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Complex Symmetric Partial Isometries

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COMPLEX SYMMETRIC PARTIAL ISOMETRIES

STEPHAN RAMON GARCIA AND WARREN R. WOGEN

Abstract. An operator $T \in B(\mathcal{H})$ is complex symmetric if there exists a conjugate-linear, isometric involution $C : \mathcal{H} \to \mathcal{H}$ so that $T = CT^*C$. We provide a concrete description of all complex symmetric partial isometries. In particular, we prove that any partial isometry on a Hilbert space of dimension $\leq 4$ is complex symmetric.

1. Introduction

The aim of this note is to complete the classification of complex symmetric partial isometries which was started in [10]. In particular, we give a concrete necessary and sufficient condition for a partial isometry to be a complex symmetric operator.

Before proceeding any further, let us first recall a few definitions. In the following, $\mathcal{H}$ denotes a separable, complex Hilbert space and $B(\mathcal{H})$ denotes the collection of all bounded linear operators on $\mathcal{H}$.

Definition. A conjugation is a conjugate-linear operator $C : \mathcal{H} \to \mathcal{H}$, which is both involutive (i.e., $C^2 = I$) and isometric (i.e., $\langle Cx, Cy \rangle = \langle y, x \rangle$).

Definition. We say that $T \in B(\mathcal{H})$ is $C$-symmetric if $T = CT^*C$. We say that $T$ is complex symmetric if there exists a conjugation $C$ with respect to which $T$ is $C$-symmetric.

It is straightforward to show that if $\dim \ker T \neq \dim \ker T^*$, then $T$ is not a complex symmetric operator. For instance, the unilateral shift is perhaps the most ubiquitous example of a partial isometry which is not complex symmetric (see [7, Prop. 1], [9, Ex. 2.14], [6, Cor. 7]). On the other hand, we have the following theorem from [11]:

Theorem 1. Let $T \in B(\mathcal{H})$ be a partial isometry.

(i) If $\dim \ker T = \dim \ker T^* = 1$, then $T$ is a complex symmetric operator,

(ii) If $\dim \ker T \neq \dim \ker T^*$, then $T$ is not a complex symmetric operator.

(iii) If $2 \leq \dim \ker T = \dim \ker T^* \leq \infty$, then either possibility can (and does) occur.

Although these results are the sharpest possible statements that can be made given only the data $(\dim \ker T, \dim \ker T^*)$, they are in some sense unsatisfactory. For instance, it is known that partial isometries on $\mathcal{H}$ that are not complex symmetric exist if $\dim \mathcal{H} \geq 5$ and that every partial isometry on $\mathcal{H}$ is complex symmetric if $\dim \mathcal{H} \leq 3$, the authors were unable to answer the corresponding question if

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dim $\mathcal{H} = 4$. To be more specific, the techniques used in [10] were insufficient to resolve the question in the case where dim $\mathcal{H} = 4$ and dim ker $T = 2$. Significant numerical evidence in favor of the assertion that all partial isometries on a four-dimensional Hilbert space are complex symmetric has recently been produced by J. Tener [13].

Suppose that $T$ is a partial isometry on $\mathcal{H}$ and let

$$\mathcal{H}_1 = (\ker T)^\perp = \text{ran } T^*$$

(1)
denote the initial space of $T$ and $\mathcal{H}_2 = (\mathcal{H}_1)^\perp = \ker T$ denote its orthogonal complement (see [12, Pr. 127] or [2, Ch. VIII, Sect. 3] for terminology). With respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, we have

$$T = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$$

(2)

where $A : \mathcal{H}_1 \to \mathcal{H}_1$ and $B : \mathcal{H}_1 \to \mathcal{H}_2$. Furthermore, the fact that $T^* T$ is the orthogonal projection onto $\mathcal{H}_1$ yields the identity

$$A^* A + B^* B = I,$$

(3)

where $I$ denotes the identity operator on $\mathcal{H}_1$. Finally, observe that the operator $A \in B(\mathcal{H}_1)$ is simply the compression of the partial isometry $T$ to its initial space.

The main result of this note is the following concrete description of complex symmetric partial isometries:

**Theorem 2.** Let $T \in B(\mathcal{H})$ be a partial isometry. If $A$ denotes the compression of $T$ to its initial space, then $T$ is a complex symmetric operator if and only if $A$ is a complex symmetric operator.

Due to its somewhat lengthy and computational proof, we defer the proof of the preceding theorem until Section 3. We remark that Theorem 2 remains true if one instead considers the final space of $T$. Indeed, simply apply the theorem with $T^*$ in place of $T$ and then take adjoints.

**Corollary 1.** Every partial isometry of rank $\leq 2$ is complex symmetric.

**Proof.** Let $T \in B(\mathcal{H})$ be a partial isometry such that rank $T \leq 2$. If rank $T = 0$, then $T = 0$ and there is nothing to prove. If rank $T = 1$, then this is handled in [10]. In the case rank $T = 2$, we may write

$$T = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$$

where $A$ is an operator on a two-dimensional space. Since every operator on a two-dimensional Hilbert space is complex symmetric (see [11, Cor. 3], [11, Cor. 3.3], [7, Ex. 6], [10, Cor. 1], [14, Cor. 3]), the desired conclusion follows from Theorem 2. □

**Corollary 2.** Every partial isometry on a Hilbert space of dimension $\leq 4$ is complex symmetric.

**Proof.** As mentioned earlier, the results of [10] indicate that only the case dim $\mathcal{H} = 4$ and dim ker $T = 2$ requires resolution. The corollary is now immediate consequence of Theorem 2 and the fact that every operator on a two-dimensional Hilbert space is complex symmetric. □
We conclude this section with the following theorem, which asserts that each $C$-symmetric partial isometry can be extended to a $C$-symmetric unitary operator on the whole space (the significance lies in the fact that the corresponding conjugations for these two operators are the same).

**Theorem 3.** If $T$ is a $C$-symmetric partial isometry, then there exists a $C$-symmetric unitary operator $U$ and an orthogonal projection $P$ such that $T = UP$.

**Proof.** Since $T$ is a $C$-symmetric partial isometry, it follows that $|T| = P$ is an orthogonal projection and that $T = CJP$ where $J$ is a conjugation supported on $\text{ran } P$ which commutes with $P$ [8, Sect. 2.2]. We may extend $J$ to a conjugation $\tilde{J}$ on all of $\mathcal{H}$ by forming the internal direct sum $J \oplus J'$ where $J'$ is a partial conjugation supported on $\ker P$. The operator $U = CJ$ is a $C$-symmetric unitary operator. \qed

2. Partial isometries and the norm closure problem

Partial isometries on infinite-dimensional spaces often provide examples of note. For instance, one can give a simple example of a partial isometry $T$ satisfying $\dim \ker T = \dim \ker T^* = \infty$ which is not a complex symmetric operator:

**Example 1.** Let $S$ denote the unilateral shift on $l^2(\mathbb{N})$. Although $S$ is certainly not a complex symmetric operator (by (ii) of Theorem 1, see also [9, Ex. 2.14], or [6, Cor. 7]), part (i) of Theorem 1 does ensure that the partial isometry $S \oplus S^*$ is complex symmetric. Indeed, simply take $N$ to be the bilateral shift on $l^2(\mathbb{Z})$ and note that $S \oplus S^*$ is unitarily equivalent to $N - Ne_0 \otimes e_0$. That $S \oplus S^*$ is complex symmetric can also be verified by a direct computation [8, Ex. 5]. On the other hand, the partial isometry $T = S \oplus 0$ on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ is not a complex symmetric operator by Lemma [1].

Let $S(\mathcal{H})$ denote the subset of $B(\mathcal{H})$ consisting of all bounded complex symmetric operators on $\mathcal{H}$. There are several ways to think about $S(\mathcal{H})$. By definition, we have

$$S(\mathcal{H}) = \{ T \in B(\mathcal{H}) : \exists \text{ a conjugation } C \text{ s.t. } T = CT^*C \}.$$ 

If $C$ is a fixed conjugation on $\mathcal{H}$, then we also have

$$S(\mathcal{H}) = \{ UTU^* : T = CT^*C, \ U \text{ unitary} \}.$$ 

Thus if we identify $\mathcal{H}$ with $l^2(\mathbb{N})$ and $C$ denotes the canonical conjugation on $l^2(\mathbb{N})$ (i.e., entry-by-entry complex conjugation), we can think of $S(\mathcal{H})$ as being the unitary orbit of the set of all bounded (infinite) complex symmetric matrices.

The following example shows that the set $S(\mathcal{H})$ is not closed in the strong operator topology (SOT):

**Example 2.** We maintain the notation of Example [1]. For $n \in \mathbb{N}$, let $P_n$ denote the orthogonal projection onto the span of the basis vectors $\{ e_i : i \geq n \}$ of $l^2(\mathbb{N})$. Now observe that each operator $T_n = P_n S \oplus S^*$ is unitarily equivalent to $S \oplus 0_n \oplus S^*$ where $0_n$ denotes the zero operator on an $n$-dimensional Hilbert space. Each $T_n$ is complex symmetric since $S \oplus S^*$ is complex symmetric (by Lemma [1]). On the other hand, since $P_n S$ is SOT-convergent to 0, it follows that the SOT-limit of the sequence $T_n$ is $0 \oplus S^*$, which is not a complex symmetric operator (by Lemma [1]).
The preceding example demonstrates that the set of all complex symmetric operators (on a fixed, infinite-dimensional Hilbert space $\mathcal{H}$) is not SOT-closed. We also remark that the conjugations corresponding to the operators $T_n$ from Example 1 depend on $n$. In contrast, if we fix a conjugation $C$, then it is elementary to see that the set of $C$-symmetric operators is a SOT-closed subspace of $B(\mathcal{H})$.

We conclude with a related question, which we have been unable to resolve:

**Question.** Is $S(\mathcal{H})$ norm closed?

### 3. Proof of Theorem 2

This entire section is devoted to the proof of Theorem 2. We first require the following lemma:

**Lemma 1.** If $\mathcal{H}, \mathcal{K}$ are separable complex Hilbert spaces, then $T \in B(\mathcal{H})$ is a complex symmetric operator if and only if $T \oplus 0 \in B(\mathcal{H} \oplus \mathcal{K})$ is a complex symmetric operator.

**Proof.** If $T$ is a $C$-symmetric operator on $\mathcal{H}$, then it is easily verified that $T \oplus 0$ is $(C \oplus J)$-symmetric on $\mathcal{H} \oplus \mathcal{K}$ for any conjugation $J$ on $\mathcal{K}$. The other direction is slightly more difficult to prove.

Suppose that $S = T \oplus 0$ is a complex symmetric operator on $\mathcal{H} \oplus \mathcal{K}$. Before proceeding any further, let us remark that it suffices to consider the case where

$$\mathcal{H} = \text{ran} T + \text{ran} T^*.$$  \(\text{(4)}\)

Otherwise let $\mathcal{H}_1 = \text{ran} T + \text{ran} T^*$ and note that $\mathcal{H}_1$ is a reducing subspace of $\mathcal{H}$. If $\mathcal{H}_2$ denotes the orthogonal complement of $\mathcal{H}_1$ in $\mathcal{H}$, then with respect to the orthogonal decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{K}$, the operator $S$ has the form $T' \oplus 0 \oplus 0$, where $T'$ denotes the restriction of $T$ to $\mathcal{H}_1$. By now considering $S$ with respect to the orthogonal decomposition $\mathcal{H} \oplus \mathcal{K} = \mathcal{H}_1 \oplus (\mathcal{H}_2 \oplus \mathcal{K})$, it follows that we need only consider the case where \(\text{(4)}\) holds.

Suppose now that \(\text{(4)}\) holds and that $S$ is $C$-symmetric where $C$ denotes a conjugation on $\mathcal{H} \oplus \mathcal{K}$. Writing the equations $CS = S^*C$ and $CS^* = SC$ in terms of the $2 \times 2$ block matrices

$$S = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$ \(\text{(5)}\)

(the entries $C_{ij}$ of $C$ are conjugate-linear operators), we find that

$$C_{11}T = T^*C_{11},$$  \(\text{(6)}\)

$$C_{21}T = C_{21}T^* = 0,$$  \(\text{(7)}\)

$$T^*C_{12} = TC_{12} = 0.$$  \(\text{(8)}\)

Since $C_{21}T = C_{21}T^* = 0$, it follows that $C_{21}$ vanishes on $\text{ran} T + \text{ran} T^*$ and hence on $\mathcal{H}$ itself by \(\text{(4)}\). On the other hand, \(\text{(8)}\) implies that $C_{12}$ vanishes on the orthogonal complements of $\text{ker} T$ and $\text{ker} T^*$ in $\mathcal{H}$. By \(\text{(4)}\), this implies that $C_{12}$ vanishes identically.

It follows immediately from \(\text{(6)}\) that $C_{11}$ and $C_{22}$ must be conjugations on $\mathcal{H}$ and $\mathcal{K}$, respectively, whence $T$ is $C_{11}$-symmetric by \(\text{(6)}\). This concludes the proof of the lemma. \(\square\)
Now let us suppose that \( T \) is a partial isometry on \( \mathcal{H} \) and let
\[
\mathcal{H}_1 = (\ker T)^\perp = \operatorname{ran} T^*.
\]
and \( \mathcal{H}_2 = \ker T \). With respect to the decomposition \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \), it follows that
\[
T = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}
\]
where \( A : \mathcal{H}_1 \to \mathcal{H}_1 \), \( B : \mathcal{H}_1 \to \mathcal{H}_2 \), and
\[
A^* A + B^* B = I.
\]

\((\Rightarrow)\) Suppose that \( T \) is a complex symmetric operator. For an operator with polar decomposition \( T = U|T| \) (i.e., \( U \) is the unique partial isometry satisfying \( \ker U = \ker T \) and \( |T| \) denotes the positive operator \( \sqrt{T^*T} \)), the Aluthge transform of \( T \) is defined to be the operator \( \tilde{T} = |T|^{1/2} U |T|^{1/2} \). Noting that
\[
T^* T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},
\]
we find that
\[
\tilde{T} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.
\]
By [5, Thm. 1], we know that the Aluthge transform of a complex symmetric operator is complex symmetric. Applying Lemma 1 to \( \tilde{T} \), we conclude that \( A \) is complex symmetric, as desired.

\((\Leftarrow)\) Let us now consider the more difficult implication of Theorem 2, namely that if \( A \) is a complex symmetric operator, then \( T \) is as well. We claim that it suffices to consider the case where \( \operatorname{ran} B = \mathcal{H}_2 \). In other words, we argue that if
\[
K = \operatorname{ran} T + \operatorname{ran} T^*,
\]
then we may suppose that \( K = \mathcal{H} \). Indeed, \( K \) is a reducing subspace for \( T \) and \( T = 0 \) on \( K^\perp \). By Lemma 1 if \( T|_K \) is a complex symmetric operator, then so is \( T \).

Write \( B = V|B| \) where \( V : \mathcal{H}_1 \to \mathcal{H}_2 \) is a partial isometry with initial space \( (\ker B)^\perp \subseteq \mathcal{H}_1 \) and final space \( \mathcal{H}_2 \) (since \( \operatorname{ran} B = \mathcal{H}_2 \)). In particular, we have the relations
\[
V^* B = |B| = B^* V, \quad |B| = \sqrt{I - A^* A}.
\]
By hypothesis, the operator \( A \in B(\mathcal{H}_1) \) is complex symmetric. Therefore suppose that \( K \) is a conjugation on \( \mathcal{H}_1 \) such that \( KA = A^* K \) and observe that the equations
\[
A \sqrt{I - A^* A} = \sqrt{I - AA^* A}, \quad A^* \sqrt{I - AA^*} = \sqrt{I - A^* AA^*}, \quad K \sqrt{I - A^* A} = \sqrt{I - AA^* K}, \quad K \sqrt{I - AA^*} = \sqrt{I - A^* AK},
\]
follow from a standard polynomial approximation argument (i.e., if \( p(x) \in \mathbb{R}[x] \), then \( Ap(A^* A) = p(AA^*)A \) and \( Kp(A^* A) = p(AA^*)K \) hold whence the desired identities follow upon passage to the strong operator limit). In particular, it follows from the preceding that
\[
(KA) \sqrt{I - A^* A} = \sqrt{I - A^* A(KA)},
\]
that is
\[ KA|B| = |B|KA, \quad A^*K|B| = |B|A^*K. \tag{11} \]

Let us now define a conjugate-linear operator \( C \) on \( \mathcal{H} \) by the formula
\[
C = \begin{pmatrix}
AK & KB^*\\
BK & -VA^*KV^*
\end{pmatrix}.
\tag{12}
\]

Assuming for the moment that \( C \) is a conjugation on \( \mathcal{H} \), we observe that
\[
\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}^T \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.
\]

Since it is clear that \( J \) is a partial conjugation which is supported on the range of \(|T|\) and which commutes with \(|T|\), it follows immediately that \( T \) is a \( C \)-symmetric operator (see [8, Thm. 2]).

To complete the proof of Theorem 2, we must therefore show that \( C \) is a conjugation on \( \mathcal{H} \). In other words, we must check that \( C^2 \) is the identity operator on \( \mathcal{H} \) and that \( C \) is isometric. Since these computations are somewhat lengthy, we perform them separately:

**Claim:** \( C^2 = I \).

**Pf. of Claim.** We first expand out \( C^2 \) as a \( 2 \times 2 \) block matrix:
\[
C^2 = \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} = \begin{pmatrix} AA^* + KB^*BK & AB^* - KA^*KV^* \\ BA^* - VA^*KV^*BK & BB^* + VA^*KV^*VA^*KV^* \end{pmatrix}.
\]

To obtain the preceding line, we used the fact that \( K \) is a conjugation and \( A \) is \( K \)-symmetric. Letting \( E_{ij} \) denote the entries of the preceding block matrix we find that
\[
E_{11} = AA^* + KB^*BK = AA^* + K(I - A^*A)K = AA^* + (I - AA^*) = I.
\]
\[
E_{12} = AB^* - KB^*VA^*KV^* = AB^* - |B|A^*KV^* \quad \text{by (11)}
\]
\[
= AB^* - KA^*K|B|V^* \quad \text{by (11)}
\]
\[
= AB^* - A|B|V^* = AB^* - AB^* \quad \text{since } B^* = |B|V = 0.
\]
\[
E_{21} = BA^* - VA^*KV^*BK
\]
\[ \begin{align*}
= BA^* - YA^* K |B| K & \quad \text{since } V^* B = |B| \\
= BA^* - V |B| A^* K K & \quad \text{by (11)} \\
= BA^* - V |B| A^* & \quad \text{by (11)} \\
= 0.
\end{align*} \]

As for \( E_{22} \), it suffices to show that \( E_{22} \) agrees with \( I \) (the identity operator on \( \mathcal{H}_2 \)) on the range of \( B \), which is dense in \( \mathcal{H}_2 \). In other words, we wish to show that

\[ E_{22}Bx = BB^*Bx + VA^*KV^*V A^* KV^* Bx = Bx \quad \text{(13)} \]

for all \( x \in \mathcal{H}_2 \). Let us investigate the second term of (13):

\[ \begin{align*}
VA^*KV^*VA^* KV^* Bx &= VA^*KV^*V |B| A^* K x & \quad \text{by (10)} \\
&= VA^*KV^*V |B| A^* K x & \quad \text{by (11)} \\
&= V |B| A^* K A^* K x & \quad \text{by (11)} \\
&= BA^*KA^* K x & \quad \text{since } B = V |B| \\
&= BA^* A x & \quad \text{since } A^* A + B^* B = I \\
&= B (I - B^* B) x & \quad \text{since } A^* A + B^* B = I \\
&= Bx - BB^* Bx.
\end{align*} \]

Putting this together with (13), we find that \( E_{22}Bx = Bx \) for all \( x \in \mathcal{H}_2 \) whence \( E_{22} = I \), as claimed. \( \square \)

Claim: \( C \) is isometric.

Pf. of Claim. The proof requires three steps:

(i) Show that \( C \) is isometric on \( \mathcal{H}_1 \),

(ii) Show that \( C \) is isometric on \( B \mathcal{H}_1 \), which is dense in \( \mathcal{H}_2 \),

(iii) Show that \( C \mathcal{H}_1 \perp C(B \mathcal{H}_1) \).

For the first portion, observe that

\[ \begin{align*}
\left\| C \begin{pmatrix} x \\ 0 \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} AK & KB^* \\ BK & -VA^* KV^* \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} \right\|^2 \\
&= \left\| \begin{pmatrix} AK x \\ BK x \end{pmatrix} \right\|^2 \\
&= (AK x, AK x) + (BK x, BK x) \\
&= (A^* AK x, K x) + (B^* BK x, K x) \\
&= (A^* A + B^* B) K x, K x \\
&= (K x, K x) \\
&= \|K x\|^2 \\
&= \|x\|^2.
\end{align*} \]

Thus (i) holds.
Now for (ii):

\[
\left\| C \begin{pmatrix} 0 \\ Bx \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} 0 \\ Bx \end{pmatrix} \right\|^2 \\
= \left\| \begin{pmatrix} KB^*Bx \\ -VA^*KV^*Bx \end{pmatrix} \right\|^2 \\
= \|KB^*Bx\|^2 + \|VA^*KV^*Bx\|^2 \\
= \|B^*Bx\|^2 + \|VA^*K\|B\|x\|^2 \\
= \|B^*Bx\|^2 + \|VA\|Kx\|^2 \\
= \|B^*Bx\|^2 + \langle BA^*K, BA^*K \rangle \\
= \|B^*Bx\|^2 + \langle BA^*Kx, BA^*Kx \rangle \\
= \|B^*Bx\|^2 + \langle (I - A^*A)A^*Kx, A^*Kx \rangle \\
= \|B^*Bx\|^2 + \langle A^*K(I - A^*A)x, A^*Kx \rangle \\
= \|B^*Bx\|^2 + \langle K(I - A^*A)x, AA^*Kx \rangle \\
= \langle B^*Bx, B^*Bx \rangle + \langle KAA^*Kx, (I - A^*A)x \rangle \\
= \langle (I - A^*A)x, (I - A^*A)x \rangle + \langle A^*Ax, (I - A^*A)x \rangle \\
= \langle x, (I - A^*A)x \rangle - \langle A^*Ax, (I - A^*A)x \rangle + \langle A^*Ax, (I - A^*A)x \rangle \\
= \langle x, (I - A^*A)x \rangle \\
= \langle x, B^*Bx \rangle \\
= \langle Bx, Bx \rangle \\
= \|Bx\|^2.
\]

Thus (ii) holds.

Now for (iii):

\[
\left\langle C \begin{pmatrix} x \\ 0 \end{pmatrix}, C \begin{pmatrix} 0 \\ By \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} 0 \\ By \end{pmatrix} \right\rangle \\
= \left\langle \begin{pmatrix} AKx \\ BKx \end{pmatrix}, \begin{pmatrix} KB^*By \\ -VA^*KV^*By \end{pmatrix} \right\rangle \\
= \langle AKx, KB^*By \rangle - \langle BKx, VA^*KV^*By \rangle \\
= \langle B^*By, KA^*Kx \rangle - \langle BKx, VA^*K\|B\|y \rangle \\
= \langle B^*By, A^*x \rangle - \langle BKx, VB^*A^*Ky \rangle \\
= \langle AB^*By, x \rangle - \langle BKx, BA^*Ky \rangle \\
= \langle AB^*By, x \rangle - \langle B^*BKx, A^*Ky \rangle \\
= \langle AB^*By, x \rangle - \langle (I - A^*A)Kx, A^*Ky \rangle \\
= \langle AB^*By, x \rangle - \langle K(I - AA^*)x, A^*Ky \rangle \\
= \langle AB^*By, x \rangle - \langle KA^*Ky, (I - AA^*)x \rangle \\
= \langle AB^*By, x \rangle - \langle Ay, (I - AA^*)x \rangle 
\]
\[
= \langle AB^* By, x \rangle - \langle (I - AA^*)Ay, x \rangle \\
= \langle AB^* By, x \rangle - \langle A(I - A^*A)y, x \rangle \\
= \langle AB^* By, x \rangle - \langle AB^* By, x \rangle \\
= 0.
\]

By the polarization identity, it follows that
\[
\left\langle C \begin{pmatrix} x_1 \\ Bx_2 \end{pmatrix}, C \begin{pmatrix} y_1 \\ By_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} x_2 \\ By_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ By_1 \end{pmatrix} \right\rangle
\]
holds for all \( x_1, x_2, y_1, y_2 \in \mathcal{H} \) whence \( C \) is isometric on \( \mathcal{H} \). \( \square \)

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