmetric operator if and only if $A$ is a complex symmetric operator.

Due to its somewhat lengthy and computational proof, we defer the proof of the preceding theorem until Section 3. We remark that Theorem 2 remains true if one instead considers the final space of $T$. Indeed, simply apply the theorem with $T^*$ in place of $T$ and then take adjoints.

**Corollary 1.** Every partial isometry of rank $\leq 2$ is complex symmetric.

**Proof.** Let $T \in B(\mathcal{H})$ be a partial isometry such that $\mathrm{rank} T \leq 2$. If $\mathrm{rank} T = 0$, then $T = 0$ and there is nothing to prove. If $\mathrm{rank} T = 1$, then this is handled in [10]. In the case $\mathrm{rank} T = 2$, we may write

$$T = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$$

where $A$ is an operator on a two-dimensional space. Since every operator on a two-dimensional Hilbert space is complex symmetric (see [1, Cor. 3], [3, Cor. 3.3], [7, Ex. 6], [10, Cor. 1], [12, Cor. 3]), the desired conclusion follows from Theorem 2. □

**Corollary 2.** Every partial isometry on a Hilbert space of dimension $\leq 4$ is complex symmetric.

**Proof.** As mentioned earlier, the results of [10] indicate that only the case $\dim \mathcal{H} = 4$ and $\dim \ker T = 2$ requires resolution. The corollary is now immediate consequence of Theorem 2 and the fact that every operator on a two-dimensional Hilbert space is complex symmetric. □
We conclude this section with the following theorem, which asserts that each \( C \)-symmetric partial isometry can be extended to a \( C \)-symmetric unitary operator on the whole space (the significance lies in the fact that the corresponding conjugations for these two operators are the same).

**Theorem 3.** If \( T \) is a \( C \)-symmetric partial isometry, then there exists a \( C \)-symmetric unitary operator \( U \) and an orthogonal projection \( P \) such that \( T = UP \).

**Proof.** Since \( T \) is a \( C \)-symmetric partial isometry, it follows that \( |T| = P \) is an orthogonal projection and that \( T = CJP \) where \( J \) is a conjugation supported on \( \text{ran} P \) which commutes with \( P \) \[8, Sect. 2.2\]. We may extend \( J \) to a conjugation \( \tilde{J} \) on all of \( \mathcal{H} \) by forming the internal direct sum \( J \oplus J' \) where \( J' \) is a partial conjugation supported on \( \ker P \). The operator \( U = CJ \) is a \( C \)-symmetric unitary operator.

\[ \square \]

2. Partial isometries and the norm closure problem

Partial isometries on infinite-dimensional spaces often provide examples of note. For instance, one can give a simple example of a partial isometry \( T \) satisfying \( \dim \ker T = \dim \ker T^* = \infty \) which is not a complex symmetric operator:

**Example 1.** Let \( S \) denote the unilateral shift on \( l^2(\mathbb{N}) \). Although \( S \) is certainly not a complex symmetric operator (by (ii) of Theorem 1, see also [9, Ex. 2.14], or [6, Cor. 7]), part (i) of Theorem 1 does ensure that the partial isometry \( S \oplus 0 \) is complex symmetric. Indeed, simply take \( N \) to be the bilateral shift on \( l^2(\mathbb{Z}) \) and note that \( S \oplus 0_\infty \oplus e_0 \). That \( S \oplus S^* \) is also complex symmetric can also be verified by a direct computation \[8, Ex. 5\]. On the other hand, the partial isometry \( T = S \oplus 0 \) on \( l^2(\mathbb{N}) \oplus l^2(\mathbb{N}) \) is not a complex symmetric operator by Lemma 1.

Let \( S(\mathcal{H}) \) denote the subset of \( B(\mathcal{H}) \) consisting of all bounded complex symmetric operators on \( \mathcal{H} \). There are several ways to think about \( S(\mathcal{H}) \). By definition, we have

\[ S(\mathcal{H}) = \{ T \in B(\mathcal{H}) : \exists \text{ a conjugation } C \text{ s.t. } T = CT^*C \}. \]

If \( C \) is a fixed conjugation on \( \mathcal{H} \), then we also have

\[ S(\mathcal{H}) = \{ UTU^* : T = CT^*C, \text{ U unitary} \}. \]

Thus if we identify \( \mathcal{H} \) with \( l^2(\mathbb{N}) \) and \( C \) denotes the canonical conjugation on \( l^2(\mathbb{N}) \) (i.e., entry-by-entry complex conjugation), we can think of \( S(\mathcal{H}) \) as being the unitary orbit of the set of all bounded (infinite) complex symmetric matrices.

The following example shows that the set \( S(\mathcal{H}) \) is not closed in the strong operator topology (SOT):

**Example 2.** We maintain the notation of Example 1. For \( n \in \mathbb{N} \), let \( P_n \) denote the orthogonal projection onto the span of the basis vectors \( \{ e_i : i \geq n \} \) of \( l^2(\mathbb{N}) \). Now observe that each operator \( T_n = P_nS \oplus S^* \) is unitarily equivalent to \( S \oplus 0_\infty \oplus S^* \) where \( 0_\infty \) denotes the zero operator on an \( n \)-dimensional Hilbert space. Each \( T_n \) is complex symmetric since \( S \oplus S^* \) is complex symmetric (by Lemma 1). On the other hand, since \( P_nS \) is SOT-convergent to 0, it follows that the SOT-limit of the sequence \( T_n \) is \( 0 \oplus S^* \), which is not a complex symmetric operator (by Lemma 1).
The preceding example demonstrates that the set of all complex symmetric operators (on a fixed, infinite-dimensional Hilbert space $H$) is not SOT-closed. We also remark that the conjugations corresponding to the operators $T_n$ from Example 2 depend on $n$. In contrast, if we fix a conjugation $C$, then it is elementary to see that the set of $C$-symmetric operators is a SOT-closed subspace of $B(H)$.

We conclude with a related question, which we have been unable to resolve:

**Question.** Is $S(H)$ norm closed?

### 3. Proof of Theorem 2

This entire section is devoted to the proof of Theorem 2. We first require the following lemma:

**Lemma 1.** If $H, K$ are separable complex Hilbert spaces, then $T \in B(H)$ is a complex symmetric operator if and only if $T \oplus 0 \in B(H \oplus K)$ is a complex symmetric operator.

**Proof.** If $T$ is a $C$-symmetric operator on $H$, then it is easily verified that $T \oplus 0$ is $(C \oplus J)$-symmetric on $H \oplus K$ for any conjugation $J$ on $K$. The other direction is slightly more difficult to prove.

Suppose that $S = T \oplus 0$ is a complex symmetric operator on $H \oplus K$. Before proceeding any further, let us remark that it suffices to consider the case where

$$H = \text{ran} T + \text{ran} T^*.$$  \hfill (4)

Otherwise let $H_1 = \text{ran} T + \text{ran} T^*$ and note that $H_1$ is a reducing subspace of $H$. If $H_2$ denotes the orthogonal complement of $H_1$ in $H$, then with respect to the orthogonal decomposition $H_1 \oplus H_2 \oplus K$, the operator $S$ has the form $T' \oplus 0 \oplus 0$, where $T'$ denotes the restriction of $T$ to $H_1$. By now considering $S$ with respect to the orthogonal decomposition $H \oplus K = H_1 \oplus (H_2 \oplus K)$, it follows that we need only consider the case where (4) holds.

Suppose now that (4) holds and that $S$ is $C$-symmetric where $C$ denotes a conjugation on $H \oplus K$. Writing the equations $CS = S^*C$ and $CS^* = SC$ in terms of the $2 \times 2$ block matrices

$$S = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$  \hfill (5)

(the entries $C_{ij}$ of $C$ are conjugate-linear operators), we find that

$$C_{11}T = T^*C_{11},$$  \hfill (6)
$$C_{21}T = C_{21}T^* = 0,$$  \hfill (7)
$$T^*C_{12} = TC_{12} = 0.$$  \hfill (8)

Since $C_{21}T = C_{21}T^* = 0$, it follows that $C_{21}$ vanishes on $\text{ran} T + \text{ran} T^*$ and hence on $H$ itself by (4). On the other hand, (8) implies that $C_{12}$ vanishes on the orthogonal complements of $\ker T$ and $\ker T^*$ in $H$. By (4), this implies that $C_{12}$ vanishes identically.

It follows immediately from (5) that $C_{11}$ and $C_{22}$ must be conjugations on $H$ and $K$, respectively, whence $T$ is $C_{11}$-symmetric by (6). This concludes the proof of the lemma. \hfill $\square$
Now let us suppose that $T$ is a partial isometry on $\mathcal{H}$ and let
\[ \mathcal{H}_1 = (\ker T)^\perp = \text{ran} \, T^*. \]
and $\mathcal{H}_2 = \ker T$. With respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, it follows that
\[ T = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \]
where $A : \mathcal{H}_1 \to \mathcal{H}_1$, $B : \mathcal{H}_1 \to \mathcal{H}_2$, and
\[ A^* A + B^* B = I. \tag{9} \]

$(\Rightarrow)$ Suppose that $T$ is a complex symmetric operator. For an operator with polar decomposition $T = U |T|$ (i.e., $U$ is the unique partial isometry satisfying $\ker U = \ker T$ and $|T|$ denotes the positive operator $\sqrt{T^* T}$), the Aluthge transform of $T$ is defined to be the operator $\tilde{T} = |T|^\frac{1}{2} U |T|^\frac{1}{2}$. Noting that
\[ T^* T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \]
we find that
\[ \tilde{T} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}. \]

By [5, Thm. 1], we know that the Aluthge transform of a complex symmetric operator is complex symmetric. Applying Lemma 1 to $\tilde{T}$, we conclude that $A$ is complex symmetric, as desired.

$(\Leftarrow)$ Let us now consider the more difficult implication of Theorem 2, namely that if $A$ is a complex symmetric operator, then $T$ is as well. We claim that it suffices to consider the case where $\text{ran} \, B = \mathcal{H}_2$. In other words, we argue that if $K = \text{ran} \tilde{T} + \text{ran} \tilde{T}^*$, then we may suppose that $K = \mathcal{H}$. Indeed, $K$ is a reducing subspace for $T$ and $T = 0$ on $K^\perp$. By Lemma 1 if $T|_K$ is a complex symmetric operator, then so is $T$.

Write $B = V |B|$ where $V : \mathcal{H}_1 \to \mathcal{H}_2$ is a partial isometry with initial space $(\ker B)^\perp \subseteq \mathcal{H}_1$ and final space $\mathcal{H}_2$ (since $\text{ran} \, B = \mathcal{H}_2$). In particular, we have the relations
\[ V^* B = |B| = B^* V, \quad |B| = \sqrt{I - A^* A}. \tag{10} \]

By hypothesis, the operator $A \in B(\mathcal{H}_1)$ is complex symmetric. Therefore suppose that $K$ is a conjugation on $\mathcal{H}_1$ such that $KA = A^* K$ and observe that the equations
\[ A \sqrt{I - A^* A} = \sqrt{I - AA^*}, \]
\[ A^* \sqrt{I - AA^*} = \sqrt{I - A^* AA^*}, \]
\[ K \sqrt{I - A^* A} = \sqrt{I - AA^* K}, \]
\[ K \sqrt{I - AA^*} = \sqrt{I - A^* AK}, \]
follow from a standard polynomial approximation argument (i.e., if $p(x) \in \mathbb{R}[x]$, then $Ap(A^* A) = p(AA^*)A$ and $Kp(A^* A) = p(AA^*)K$ hold whence the desired identities follow upon passage to the strong operator limit). In particular, it follows from the preceding that
\[ (KA) \sqrt{I - A^* A} = \sqrt{I - A^* A} (KA), \]
that is
\[ KA|B| = |B|KA, \quad A^*K|B| = |B|A^*K. \]  
(11)

Let us now define a conjugate-linear operator \( C \) on \( H \) by the formula
\[
C = \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix}.
\]  
(12)

Assuming for the moment that \( C \) is a conjugation on \( H \), we observe that
\[
\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.
\]

Since it is clear that \( J \) is a partial conjugation which is supported on the range of \( |T| \) and which commutes with \( |T| \), it follows immediately that \( T \) is a \( C \)-symmetric operator (see [8, Thm. 2]).

To complete the proof of Theorem 2, we must therefore show that \( C \) is a conjugation on \( H \). In other words, we must check that \( C^2 \) is the identity operator on \( H \) and that \( C \) is isometric. Since these computations are somewhat lengthy, we perform them separately:

**Claim:** \( C^2 = I \).

**Pf. of Claim.** We first expand out \( C^2 \) as a \( 2 \times 2 \) block matrix:
\[
C^2 = \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix}
= \begin{pmatrix} AKAK + KB^*BK & AKB^* - KB^*VA^*KV^* \\ BKAK - VA^*KV^*BK & BKKB^* + VA^*KV^*VA^*KV^* \end{pmatrix}
= \begin{pmatrix} AA^* + KB^*BK & AB^* - KB^*VA^*KV^* \\ BA^* - VA^*KV^*BK & BB^* + VA^*KV^*VA^*KV^* \end{pmatrix}.
\]

To obtain the preceding line, we used the fact that \( K \) is a conjugation and \( A \) is \( K \)-symmetric. Letting \( E_{ij} \) denote the entries of the preceding block matrix we find that
\[
E_{11} = AA^* + KB^*BK
= AA^* + K(|I - A^*A)K
= AA^* + (I - AA^*)
= I.
\]

\[
E_{12} = AB^* - KB^*VA^*KV^*
= AB^* - K|B|A^*KV^* \quad \text{by (11)}
= AB^* - KA^*K|B|V^* \quad \text{by (11)}
= AB^* - A|B|V^*
= AB^* - AB^* \quad \text{since } B^* = |B|V
= 0.
\]

\[
E_{21} = BA^* - VA^*KV^*BK
\]
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\[ BA^* - V A^* K |B| K \]
\[ = BA^* - V |B| A^* K K \]
\[ = BA^* - V |B| A^* \]
\[ = BA^* - BA \]
\[ = 0. \]

As for \( E_{22} \), it suffices to show that \( E_{22} \) agrees with \( I \) (the identity operator on \( H_2 \)) on the range of \( B \), which is dense in \( H_2 \). In other words, we wish to show that \( E_{22} B x = B x \) for all \( x \in H_2 \), which is equivalent to showing that
\[ E_{22} B x = BB^* B x + VA^* KV^* VA^* KV^* B x = B x \quad (13) \]
for all \( x \in H_2 \). Let us investigate the second term of (13):
\[ VA^* KV^* VA^* KV^* B x = VA^* KV^* VA^* K |B| x \]
\[ = VA^* KV^* V |B| A^* K x \]
\[ = VA^* K |B| A^* K x \]
\[ = V |B| A^* K A^* K x \]
\[ = BA^* K A^* K x \]
\[ = BA^* Ax \]
\[ = B(I - B^* B)x \]
\[ = Bx - BB^* Bx. \]

Putting this together with (13), we find that \( E_{22} B x = B x \) for all \( x \in H_2 \) whence \( E_{22} = I \), as claimed. \( \square \)

Claim: \( C \) is isometric.

Pf. of Claim. The proof requires three steps:

(i) Show that \( C \) is isometric on \( H_1 \),

(ii) Show that \( C \) is isometric on \( BH_1 \), which is dense in \( H_2 \),

(iii) Show that \( C H_1 \perp C(BH_1) \).

For the first portion, observe that
\[ \left\| C \begin{pmatrix} x \\ 0 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} AKx \\ BK \end{pmatrix} \begin{pmatrix} KB^* \\ -VA^* KV^* \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} \right\|^2 \\
= \left\| \begin{pmatrix} AKx \\ BKx \end{pmatrix} \right\|^2 \\
= (AKx, AKx) + (BKx, BKx) \\
= (A^* AKx, Kx) + (B^* BKx, Kx) \\
= ((A^* A + B^* B)Kx, Kx) \\
= (Kx, Kx) \\
= \|Kx\|^2 \\
= \|x\|^2. \]
Thus (i) holds.
Now for (ii):

\[
\begin{align*}
\| C \begin{pmatrix} 0 \\ Bx \end{pmatrix} \|^2 &= \left\| \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} 0 \\ Bx \end{pmatrix} \right\|^2 \\
&= \left\| \begin{pmatrix} KB^*Bx \\ -VA^*KV^*Bx \end{pmatrix} \right\|^2 \\
&= \|KB^*Bx\|^2 + \|VA^*KV^*Bx\|^2 \\
&= \|B^*Bx\|^2 + \|VA^*K|B|x\|^2 \\
&= \|B^*Bx\|^2 + \|VA^*K|x\|^2 \\
&= \|B^*Bx\|^2 + \|BA^*K|x\|^2 \\
&= \|B^*Bx\|^2 + \langle BA^*K|x, BA^*K|x \rangle \\
&= \|B^*Bx\|^2 + \langle B^*BA^*K|x, A^*K|x \rangle \\
&= \|B^*Bx\|^2 + \langle (I - A^*A)x, A^*K|x \rangle \\
&= \|B^*Bx\|^2 + \langle A^*K(I - A^*A)x, A^*K|x \rangle \\
&= \|B^*Bx\|^2 + \langle K(I - A^*A)x, A^*K|x \rangle \\
&= \langle B^*Bx, B^*Bx \rangle + \langle KAA^*K|x, (I - A^*A)x \rangle \\
&= \langle (I - A^*A)x, (I - A^*A)x \rangle + \langle A^*Ax, (I - A^*A)x \rangle \\
&= \langle x, (I - A^*A)x \rangle - \langle A^*Ax, (I - A^*A)x \rangle + \langle A^*Ax, (I - A^*A)x \rangle \\
&= \langle x, (I - A^*A)x \rangle \\
&= \langle x, B^*Bx \rangle \\
&= \langle Bx, Bx \rangle \\
&= \|Bx\|^2.
\end{align*}
\]

Thus (ii) holds.

Now for (iii):

\[
\begin{align*}
\left\langle C \begin{pmatrix} x \\ 0 \end{pmatrix}, C \begin{pmatrix} 0 \\ By \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} 0 \\ By \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} AKx \\ BKx \end{pmatrix}, \begin{pmatrix} KB^*By \\ -VA^*KV^*By \end{pmatrix} \right\rangle \\
&= \langle AKx, KB^*By \rangle - \langle BKx, VA^*KV^*By \rangle \\
&= \langle B^*By, KAx \rangle - \langle BKx, VA^*K|By \rangle \\
&= \langle B^*By, A^*x \rangle - \langle BKx, V|B|A^*Ky \rangle \\
&= \langle AB^*By, x \rangle - \langle BKx, BA^*Ky \rangle \\
&= \langle AB^*By, x \rangle - \langle B^*BKx, A^*Ky \rangle \\
&= \langle AB^*By, x \rangle - \langle (I - A^*A)Kx, A^*Ky \rangle \\
&= \langle AB^*By, x \rangle - \langle K(I - AA^*)x, A^*Ky \rangle \\
&= \langle AB^*By, x \rangle - \langle KA^*Ky, (I - AA^*)x \rangle \\
&= \langle AB^*By, x \rangle - \langle Ay, (I - AA^*)x \rangle.
\end{align*}
\]
\[
= \langle AB^*By, x \rangle - \langle (I - AA^*)Ay, x \rangle \\
= \langle AB^*By, x \rangle - \langle A(I - A^*A)y, x \rangle \\
= \langle AB^*By, x \rangle - \langle AB^*By, x \rangle \\
= 0.
\]

By the polarization identity, it follows that

\[
\left\langle C \left( \begin{array}{c} x_1 \\ Bx_2 \end{array} \right), C \left( \begin{array}{c} y_1 \\ By_2 \end{array} \right) \right\rangle = \left\langle \left( \begin{array}{c} x_2 \\ By_2 \end{array} \right), \left( \begin{array}{c} x_1 \\ By_1 \end{array} \right) \right\rangle
\]

holds for all \( x_1, x_2, y_1, y_2 \in \mathcal{H} \) whence \( C \) is isometric on \( \mathcal{H} \). \( \square \)

References


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