9-1-1987

Ordered Products of Topological Groups

Melvin Henriksen
Harvey Mudd College

Ralph Kopperman
City College of New York

Frank A. Smith
Kent State University

Recommended Citation
Henriksen, M., R. Kopperman, and F. A. Smith. "Ordered products of topological groups." Mathematical Proceedings of the Cambridge Philosophical Society 102.2 (September 1987): 281-295. DOI: 10.1017/S030500410006730X

Math. Proc. Camb. Phil. Soc. (1987), **102**, 281
Printed in Great Britain

Ordered products of topological groups

BY M. HENRIKSEN
Harvey Mudd College, Claremont, CA 91711, *U.S.A.*
R. KOPPERMAN
City College of New York, New York, NY 10031, *U.S.A.*
AND F. A. SMITH
Kent State University, Kent, OH 44242, *U.S.A.*

(*Received* 25 *June* 1986; *revised* 5 *January* 1987)

1. *Introduction*

The topology most often used on a totally ordered group $(G, <)$ is the interval topology. There are usually many ways to totally order $G \times G$ (e.g., the lexicographic order) but the interval topology induced by such a total order is rarely used since the product topology has obvious advantages. Let $\mathbb{R}(+)$ denote the real line with its usual order and $Q(+)$ the subgroup of rational numbers. There is an order on $Q \times Q$ whose associated interval topology is the product topology, but no such order on $\mathbb{R} \times \mathbb{R}$ can be found. In this paper we characterize those pairs G, H of totally ordered groups such that there is a total order on $G \times H$ for which the interval topology is the product topology.

Throughout $(G, <_G)$ will denote a group G with identity element e that is totally ordered by a relation $<_G$ (abbreviated by $<$ whenever the group G is clear from the context) compatible with the multiplication of G. More precisely, if we let $P(G) = \{g \in G: e <_G g\}$, we require that

(α) $P(G)\,P(G) \subseteq P(G)$
(β) $P(G) \cap P(G)^{-1} = \varnothing$
(γ) $P(G)\,g = gP(G)$ for each $g \in G$, and
(δ) $P(G) \cup P(G)^{-1} \cup \{e\} = G$.

Also $a <_G b$ if and only if $a^{-1}b$ or ab^{-1} is in $P(G)$. Any such order $<_G$ is called a *group order* on G. If a subset P of G satisfies (α), (β), (γ) and (δ), and we let $a <_G b$ mean $a^{-1}b \in P$, then $<_G$ is a group order on G for which $P = P(G)$. See [**3**] or [**1**], where the above is formulated in terms of $G^+ = P(G) \cup \{e\}$.

Suppose $<_G$ is a group order on G and $<_H$ is a group order on H. A group order $<$ on $G \times H$ such that $(e, e) < (a, e)$ if and only if $e <_G a$ and $(e, e) < (e, b)$ if and only if $e <_H b$ is said to *extend the orders of G and H*. Note that if $<$ extends the orders on G and H, then $P(G) \times P(H) \subseteq P(G \times H)$. If $<_G$ is a group order on G, then the collection of all open intervals $\langle g_1, g_2\rangle$ of G where $g_1 < g_2$ are in G, forms a base for a topology $\tau(<_G) = \tau(G)$, called the *interval topology* on G. Note that the symmetric open intervals $\{\langle g^{-1}, g\rangle : e < g\}$, form a base of neighbourhoods of e, and that the map $(a, b) \to ab^{-1}$ on $G \times G$ to G is continuous, whence $(G, \tau(<_G))$ is a topological group. See [**7**], chapter VII.

If the group order $<$ on $G\times H$ extends the orders of G and H, and if the restriction of the interval topology $\tau(<)$ on $G\times H$ to $G\times\{e\}$ is homeomorphic to the topology $\tau(<_G)$ under the map $(g,e)\to g$, we say that $<$ is *topologically compatible with the order of G*. Topological compatibility with the order of H is defined similarly.

In this paper, we determine precisely when the product of two totally ordered groups admits an order topologically compatible with each of the factors.

A totally ordered group $(G, <)$ is said to be *densely ordered* if $g_1 < g_2$ in G implies there is a $g_3 \in G$ such that $g_1 < g_3 < g_2$. Since $g_1 < g_2$ if and only if $e < g_1^{-1}g_2$, it is clear that $(G, <)$ is densely ordered if and only if $P(G)$ has no least element. If $(G, <)$ is not densely ordered, it is said to be *discretely ordered.* It is easy to show that $\tau(G)$ is the discrete topology if and only if the order on G is discrete.

It turns out that if either G or H is discretely ordered, then $G\times H$ admits an order topologically compatible with the orders of G and H. If the orderings on G and H are dense and archimedean, then we may identify G and H with subgroups of the additive group $\mathbb{R}(+)$ of real numbers. We show below that under these hypotheses, $G\times H$ admits an order topologically compatible with the orders of G and H if and only if not every real number is of the form g/h, where $g\in G$ and $0 \neq h \in H$. We use this latter result to characterize, more generally, those densely ordered groups G, H for which $G\times H$ admits an order topologically compatible with the orders of G and H, but this result is too complicated to state at this point; see Section 4.

2. *Preliminary results and the hiding maps*

If every element of the set A is also in the set B, we write $A \subseteq B$, and if the inclusion is proper we write $A \subset B$.

The *lexicographic order* on $G\times H$ *with G dominating* is the order $<$ such that $(g_1,h_1) < (g_2,h_2)$ if $g_1 <_G g_2$ or $g_1 = g_2$ and $h_1 <_H h_2$. The *lexicographic order* on $G\times H$ *with H dominating* is defined similarly. Note that each of these orders extends the orders on G and H.

2·1. PROPOSITION. *If $(G, <_G)$ and $(H, <_H)$ are totally ordered groups, one of which is discretely ordered, then $G\times H$ admits an order $<$ topologically compatible with the orders of G and H.*

Proof. Suppose $<$ is the lexicographic order on $G\times H$ with G dominating, where $<_G$ is discrete. If l is the least element of $P(G)$, then (e,e) is the only element of the open interval $\langle(l^{-1},e),(l,e)\rangle$ of $(G\times\{e\}, <)$. So $<$ induces the discrete topology on $G\times\{e\}$. Since $\langle l^{-1}, l\rangle = \{e\}$, $<_G$ also induces the discrete topology on G. Thus $\tau(<_G)$ is homeomorphic to $\tau(<)$ restricted to $G\times\{e\}$. Since $\{e\}\times H$ is a convex subgroup of $G\times H$, the order $<'$ obtained by restricting $<$ to $\{e\}\times H$ is such that $(\{e\}\times H, <')$ and $(H, <_H)$ are order isomorphic. Hence $<$ is topologically compatible with the orders of G and H. In the case when, instead, $<_H$ is discrete, the proof is similar.

Dense orders on groups are characterized as follows.

$$(G, <) \text{ is densely ordered if and only if } [P(G)]^2 = P(G). \tag{1}$$

To see this, assume first that $P^2 = P$ and $g\in P$. Then $g = pq$ for some $p, q\in P$. Since $e < p$, we have $e < q < pq = g$, so P has no least element and $<$ is a dense order on G. Conversely, if P has no least element and $g\in P$, there is an $f\in P$ such that $f < g$. Then $e < f^{-1}g$ and $g = f(f^{-1}g)\in P^2$. So $P = P^2$ and (1) holds.

An *upper filter* in a densely ordered group $(G, <)$ is a subset U of G such that $UP = U$. Thus $\varnothing$ and G are always upper filters in G, as is gP for each $g \in G$ by (1). Let $\mathscr{U}(G)$ denote the set of upper filters on G.

It is an exercise to verify

$$g_1 < g_2 \quad \text{if and only if} \quad g_2 \in g_1 P. \tag{2}$$

Hence $$\text{If} \quad U \leqslant \mathscr{U}(G), g_1 \in U \quad \text{and} \quad g_1 < g_2 \quad \text{then} \quad g_2 \in U. \tag{3}$$

An upper filter of the form gP for some $g \in P$ is called a *principal upper filter*. Since each non-empty $U \in \mathscr{U}(G)$ is the union of principal upper filters,

$$\text{Each} \quad U \in \mathscr{U}(G) \quad \text{is open in the interval topology of} \quad G. \tag{4}$$

If $(G, <)$ is densely ordered, then the set of principal upper filters of G is dense in $\mathscr{U}(G)$ in the following sense.

2·2. Lemma. *Suppose $(G, <)$ is densely ordered, and U and V are distinct elements of $\mathscr{U}(G)$. Then*

(*a*) *$U \subset V$ or $V \subset U$, and*

(*b*) *there is a principal upper filter of G strictly between U and V.*

Proof. (*a*) Either $U \subset V$ or there is a $u \in U \backslash V$. By (3), if the latter holds, we cannot have $u \geqslant v$ for any $v \in V$. So $u < v$ for each $v \in V$, and hence $V \subseteq uP \subseteq UP = U$. Hence $V \subset U$ since $V \neq U$.

(*b*) Suppose $V \subset U$ and $g \in U \backslash V$. Then $gP \subseteq UP = U$ and $g \in U \backslash gP$. Thus $V \subseteq gP \subseteq U$. If $V \neq gP$, we are done; otherwise, since G is densely ordered, $gP \cup \{g\}$ is not open. So there is an $m \in U \backslash (gP \cup \{g\})$. Thus $mP \subseteq U$, $m \in U \backslash mP$, and $m < g$. So $g \in mP$ and hence mP lies properly between V and U since $g \in mP \backslash V$.

Our last lemma showed that $\mathscr{U}(G)$ is totally ordered under set inclusion. Although it is an abuse of notation, we let $(\mathscr{U}(G), <_G)$ denote $\mathscr{U}(G)$ under the ordering defined by letting $U <_G V$ mean $V \subset U$.

2·3. Proposition. *If $(G, <)$ is a totally ordered group, then under the operation of set multiplication $(\mathscr{U}(G), <_G)$ is a totally ordered monoid with identity element eP. Moreover, the map $\alpha: G \to \mathscr{U}(G)$ given by $\alpha(g) = gP$ is an order-preserving monomorphism of G onto a dense subset of $\mathscr{U}(G)$.*

Proof. If U and V are in $\mathscr{U}(G)$, then $(UV)P = U(VP) = UV$, and $(eP)U = PU = UP = U$ by (γ) of Section 1. So $\mathscr{U}(G)$ is a monoid. Clearly $U \subseteq U'$ and $V \subseteq V'$ imply $UV \subseteq U'V'$, whence $\mathscr{U}(G)$ is a totally ordered monoid.

If $g_1 < g_2$ in G, then $g_2 \in g_1 P$ by (2), whence by (1), $g_2 P \subseteq g_1 P^2 = g_1 P$, so α is order preserving. But $g_2 \notin g_2 P$, so $g_2 P \neq g_1 P$ and α is a monomorphism. It is immediate from Lemma 2·2(*b*) that $\alpha[G]$ is dense in $\mathscr{U}(G)$.

The following characterization of topological compatibility is the major tool in solving the problem posed in the introduction.

2·4. Theorem. *Suppose $<$ is a group order on the product $G \times H$ of two densely ordered groups that extends the orders of G and H. Then:*

(*a*) *$\tau(G)$ and $\tau(H)$ are weaker than the order topologies induced on $G \times \{e\}$ and $\{e\} \times H$ by $<$;*

(*b*) *$<$ is topologically compatible with the orders on G and H if and only if, whenever*

$(g,h)\in P(G\times H)$, there are $g^*\in G$ and $h^*\in H$ such that $(e,e) < (g^*,e) < (g,h)$ and $(e,e) < (e,h^*) < (g,h)$;

(*c*) *If $<$ is topologically compatible with the orders of G and H, then G and H are totally disconnected.*

Proof. (*a*) If $g_1 < g_2$ in G, then $\langle g_1, g_2\rangle \times \{e\} = \langle (g_1,e),(g_2,e)\rangle \cap (G\times\{e\})$, so $\tau(G)$ is weaker than the order topology induced on $G\times\{e\}$ by $<$. The proof for $\tau(H)$ is similar.

(*b*) Suppose $<$ is topologically compatible with the orders of G and H and $(e,e) < (g,h)$. Since (e,e) is in the open interval $\langle (g^{-1},h^{-1}),(g,h)\rangle$, the topological compatibility implies there are g_1^*, g_2^* in G such that $(e,e)\in\langle g_1^*, g_2^*\rangle\times\{e\} \subset \langle (g^{-1},h^{-1}),(g,h)\rangle$. Thus in particular, $(e,e) < (g_2^*,e) \leqslant (g,h)$. Since $<_G$ is a dense order, there is a $g^*\in G$ such that $e < g^* < g_2^*$. Then $(g^*,e)\in\langle (g^{-1},h^{-1}),(g,h)\rangle$. Thus $(e,e) < (g^*,e) < (g,h)$. An element $h^*\in H$ such that $(e,e) < (e,h^*) < (g,h)$ can be produced similarly.

Suppose next that whenever $(e,e) < (g,h)$, there are $g^*\in G$, $h^*\in H$ satisfying the inequalities in (*b*), and suppose $(e,e)\in\langle (g_1,h_1),(g_2,h_2)\rangle$. By assumption, there is a $g_2^*\in G$ such that $(e,e) < (g_2^*,e) < (g_2,h_2)$ and a $g_1^*\in G$ such that $(e,e) < (g_1^{*-1},e) < (g_1^{-1},h_1^{-1})$. Thus $(e,e)\in\langle (g_1^*,e),\ (g_2^*,e)\rangle \subset \langle (g_1,h_1),\ (g_2,h_2)\rangle$. So the restriction of $\tau(<)$ to $G\times\{e\}$ is weaker than $\tau(G)$. Similarly, it is weaker than $\tau(H)$. Thus, by (*a*), $<$ is topologically compatible with the orders on G and H.

(*c*) It suffices to show that the component of e in each of G and H is $\{e\}$. If $e <_G g$, then there is an $h\in H$ such that $(e,e) < (e,h) < (g,e)$. Thus $e\in\{k\in G\colon (k,e) < (e,h)\}$ and $g\in\{k\in G\colon (k,e) > (e,h)\}$, so there is a partition of G into disjoint open sets, one containing e and the other g. Thus the complement K of e contains no positive element and it follows that $K=\{e\}$. Similarly, the component of e in H is $\{e\}$.

For any set A, let $\exp A$ denote the family of all subsets of A.

2·5. DEFINITION. *Suppose $(G,<_G)$ and $(H,<_H)$ are densely ordered groups and $<$ is an order on $G\times H$ that extends the orders on G and H. For each $a\in G$, let $\phi(a) = \{h\in H\colon (e,e) < (a,h)\}$. Then $\phi\colon G\to\exp H$ is called the map that hides G from H in $\exp H$, or the hiding map.*

If $g,a\in G$, we abbreviate $a^{-1}ga$ by g^a. The terminology 'hiding map' will be justified in part (*c*) of the following lemma.

2·6. LEMMA. *Suppose $<$ is a group order on the product $G\times H$ of two densely ordered groups. Then:*

(*a*) *If $<$ extends the orders of G and H, and $\phi\colon G\to\exp H$ is the hiding map, then for $a,b\in G$ and $h\in H$*

(i) *$\phi(a^{-1})\cup\phi(a)^{-1} = H$ if $a\neq e$, and $\phi(e)\cup\phi(e)^{-1} = H\backslash\{e\}$,*

(ii) *$\phi(a^{-1})\cap\phi(a)^{-1}$ is empty,*

(iii) *$\phi(a)\phi(b)\subseteq\phi(ab)$,*

(iv) *$\phi(a)^h = \phi(a) = \phi(a^b)$,*

(v) *$\phi(e) = P(H)$,*

(vi) *$a\in P(G)$ implies $e\in\phi(a)$, and*

(vii) *$a <_G b$ implies $\phi(a)\subseteq\phi(b)$;*

(*b*) *If $\phi\colon G\to\exp H$ satisfies* (i) *through* (vi), *and we let $P(G\times H) = \{(a,h)\in G\times H\colon h\in\phi(a)\}$, then $P(G\times H)$ defines a group order $<$ on $G\times H$ that extends the orders of G and H;*

(c) *If $<$ is topologically compatible with the orders of G and H, then $\phi(g)$ is never a principal upper filter unless $g = e$. That is, $\phi[G] \cap \alpha[H] = \{P(H)\}$.*

Proof. (a) It is clear from the definition of ϕ that $\phi(e) = P(H)$, so (v) holds. If $a \neq e$, then $(e,e) < (a^{-1},h)$ or $(e,e) < (a^{-1},h)^{-1} = (a,h^{-1})$, so $h \in \phi(a^{-1}) \cup \phi(a)^{-1}$. Since $\phi(e) = P(H)$, $\phi(e) \cup \phi(e)^{-1} = H\backslash\{e\}$, and (i) holds.

If $h \in \phi(a^{-1}) \cap \phi(a)^{-1}$, then $(e,e) < (a^{-1},h)$ and $(e,e) < (a,h^{-1}) = (a^{-1},h)^{-1}$, contrary to ($\beta$) of Section 1. So (ii) holds.

If $h \in \phi(a)$ and $j \in \phi(b)$, then $(e,e) < (a,h)$ and $(e,e) < (b,j)$, whence $(e,e) < (a,h)(b,j) = (ab,hj)$. Thus (iii) holds.

To see (iv), note that for $k \in G$, $k \in \phi(a)$ if and only if $(e,e) < (a,k)$ if and only if $(e,e) < (a,k)^{(b,h^{-1})}$. This is ($\gamma$) of Section 1. Thus

$$\phi(a) = \phi(a^b)^{h^{-1}}. \tag{5}$$

Letting successively $b = e$ and $h = e$ in (5) yields (iv).

Since the order of $G \times H$ extends the order of H, $\phi(e) = P(H)$, and (vi) restates the assumption that $<$ extends the order of G. So (vi) holds.

If $a <_G b$, then $(a,e) < (b,e)$ since $<$ extends the order of G. So if $h \in \phi(a)$, then $(e,e) < (a,h) = (a,e)(e,h) < (b,e)(e,h) = (b,h)$ whence $h \in \phi(b)$. Thus (vii) holds and the proof of (a) is complete.

(b) To show that $<$ is a group order, we will verify that (α), (β), (γ) and (δ) of Section 1 hold. Suppose (a,h) and (b,k) are in $P(G \times H)$; then $h \in \phi(a)$ and $k \in \phi(b)$, so by (iii), $hk \in \phi(a)\phi(b) \subseteq \phi(ab)$. Thus $(ab,hk) \in P(G \times H)$ and (α) holds.

If $(a,h) \in P(G \times H)$, then $h \in \phi(a)$, so $h^{-1} \in \phi(a)^{-1}$. If also $(a,h) \in P(G \times H)^{-1}$, then $(a^{-1},h^{-1}) \in P(G \times H)$, whence $h^{-1} \in \phi(a^{-1})$ as well as $\phi(a)^{-1}$, contrary to (ii). This contradiction establishes (β).

That (γ) holds follows from (5), and that (i) implies (δ) is an exercise.

By (v), $e <_H h$ if and only if $h \in \phi(e)$ if and only if $(e,e) < (e,h)$. So $<$ extends the order of H. Also, if $e <_G a$, then $e \in \phi(a)$ by (vi), so $(e,e) < (a,e)$. Thus $<$ extends the order of G as well as that of H. This completes the proof of (b).

(c) By (vi) and the definition of α, $\phi(e) = \alpha(e) = eP(H) = P(H)$, so $P(H) \in \phi[G] \cap \alpha[H]$. Clearly h is the greatest lower bound of $\alpha(h) = hP(H)$, while, as will be shown next, $\phi(g)$ fails to have a greatest lower bound if $g \neq e$.

For, if $h \in \phi(g)$, then $(e,e) < (g,h)$, and by Theorem 2·4(b), there is an $h^* \in H$ such that $(e,e) < (e,h^*) < (g,h)$. Thus $(e,e) < (g,h(h^*)^{-1})$. So $h(h^*)^{-1} \in \phi(g)$, and $h(h^*)^{-1} < h$. Thus h is not a lower bound of $\phi(g)$. If $h \notin \phi(g)$, then, since $g \neq e$, $(e,e) < (g,h)^{-1} = (g^{-1},h^{-1})$. Using Theorem 2·4(b) again, there is an $h^* \in H$ such that $(e,e) < (e,h^*) < (g^{-1},h^{-1})$. So $(e,e) < (g^{-1},h^{-1}(h^*)^{-1}) = (g,h^*h)^{-1}$. Thus $h^*h \notin \phi(g)$. Also $h < h^*h$, showing h is not a greatest lower bound for $\phi(g)$. Thus no $h \in H$ can be a greatest lower bound for $\phi(g)$.

Hence (c) holds and the proof of the lemma is complete.

This next example illustrates that the hiding map may assume H or $\varnothing$ as values, that not every value need be open, and that $\phi(a)\phi(b)$ need not equal $\phi(ab)$ even if both $\phi(a)$ and $\phi(b)$ are non-empty.

2·7. *Example.* Let $\mathbb{R}_1(+)$ and $\mathbb{R}_2(+)$ denote two copies of the additive group of real numbers with its usual order, and let $<$ denote the lexicographic order of $\mathbb{R}_1 \times \mathbb{R}_2$ with $\mathbb{R}_1$ dominating (clearly $<$ extends the orders of $\mathbb{R}_1$ and $\mathbb{R}_2$). For each $g \in \mathbb{R}_1$, let

$\phi_1(g) = \{h \in \mathbb{R}_2 : (0,0) < (g,h)\}$, so ϕ_1 is the hiding map of $\mathbb{R}_1$ in $\exp \mathbb{R}_2$. Routine calculations show that $\phi_1(g) = \mathbb{R}_2$ if $g > 0$, $\phi_1(0) = P(\mathbb{R}_2)$, and $\phi_1(g) = \varnothing$ if $g < 0$.

Let $\phi_2 : \mathbb{R}_2 \to \exp \mathbb{R}_1$ denote the hiding map of $\mathbb{R}_2$ into $\exp \mathbb{R}_1$, so for each $h \in \mathbb{R}_2$, $\phi_2(h) = \{g \in \mathbb{R}_1 : (0,0) < (g,h)\}$. It is easy to see that $\phi_2(h) = G^+ = P(G) \cup \{0\}$ if $h > 0$, and $\phi_2(h) = P(G)$ if $h \leqslant 0$. In particular, $\phi_2(h)$ fails to be open if $h > 0$. Moreover, $\phi_2(-1) + \phi_2(2) = P(G) + G^+ = P(G) \subset G^+ = \phi_2(1)$.

Much more can be said about the hiding map when the order on $G \times H$ is topologically compatible with the order of each of its factors.

2·8. THEOREM. *If $(G, <_G)$ and $(H, <_H)$ are densely ordered groups, $<$ is an order on $G \times H$ that extends the orders on G and H, and $\phi : G \to \exp H$ is the hiding map, then the following are equivalent*:

(i) *$<$ is topologically compatible with the orders on G and H*;

(ii) *$\phi[G] \subset \mathscr{U}(H)$ and ϕ is continuous with respect to the interval topologies on G and $\mathscr{U}(H)$.*

Moreover, if (ii) *holds, then there is a $g \neq e$ in G such that $\phi(g)$ is a non-empty proper subset of H.*

Proof. Suppose (ii) holds and $(e,e) < (g,h)$. Elements g^* and h^* satisfying the conditions of Theorem 2·4(*b*) will be produced. Since $\phi(g) \in \mathscr{U}(H)$, $h \in \phi(g) = \phi(g)P$, so $h = kh^*$ for some $k \in \phi(g)$ and $h^* \in P$. Hence $(e,e) < (e,h^*) < (g,k)(e,h^*) = (g,h)$. By the continuity of ϕ, since $hP \subset \phi(g)P = \phi(g)$, there is a neighbourhood $\langle g_1, g_2 \rangle$ of g in G such that if $g_1 < g' < g_2$, then $\phi(g') \supset hP$. Thus $q \in \phi(g')$ for some $q \leqslant_H h$. If $q = h$, then $h \in \phi(g')$. If $q <_H h$, then $h = qp$ for some $p \in P$. Hence $h \in qP \subseteq \phi(g')P = \phi(g')$, and we have $h \in \phi(g')$. Since $<_G$ is a dense order, there is a $k \in G$ such that $g_1 <_G k <_G g$. Thus $e <_G gk^{-1} = g^*$; and $(e,e) < (k,h)$ whence $(e,e) < (g^*,e) < (g^*,e)(k,h) = (g,h)$. So, by Theorem 2·4(*b*), $<$ is topologically compatible with the orders of G and H.

In the proof of 2·6(*c*), it was shown that $\phi(g) \in \mathscr{U}(H)$.

To establish the continuity of ϕ, we begin by showing:

$$\text{if } (e,e) \in \langle (g,h_1),(g,h_2) \rangle, \text{ then there are } g_1, g_2 \text{ in } G \text{ such that}$$
$$g \in \langle g_1, g_2 \rangle \text{ and if } k \in \langle g_1, g_2 \rangle, \text{ then } (e,e) \in \langle (k,h_1),(k,h_2) \rangle. \quad (6)$$

To establish (6), we begin by using Theorem 2·4(*b*) to find $g_1^* < g_2^*$ in G such that $(g,h_1) < (g_1^*,e) < (e,e) < (g_2^*,e) < (g,h_2)$. Let $g_1 = gg_2^{*-1}$ and $g_2 = gg_1^{*-1}$. Since $g_1^* <_G e <_G g_2^*$, $g_1 = gg_2^{*-1} <_G g <_G gg_1^{*-1} = g_2$. If $g_1 <_G k <_G g_2$, then $(k,h_1) < (g_2,h_1) = (g,h_1)(g_2^{*-1},e) < (e,e) < (g,h_2)(g_1^{*-1},e) = (g_1,h_2) < (k,h_2)$, and (6) holds.

Now suppose $\langle U_1, U_2 \rangle$ is a neighbourhood of $\phi(g)$ in $\mathscr{U}(H)$; that is suppose $U_2 \subset \phi(g) \subset U_1$. We wish to find a neighbourhood $\langle g_1, g_2 \rangle$ of g in G such that if $g_1 < k < g_2$, then $\phi(k) \in \langle U_1, U_2 \rangle$. Choose $h_2 \in \phi(g) \backslash U_2$, whence $(e,e) < (g,h_2)$. If $h_1' \in U_1 \backslash \phi(g)$, then $(g,h_1') \leqslant (e,e)$. If $g \neq e$, then $(g,h_1') < (e,e)$. If $g = e$, there is an $r \in U_1 = U_1 P$ such that $r \leqslant_H e$. Then $r = h_1 p$ for some $h_1 \in U$ and $p \in P$, and $(g,h_1) < (e,e) < (g,h_2)$. By (6), there is a neighbourhood $\langle g_1, g_2 \rangle$ of g such that if $g_1 < k < g_2$, then $(k,h_1) < (e,e) < (k,h_2)$; that is, $h_1 \in U_1 \backslash \phi(k)$ and $h_2 \in \phi(k) \backslash U_2$, whence $\phi(k) \in \langle U_1, U_2 \rangle$. Thus ϕ is continuous at g, and the equivalence of (i) and (ii) is established.

If (ii) holds and $h \in P(H)$, then $h^{-1} <_H e <_H h$, whence $hP \subseteq eP = \phi(e) \subseteq h^{-1}P$. Now $e = h^{-1}h \in h^{-1}P$ and $e \notin eP$, so $\phi(e) \neq h^{-1}P$. Also, since $h \in \phi(e)$ and $h \notin hP$, the latter is included properly in $\phi(e)$. Hence $\phi(e) \in \langle h^{-1}P, hP\rangle$, which we call U. Since ϕ is continuous, there is a $k \in G$ such that $V = \langle k^{-1}, k\rangle$ is a neighbourhood of e and $g \in V$ implies $\phi(g) \in U$. Clearly $\phi(g)$ is a non-empty proper subset of H and since $<_G$ is a dense order, we may assume that $g \neq e$. This completes the proof of Theorem 2·8.

3. *Topologically compatible pairs; the Archimedean case*

Recall that a totally ordered group G is said to be *archimedean* if $a \in P(G)$ implies $\{a^n : n = 1, 2, 3, \ldots\}$ has no upper bound.

3·1. PROPOSITION. *If $<$ is an order on the product $G \times H$ of two densely ordered archimedean groups that is topologically compatible with the orders of G and H, and if $\phi : G \to \mathscr{U}(H)$ is the hiding map, then ϕ is a monomorphism of G onto a subgroup of $\mathscr{U}(H)$.*

Proof. By Theorem 2·8, $\phi[G] \subset \mathscr{U}(H)$, and by Lemma 2·6 (*a*) and Theorem 2·8 again, there is an $a \in G$ such that both $\phi(a)$ and $\phi(a^{-1})$ are non-empty proper subsets of H. Choose $h \in \phi(a^{-1})\,\phi(a)$. It will be shown by induction that

$$\text{if } \phi(a) \text{ and } \phi(a^{-1}) \text{ are non-empty, then for each positive integer } m, \text{ there is a } p \in \phi(a^{-1})\,\phi(a) \text{ such that } p^m \leqslant h. \tag{7}$$

Note first that $\phi(a^{-1})\,\phi(a) \subseteq \phi(e) = P(H)$ by Lemma 2·6 (*a*).

If $m = 1$, take $p = h$.

Next assume that (7) holds for the positive integer m; more precisely pick $j \in \phi(a^{-1})$, $k \in \phi(a)$ such that $(jk)^m \leqslant h$. Then $jk \in P(H)$, and by (1), there are $p, q \in P(H)$ such that $jk = pq$. If also $p \leqslant q$ then $p^2 \leqslant jk$. Since H is archimedean, there is a positive integer s such that $jk \leqslant_H p^s$, whence $p^s k^{-1} \geqslant j = \phi(a^{-1})$. Since $\phi(a^{-1}) \in \mathscr{U}(H)$, $p^s k^{-1} \in \phi(a^{-1})$. Now $p^0 k^{-1} = k^{-1} \in \phi(a)^{-1}$, so by Lemma 2·6 (*a*), $p^0 k^{-1} \notin \phi(a^{-1})$. Hence there is a least positive integer r such that $p^r k^{-1} \in \phi(a^{-1})$. Then $p^{r-1}k^{-1} \in \phi(a)^{-1}$ and $(p^{r-1}k^{-1})^{-1} \in \phi(a)$, so $p = (p^r k^{-1})(p^{r-1}k^{-1})^{-1} \in \phi(a^{-1})\,\phi(a)$, and $p^{m+1} \leqslant p^{2m} \leqslant (jk)^m \leqslant h$. If, instead, $q < p$, then $q^2 \leqslant jk$ and a similar argument yields $q \in \phi(a^{-1})\,\phi(a)$ and $q^{m+1} \leqslant h$. Thus (7) holds.

Next, we show that

$$\phi(a^{-1})\,\phi(a) = \phi(e) \quad \text{if} \quad \phi(a) \quad \text{is a non-empty proper subset of} \quad H. \tag{8}$$

By Lemma 2·6 (*a*), $\phi(a^{-1})\,\phi(a) \subseteq \phi(e)$. Suppose $q \in \phi(e) = P(H)$. Since H is archimedean, there is a positive integer t such that $h \leqslant q^t$, and by (7), there is a $p \in \phi(a^{-1})\,\phi(a)$ such that $p^t \leqslant h \leqslant q^t$. Hence $p \leqslant q$. By Theorem 2·8 and Proposition 2·3, $\phi(a^{-1})\,\phi(a) \in \mathscr{U}(H)$, so $q \in \phi(a^{-1})\,\phi(a)$ and (8) holds.

Our next task is to verify

$$\phi(b^{-1})\,\phi(b) = \phi(e) = \phi(b)\,\phi(b^{-1}) \quad \text{for any} \quad b \in G. \tag{9}$$

By Theorem 2·8, there is an $a \in P(G)$ that satisfies the hypothesis of (8). If $b \in P(G)$, then since G is archimedean, there is a positive integer n such that $b <_G a^n$. By Lemma 2·6 (*a*), $\phi(b) \subseteq \phi(a^n)$. If $\phi(a^n) = H$, then for any $h \in H$, $(a, h)^n = (a^n, h^n) > (e, e)$, and by ([1], 12·12), $(a, h) > (e, e)$, so $\phi(a) = H$, contrary to the choice of a. Hence $\phi(b)$ is a proper subset of H and is non-empty since it contains $\phi(e) = P(H)$. So $\phi(b^{-1})\,\phi(b) = \phi(e)$ by (8). By Lemma 2·6 (*a*), $\phi(b^{-1})$ is also a non-empty proper subset of H, so (8) may also be used to show that $\phi(b)\,\phi(b^{-1}) = \phi(e)$, and may be used again to show that $\phi(b^{-1})\,\phi(b) = \phi(e) = \phi(b)\,\phi(b^{-1})$ if $b <_G e$; that these latter equalities hold if $b = e$ is the content of (1). Thus (9) holds; and we know that for each $b \in G$ $\phi(b^{-1})$ is the inverse of $\phi(b)$.

Next, suppose $a, b \in G$ are arbitrary. By Lemma 2·6(a), $\phi(a)\phi(b) \subseteq \phi(ab)$. If this inclusion is proper, and $<$ denotes the order of $\mathscr{U}(H)$, then $\phi(ab) < \phi(a)\phi(b)$, so by (9),

$$\phi(e) < \phi(a)\phi(b)\phi((ab)^{-1}) = \phi(a)\phi(b)\phi(b^{-1}a^{-1}) \leqslant \phi(a)\phi(b)\phi(b^{-1})\phi(a^{-1}) = \phi(e).$$

This contradiction shows that ϕ is a homomorphism. So if we can show

$$\phi(g) = \phi(e) \quad \text{implies} \quad g = e, \tag{10}$$

we may conclude that ϕ is a monomorphism.

If $\phi(g) = \phi(e)$, then since ϕ is a homomorphism, $\phi(e) = \phi(g)\phi(g^{-1}) = \phi(g^{-1})$, so $e \notin \phi(e) = \phi(g)^{-1} \cup \phi(g^{-1})$, contradicting Lemma 2·6(i) unless $g = e$.

By a well-known theorem of Hölder, every archimedean ordered group is isomorphic to a subgroup of $\mathbb{R}(+)$. If G and H are subgroups of $\mathbb{R}(+)$, let

$$G * H = \{g/h : g \in G,\ h \in H\backslash\{0\}\}. \tag{11}$$

3·2. THEOREM. *Suppose G and H are densely ordered subgroups of $\mathbb{R}(+)$. Then $G \times H$ admits an order $<$ topologically compatible with the orders on G and H if and only if $G * H \neq \mathbb{R}$. When $<$ is such an order, $(G \times H, <)$ is archimedean.*

Proof. Suppose first that there is an $a \in \mathbb{R}\backslash G * H$, and let $P = P(G \times H) = \{(g, h) \in G \times H : ah <_{\mathbb{R}} g\}$. We will show that P defines a group order on $G \times H$ by verifying that (α), (β), (γ) and (δ) (rewritten in additive notation) of Section 1 hold.

Suppose (g, h) and (g', h') are in P; then $0 <_{\mathbb{R}} (g - ah) + (g' - ah') = a(g + g') - a(h + h')$. So (α) holds. If $(g, h) \in P \cap (-P)$, then $ah <_{\mathbb{R}} g$ and $a(-h) <_{\mathbb{R}} (-g)$. Since this cannot hold, (β) follows. The commutativity of $\mathbb{R}(+)$ implies (γ). If $g = ah$, then $g = h = 0$ or $a \in G * H$ by (11). Hence $g <_{\mathbb{R}} ah$ or $ah <_{\mathbb{R}} g$ and (δ) holds. So $P(G \times H)$ defines a group order.

Suppose $(0, 0) < n(g, h) < (x, y)$ for some $g, x \in G$, $h, y \in H$, and $n = 1, 2, \ldots$. Then $0 >_{\mathbb{R}} (ah - g)$ and $(x - ng) >_{\mathbb{R}} a(y - nh)$ or $0 > n(ah - g) >_{\mathbb{R}} ay - x$ whenever n is positive. Since $\mathbb{R}(+)$ is archimedean, this cannot hold, so $<$ is an archimedean order on $G \times H$.

We will show that $<$ is topologically compatible with the orders of G and H by verifying the conditions of Theorem 2·4(b). If $(0, 0) < (g, h)$, then $r = g - ah \in P(\mathbb{R})$. It is routine to verify that $(0, 0) < (r/2, 0) < (g, h)$ and $(0, 0) < (0, -r/2a) < (g, h)$.

Suppose, conversely, that the order $<$ on $G \times H$ is topologically compatible with the orders on each of its factors. By Theorem 2·8 and Proposition 3·1, ϕ is a continuous monomorphism onto a subgroup of $\mathscr{U}(H)$. Thus $\phi : G \to \mathbb{R}(+) = \mathscr{U}(H)\backslash\{\varnothing, H\}$ (by the density of H). ϕ is order-preserving by 2·6 (vii), so by ([1], 12·2·1), there is an $a \in \mathbb{R}$ such that $\phi(g) = ag$ for each $g \in G$. If $a = g'/h'$ for some $g' \in G$ and $0 \neq h' \in H$, then $\phi(g') = \{h \in H : (g', h) > (0, 0)\} = \{h \in H : ah <_{\mathbb{R}} g'\}$, and clearly $h' \notin \phi(g') \cup \phi(-g')$, contrary to Lemma 2·6(a). Hence $G * H \neq \mathbb{R}$, and the proof of the Theorem is complete.

The next theorem, which is due to Fred Galvin, provides an ample supply of pairs G, H of subgroups of $\mathbb{R}(+)$ such that $G * H = \mathbb{R}$.

Let Z, respectively Q, denote the additive groups of integers, respectively rational numbers. If $a \in \mathbb{R}$, let $G_a = \{ag/n : 0 \neq n \in Z, g \in G\}$. Clearly G_a is a subgroup of $\mathbb{R}(+)$ which will contain Q if $ag = 1$ for some $g \in G$.

3·3. LEMMA. *If G, H are subgroups of $\mathbb{R}(+)$, and a, b are non-zero real numbers such that $G_a * H_b = \mathbb{R}$, then $G * H = \mathbb{R}$.*

Proof. If $x \in \mathbb{R}$, then by assumption there are non-zero $n, m \in Z$, $g \in G$, and $0 \neq h \in H$ such that

$$\frac{xa}{b} = \frac{(ag/n)}{(bh/m)} = \frac{mg}{nh}\frac{a}{b}.$$

Hence $x = mg/nh \in G * H$, so $\mathbb{R} \subseteq G * H \subseteq \mathbb{R}$, and the lemma holds.

3·4. THEOREM (Galvin). *There is a proper subgroup G of $\mathbb{R}(+)$ such that whenever H is a non-zero subgroup of $\mathbb{R}(+)$, $G * H = G * G = \mathbb{R}$.*

Proof. If t is irrational, there is by Zorn's lemma a subgroup G of $\mathbb{R}(+)$ containing Q and maximal with respect to avoiding t. We now show that $G * Q = \mathbb{R}$. Note first that $G \subseteq G * Q$ since $Q \subseteq G$. For any $x \in R/G$, there is by definition of G a non-zero $n \in Z$ and a $g \in G$ such that

$$\text{(i)} \quad nx + g = t.$$

If $2nx \in G$, then $x \in G * Q$. Otherwise, the definition of G yields an $m \neq 0$ in Z and an $h \in G$ such that

$$\text{(ii)} \quad m(2nx) + h = t.$$

Subtracting (ii) from (i) yields

$$n(1-2m) + (g-h) = t, \quad \text{so} \quad x = \frac{(h-g)}{n(1-2m)} \in G * Q.$$

Thus $G * Q = R$.

Let H denote any non-zero subgroup of $\mathbb{R}(+)$, and choose $k \neq 0$ in G. For $a = 1/k$ and $b = 1$, we have $Q \subseteq H_a$ and $G \subseteq G_b$, so $\mathbb{R} = G * Q \subseteq G_b * H_a$. Then by Lemma 3·3, $G * H = \mathbb{R} = G * G$.

4. *Topologically compatible pairs: the general case*

For the balance of this paper, G and H will denote infinite densely ordered groups unless the contrary is stated explicitly.

Recall that a subset K of G is called *convex* if $x_1 \leqslant g \leqslant x_2$, where $g \in G$ and $x_1, x_2 \in K$, implies $g \in K$. If $T \subseteq G$, let $cn(T)$ denote the intersection of all of the convex normal subgroups of G that contain T. It is not difficult to verify that $cn(T) = \{g \in G$: for some $t \in T$, $a \in G$, and positive integer $n, |g| < |t^n|^a\}$. By the set $ni(T)$ of *normal infinitesimals relative to* T, we mean the union of all the convex normal subgroups of G disjoint from T. It is an exercise to verify that $ni(T) = \{g \in G$: if $a \in G$, n is a positive integer, and $t \in T$, then $|g^n|^a < |t|\}$. By the *cardinal index of archimedeanness* $cia(G)$, we mean the least cardinal number of a subset S of G such that $ni(S) = \{e\}$. We call $\bigcap\{cn(g) : e \neq g \in G\}$ the *order kernel* $S(G)$ of G. Clearly $S(G)$ is a convex normal subgroup of G. If $F \subseteq P(G)$ is finite, then $ni(F) = ni(f)$, where f is the smallest element of F, and it follows easily that $S(G) = \{e\}$ if and only if $cia(G) > 1$. It is clear, also, that if $cia(G) = 1$, then $S(G) = cn(g)$ for any $e \neq g$ in $S(G)$. We summarize the above in the following proposition.

4·1. PROPOSITION. *If* $(G, <)$ *is a densely ordered group and* $S(G)$ *is the order kernel, then*:

(*a*) $cia(G) > 1$ *if and only if* $cia(G)$ *is infinite*;
(*b*) $S(G) = \{e\}$ *unless* $cia(G) = 1$.

For any $g \in G$, let $c(G)$ denote the smallest convex subgroup of G containing g. Note that G is archimedean if and only if $G = c(g)$ whenever $e \neq g \in G$.

The proof of the following lemma was simplified as a result of discussion with A. Rhemtulla.

4·2. LEMMA. *If a and b are distinct positive elements of the order kernel* $S(G)$ *of a densely ordered group, and* $S(G)$ *is not archimedean, then there are* y, z *in* G *such that* $a^y < b < a^z$.

Proof. Suppose

(*) there is an $x \in S(G) \cap P(G)$ such that for each $g \in G$, there is a positive integer n such that $x^g < x^n$.

Then $c(x) = cn(x) = S(G)$. If, for some $y \in S(G) \cap P(G)$, $x \notin c(y)$, then $y^m < x$ for every positive integer m; for each $g \in G$, choose n such that $x^g < x^n$. Thus $(y^{nm})^g < x^g < x^n$, so $(y^m)^g < x$, contrary to the fact that $x \in cn(y)$. This contradiction shows that $c(y) = c(x) = S(G)$ for each x, $y \in P(G) \cap S(G)$, and hence that $S(G)$ is archimedean. Thus (*) fails.

Assume without loss of generality that $a < b$. Since $b \in cn(a)$, for some positive integer m and $h \in G$, $b < (a^m)^h \leqslant (a^g)^h = a^{gh}$, where g is the element of G whose existence is guaranteed by the failure of (*). Taking $y = e$ and $z = gh$, the conclusion of the lemma follows.

Most of the remainder of this paper is devoted to establishing:

4·3. THEOREM. *If* G *and* H *are densely ordered groups with order kernels* $S(G)$ *and* $S(H)$, *then there is an order* $<$ *on* $G \times H$ *topologically compatible with the orders of* G *and* H *if and only if both of the following hold*:

(*a*) $cia(G) = cia(H)$, *and*

(*b*) $S(G)$ *and* $S(H)$ *are central*, (*thus archimedean*) *and we may identify them with subgroups of* $\mathbb{R}(+)$ *in such a way that* $S(G) * S(H) \neq \mathbb{R}$.

As in [6], pp. 266–271 and 274–275, we identify each ordinal α with its well-ordered set of predecessors, and we identify each cardinal m with the ordinal minimal with respect to being in one-one correspondence with a set of cardinality m.

To prove that if $G \times H$ admits an order topologically compatible with the orders of G and H, then $cia(G) = cia(H)$, we begin by showing:

4·4. *If both* $cia(G)$ *and* $cia(H)$ *exceed* 1, *then* $cia(G) = cia(H)$.

To verify this, begin by letting $T = \{g_\alpha : \alpha < cia(G)\} \subseteq P(G)$ be a set such that $ni(T) = \{e\}$. By Theorem 2·4(*b*), for each $\alpha < cia(G)$, there is an $h_\alpha \in H$ such that $(e, e) < (e, h_\alpha) < (g_\alpha, e)$. Suppose $cia(G) < cia(H)$. Then there is an $h \in P(H)$ such that $h < h_\alpha$ for each $\alpha < cia(G)$ since $ni(\{h_\alpha : \alpha < cia(G)\}) \neq \{e\}$. Thus $(e, e) < (e, h) < (e, h_\alpha) < (g_\alpha, e)$ for each $\alpha < cia(G)$. By Theorem 2·4(*b*), there is a $g \in G$ such that $(e, e) < (g, e) < (e, h)$, so $e <_G g <_G g_\alpha$ for each $\alpha < cia(G)$. Since $cia(G) > 1$, $ni(g) \neq \{e\}$, so for some $f \in P(G)$, $(f^n)^a < g$ for each positive integer n and $a \in G$. Thus $(f^n)^a < g_\alpha$ for each α, whence $f \in ni(T)$ contrary to the definition of T. We conclude that $cia(G) \geqslant cia(H)$ if $cia(G) > 1$. Similarly $cia(H) \geqslant cia(G)$ if $cia(H) > 1$, so 4·4 holds.

4·5. *$S(G)$ is archimedean.*

We may assume that $cia(G) = 1$. Let $\phi: G \to \mathscr{U}(H)$ denote the hiding map determined by $<$ as in Lemma 2·6 (and Theorem 2·8). By this latter theorem ϕ is continuous. Suppose $h <_H e$, in which case $h \notin \phi(e)$. By the continuity of ϕ, there is a $g \in P(G)$ such that $h \notin \phi(g)$. Since $cia(G) = 1$, there is an $a \in S(G) \cap P(G)$ such that $a \leqslant_G g$ and $h \notin \phi(a)$. If $S(G)$ fails to be archimedean, and $b \in S(G) \cap P(G)$, there are, by Lemma 4·2, y, z in G such that $a^y < b < a^z$. By Lemma 2·6, $\phi(a^y) \subseteq \phi(b) \subseteq \phi(a^z) = \phi(a)$, so $\phi(a) = \phi(b)$. It follows that if $h <_H e$, then $h \notin \phi(b)$ for any $b \in S(G) \cap P(G)$. Using Lemma 2·6 again, $e <_G b$ implies $\phi(e) \subseteq \phi(b)$, so $P(H) \subseteq \phi(b)$. Also since $<$ extends the order of G, $e \in \phi(b)$. Thus $\phi(b) = P(H) \cup \{e\}$ fails to be open, contrary to the density of the order of H. This contradiction establishes 4·5.

Next, we show

4·6. *If $(e,e) < (e,h) < (g,e)$ and $g \in S(G)$, then $h \in S(H)$; thus if $cia(G) = 1$, then $cia(H) = 1$.*

To see this, assume on the contrary that there is a $c \in ni(h) \cap P(H)$. Then $(e,e) < (e,c)$, so by Theorem 2·4(*b*), there is a $k \in G$ such that $(e,e) < (k,e) < (e,c)$. It follows from the definition of $ni(h)$ that for any $(x,y) \in G \times H$,

$$(e,e) < ((k,e)^n)^{(x,y)} < ((e,c)^n)^{(x,y)} < (e,h) < (g,e).$$

So if $x \in G$, then $(k^n)^x < g$, and hence $k \in ni(g) \cap P(G)$, contrary to the assumption that $g \in S(G)$. Thus $h \in S(H)$.

From this note that if $cia(G) = 1$ and $g \in P(G)$ is such that $ni(g) = \{e\}$, then as in the above $(e,e) < (e,h) < (g,e)$ for some $h \in H$. Thus $ni(h) = \{e\}$ and hence $cia(H) = 1$.

Clearly if $<$ is topologically compatible with the orders of G and H, then $<$ restricted to $S(G) \times S(H)$ is topologically compatible with the orders of the (archimedean) subgroups $S(G)$ and $S(H)$. So (*b*) will follow from Theorem 3·2 if we can show that each of $S(G)$ and $S(H)$ is central.

Denote by ψ the hiding map of $S(G)$ into $S(H)$. By Lemma 2·6(iv) and the definitions of ϕ and ψ, $\psi(g) = \phi(g) \cap S(H) = \phi(g^x) \cap S(H) = \psi(g^x)$ for each $g \in S(G)$ and $x \in G$. Since $S(G)$ is a normal subgroup of G, it follows from the last paragraph that the hypothesis of Proposition 3·1 is satisfied by $\psi: S(G) \to S(H)$. Thus ψ is a monomorphism and hence $g = g^x$, and we may conclude that $S(G)$ is central. A similar argument applied to the hiding map of $S(H)$ into $S(G)$ shows that $S(H)$ is also central. This completes the proof that (*b*) holds.

We turn now to establishing the sufficiency of conditions (*a*) and (*b*). Choose any subset S of $P(G)$ or cardinality $cia(G)$ such that $ni(S) = \{e\}$. First we assume $cia(G) > 1$ and establish the existence of a 'valuation'. Write $S = \{s_\alpha : \alpha < cia(G)\}$, let $N(g) = \{\alpha < cia(G) : g \in ni(s_\beta : \beta < \alpha)\}$, and define $v = v_G : G \to cia(G) \cup \{\infty\}$ by letting

$$v(g) = \begin{cases} \sup N(g) & \text{if } g \neq e \\ \infty & \text{if } g = e \end{cases}. \tag{12}$$

We establish first

4·7. *If $g \in G$, then $g \in ni(s_\alpha : \alpha < v(g))$.*

For, if $\beta < v(g)$, then $\beta < \sup\{\alpha : g \in ni(s_\beta : \beta < \alpha)\}$, so $g \in ni(s_\gamma : \gamma < \delta)$ for some $\beta < \delta$. Thus, for each $a \in G$, integer n, and $\gamma < \delta$, $(g^n)^a < s_\gamma$; in particular, for each such a and n, $(g^n)^a < s_\beta$. Since $\beta < v(g)$ is arbitrary, 4·7 holds.

4·8. LEMMA. *For any g, h in G:*

(i) $v(gh) \geqslant \min(v(g), v(h))$;
(ii) $v(g^h) = v(g)$;
(iii) $v(g) = v(g^{-1})$;
(iv) $v(g) = \infty$ *if and only if* $g = e$;
(v) *If* $v(g) < v(h)$, *then* $v(gh) = v(g)$;
(vi) *If* $g \geqslant h > e$, *then* $v(g) \leqslant v(h)$;
(vii) *If* $g > e$ *and* $gh < e$ (*or* $hg < e$), *then* $v(h) \leqslant v(g)$;
(viii) *If* $g > e$ *and* $h > e$, *then* $v(gh) = \min v(g), v(h))$.

Proof. If $\delta = \min(v(g), v(h))$, then both g and h are in the group $ni(s_\alpha\colon \alpha < \delta)$ as is their product. So $\min(v(g), v(h)) \leqslant \sup\{\alpha\colon gh \in ni(s_\beta\colon \beta < \alpha)\} = v(gh)$. So (i) holds.

To see (ii) and (iii), note first that both g^{-1} and g^h are in the normal subgroup $ni(s_\alpha\colon \alpha \leqslant v(g))$, so each of $v(g^{-1})$ and $v(g^h)$ is $\geqslant v(g)$. So $v(g) = v((g^{-1})^{-1}) \geqslant v(g^{-1})$ and $v(g) = v((g^h)^{h^{-1}}) \geqslant v(g^h)$. Thus $v(g^{-1}) = v(g) = v(g^h)$, and (ii) and (iii) hold.

By definition, $v(e) = \infty$. If $v(g) = \infty$, then by 4·7, $g \in ni(s_\alpha\colon \alpha < v(g)) = ni(S)$, so $g = e$ and (iv) holds.

Suppose $v(g) < v(h)$ and $v(g) < v(gh)$. Then, by (i) and (iii), $v(g) = v((gh)h^{-1}) \geqslant \min(v(gh), v(h^{-1})) = \min(v(gh), v(h)) > v(g)$. Hence $v(g) < v(h)$ implies $v(gh) \leqslant v(g)$, whence $v(gh) = v(g)$ by (i). Thus (v) holds.

That (vi) holds is immediate from the definition of v. If $g > e > gh$, then $h^{-1} = h^{-1}g^{-1}g > g > e$. By (iii) and (vi), $v(h) = v(h^{-1}) \leqslant v(g)$ and (vii) holds.

If g and h are in $P(G)$, then $g < gh$ and $h < gh$. So by (vi), $v(gh) \leqslant v(g)$ and $v(gh) \leqslant v(h)$. Thus by (i) $v(gh) = \min(v(g), v(h))$ and (viii) holds. This completes the proof of the lemma.

Suppose $cia(G) = cia(H) > 1$ and consider the maps $v_G\colon G \to cia(G) \cup \{\infty\}$ and $v_H\colon H \to cia(H) \cup \{\infty\}$ as defined above. We define an order $<$ on $G \times H$ as follows:

$$(e,e) < (g,h) \quad \text{if} \quad v_G(g) < v_H(h) \quad \text{and} \quad g \in P(G) \quad \text{or} \quad v_H(h) \leqslant v_G(g) \quad \text{and} \quad h \in P(H). \tag{13}$$

To show that $(G \times H, <)$ is a totally ordered group, we will verify that (α), (β), (γ) and (δ) of Section 1 hold.

Suppose $(g,h) \neq (e,e)$. By Lemma 4·8 (iv), $\min(v_G(g), v_H(h)) < \infty$. If $v_G(g) < v_H(h)$, then $v_G(g) = \min(v_G(g), v_H(h))$, so $g \neq e$. If $g \in P(G)$, then $(g,h) > (e,e)$, while if $g^{-1} \in P(G)$, then $(g,h)^{-1} = (g^{-1}, h^{-1}) > (e,e)$. We proceed similarly if $v_H(h) \leqslant v_G(g)$ and conclude that $P(G \times H) \cup P(G \times H)^{-1} \cup \{(e,e)\} = G \times H$, so (δ) holds.

If both (g,h) and $(g,h)^{-1} = (g^{-1}, h^{-1})$ are in $P(G \times H)$, and $v_G(g) < v_H(h)$, then both g and g^{-1} are in $P(G)$. Similarly, if $v_H(h) \leqslant v_G(g)$, then both h and h^{-1} would be in $P(H)$. Hence (β) holds.

That (γ) holds is an exercise.

To verify (α), we must consider several cases under the assumption that (g,h) and (g',h') are elements of $P(G \times H)$. Suppose first that $v_G(g) < v_H(h)$ and $v_G(g') < v_H(h')$; then both g and g' are in $P(G)$, and by Lemma 4·8 (viii) and (i)

$$v_G(gg') = \min(v_G(g), v_G(g')) < \min(v_H(h), v_H(h')) \leqslant v_H(hh').$$

Hence $(gg', hh') \in P(G \times H)$.

A similar argument yields the same conclusion if both $v_H(h) \leqslant v_G(g)$ and $v_H(h') \leqslant v_G(g')$.

Suppose next that $v_G(g) < v_H(h)$ and $v_H(h') \leqslant v_G(g')$, in which case $g \in P(G)$ and

$h' \in P(H)$. (The remaining case, in which these inequalities are reversed, follows from this one and (γ).)

(i) Suppose also that $v_G(gg') < v_H(hh')$. If $gg' \leqslant_G e$, then by Lemma 4·8 (iii), (vii), $v_G(g') \leqslant v_G(g)$, so $v_H(h') \leqslant v_G(g') \leqslant v_G(g) < v_H(h)$. By Lemma 4·8 (v), $v_H(hh') = v_H(h') \leqslant \min(v_G(g), v_G(g')) \leqslant v_G(gg')$. This contradiction shows that $gg' > e$; thus $(gg', hh') \in P(G \times H)$.

A similar argument applies if, instead of (i), we have

$$\text{(ii)}\quad v_H(hh') \leqslant v_G(gg').$$

So (α) holds and we conclude that $(G \times H, <)$ is a (densely) ordered group.

To show that $<$ is topologically compatible with the orders of G and H, we must by Theorem 2·4 (*b*), when given $(g, h) \in P(G \times H)$, find $g^* \in G$ and $h^* \in H$ such that $(e, e) < (g^*, e) < (g, h)$ and $(e, e) < (e, h^*) < (g, h)$. Either

$$\text{(i)}\quad v_G(g) < v_H(h) \quad \text{and} \quad g \in P(G), \quad \text{or}$$

$$\text{(ii)}\quad v_H(h) \leqslant v_G(g) \quad \text{and} \quad h \in P(H).$$

If (i) holds, then since $cia(G)$ is infinite, there is a $g^* \in P(G)$ such that $v_G(g) < v_G(g^*)$, whence $g^* < g$ by Lemma 4·8 (vi). By parts (iii) and (v) of this lemma, $v_G(gg^{*-1}) = v_G(g) < v_H(h)$. Thus $gg^{*-1} \in P(G)$, so $(g, h)(g^*, e)^{-1} = (gg^{*-1}, h) > (e, e)$, and $(e, e) < (g^*, e) < (g, h)$. Also, let $h^* \in P(H)$, $v_H(h) \leqslant v_H(h^*)$. Then $v_H(h) = v_H(h^{*-1})$, so $v_H(hh^{*-1}) \geqslant \min(v_H(h), v_H(h^*)) = v_H(h) > v_G(g)$. From the last sentence, we conclude $(e, e) < (e, h^*) < (g, h)$.

In case (ii) holds, the argument is similar, reversing the roles of g and h, and of G and H. Thus the order $<$ defined in (13) is topologically compatible with the orders of G and H in case $cia(G) = cia(H) > 1$.

To complete the proof of Theorem 4·3, assume that $cia(G) = cia(H) = 1$ and define an order on $G \times H$ as follows:

By Theorem 3·2, there is a group order $<$ of $S(G) \times S(H)$ topologically compatible with the orders induced on $S(G)$ and $S(H)$ by the ordering of $\mathbb{R}(+)$. We let

$$\begin{aligned} &(g, h) \in P(G \times H) \quad \text{if:} \\ &g \in S(G),\, h \in S(H) \quad \text{and} \quad (g, h) > (e, e), \quad \text{or} \\ &g \notin S(G) \quad \text{and} \quad g \in P(G), \quad \text{or} \\ &g \in S(G),\, h \notin S(H), \quad \text{and} \quad h \in P(H). \end{aligned} \tag{14}$$

Since this order on $G \times H$ extends the order $<$ on $S(G) \times S(H)$ given above, we will denote it by $<$ as well. To show that it is a group order, we will verify (α), (β), (γ) and (δ) of Section 1.

Suppose $(g, h) \neq (e, e)$. If $g \notin S(G)$, then $g > e$ and $(g, h) > (e, e)$, or $g^{-1} > e$ and $(g, h)^{-1} = (g^{-1}, h^{-1}) > (e, e)$. If $g \in S(G)$ and $h \notin S(H)$, a similar proof shows that $(g, h) > (e, e)$ or $(g, h)^{-1} > (e, e)$. The same conclusion holds if $g \in S(G)$ and $h \in S(H)$ since $<$ is a group order on $S(G) \times S(H)$. Hence (δ) holds.

Suppose both (g, h) and $(g, h)^{-1}$ are in $P(G \times H)$. Then $g \notin S(G)$ or $h \notin S(H)$. If $g \notin S(G)$, then both g and g^{-1} are in $P(G)$ by the definition of $P(G \times H)$. Hence $g \in S(G)$, whence $h \notin S(H)$ and the definition of $P(G \times H)$ would yield both h and h^{-1} in $S(H)$. This contradiction shows that (β) holds.

Since $S(G)$ and $S(H)$ are central subgroups of G and H respectively, it follows easily that (γ) holds.

The proof that (α) holds may be carried through by cases in a straightforward way. We omit the details since they are similar to those given for the order of (13).

Once more, we apply Theorem 2·4(b) to show that $<$ is topologically compatible with the orders of G and H. Suppose $(g,h) \in P(G \times H)$. If $g \in S(G)$ and $h \in S(H)$, there is a $g^* \in G$ such that $(e,e) < (g^*,e) < (g,h)$ by Theorem 3·2. If $g \notin S(G)$, then $g \in P(G)$. Since $cia(G) = 1$, there is a $g^* \in S(G) \cap P(G)$ by Proposition 4·1, and $(e,e) < (g^*,e) < (g,h)$ since $g \notin cn(g^*) \subset S(G)$. In case $g \in S(G)$ and $h \notin S(H)$, clearly $(e,e) < (g,e) < (g,h)$. A similar argument by cases will produce an element $h^* \in H$ such that $(e,e) < (e,h^*) < (g,h)$. This completes the proof of Theorem 4·3.

We conclude with some remarks, examples and open problems (which we confine to the case when G and H are densely ordered).

By Theorems 4·3 and 3·2, given two archimedean densely ordered groups G, H, there is an order on $G * H$ topologically compatible with the orders of G and H if and only if there are embeddings ϕ of G into $\mathbb{R}$ and ψ of H into $\mathbb{R}$ such that $\phi(G) * \psi(H) \neq \mathbb{R}$. Moreover, by ([1], 12·2·1), if this latter holds and ϕ', ψ' are embeddings of G, respectively H into $\mathbb{R}(+)$, then there are nonzero real numbers a, b such that $\phi(G) = a\phi'(G)$ and $\psi(G) = b\psi'(G)$. So, as in the argument given in the proof of Lemma 3·3, $\phi'(G) * \psi'(G) = \mathbb{R}$. This comment inspires the following:

PROBLEM. *Find internal characterizations of densely ordered archimedean groups G, H, for which there is an embedding ϕ of G into $\mathbb{R}(+)$ and ψ of H into $\mathbb{R}(+)$ such that $\phi(G) * \psi(H) \neq \mathbb{R}$. Do the same in case $\phi(G) * \phi(G) \neq \mathbb{R}$.*

In [7] an ordered group is called *0-simple* if it has no proper normal convex subgroups other than $\{e\}$. Clearly, any infinite archimedean ordered group is 0-simple. In ([7], chapter 1, section 2, example 8), an example is given of an 0-simple non-abelian ordered group, and in ([8], corollary 2·6·9), it is shown that every solvable 0-simple group is archimedean.

Clearly if G is 0-simple, then $cia(G) = 1$ and $G = S(G)$. So by Theorem 4·3, if G is 0-simple, but not archimedean, there cannot be a densely ordered group H and an order $<$ on $G \times H$ that is topologically compatible with the orders of G and H. It seems natural to ask: If $cia(G) = 1$ then must $S(G)$ be 0-simple?

A negative answer to this question follows.

4·9. *Example.* Let B denote the direct sum of countably many copies of $Q(+)$ indexed by Z, that is $B = \{f\colon Z \to Q : f(k) = 0$ for all but finitely many $k \in Z\}$. Order B lexicographically with left-most non-zero coordinate dominating. Let $\mathbf{0}$ denote the zero-function, and for any $i \in Z$, let $f_i \in B$ be defined by letting $f_i(k) = f(k-i)$ for each $k \in Z$. Let $G = \{(k,f) : k \in Z, f \in B\}$, and let $(k,f)(k',f') = (k+k', f_k + f')$. It is routine to verify that G is a group (with identity element $(0,\mathbf{0})$ and where $(k,f)^{-1} = (-k, -f_k)$; indeed G is the wreath product of $Q(+)$ and $Z(+)$; see [4]). Order G lexicographically with first coordinate dominating. It is routine to verify that $(0,\mathbf{0})$ and $\{(0,f) : f \in B\}$ are the only proper convex normal subgroups of G, so G is not 0-simple but $cia(G) = 1$.

Finally, we give an example of a totally ordered group G such that $S(G)$ is archimedean but not central.

4·10. *Example.* Let T denote a subfield of $\mathbb{R}$ and let

$$G = \left\{ \begin{bmatrix} r & a \\ 0 & 1 \end{bmatrix} : r, a \in T \quad \text{and} \quad r > 0 \right\}.$$

If we let
$$\begin{bmatrix} r & a \\ 0 & 1 \end{bmatrix} \in P(G) \quad \text{if} \quad r > 1 \quad \text{or} \quad r = 1 \quad \text{and} \quad a > 0,$$

then under the operation of matrix multiplication, G is a totally ordered group as is noted in [7], p. 4. The following facts are easily verified.

(i) $S(G) = cn\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \left\{\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a \in T\right\}$.

(ii) The map $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \to a$ is an isomorphism of $S(G)$ into $\mathbb{R}(+)$, so $S(G)$ is archimedean.

(iii) For any $r, a, b \in T$,
$$\begin{bmatrix} r & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r & ra+b \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} r & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r & a+b \\ 0 & 1 \end{bmatrix},$$

so $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ is not in the centre of G unless $a = 0$. Thus $S(G)$ is not central.

By Theorem 4·3, for any totally ordered group H, there cannot be an order on $G \times H$ topologically compatible with the orders of G and H.

We close with reference to two papers related to our work, but without any obvious relationship with the above; namely [2] and [5]. In fact, hearing a lecture by E. Hewitt inspired this work.

REFERENCES

[1] A. Bigard, K. Keimel and S. Wolfenstein. *Groupes et Anneaux Réticulés*, Lecture Notes in Math. vol. 608 (Springer-Verlag, 1977).
[2] S. Eilenberg. Ordered topological spaces. *Amer. J. Math.* **63** (1941), 39–45.
[3] L. Fuchs. *Partially Ordered Algebraic Systems* (Pergamon Press, 1963).
[4] M. Hall. *The Theory of Groups* (Macmillan, 1959).
[5] E. Hewitt and S. Koshi. Orderings in locally compact groups and the theorems of F. and M. Riesz, *Math. Proc. Cambridge Philos. Soc.* **93** (1983), 441–487.
[6] J. Kelley. *General Topology* (Van Nostrand, 1955).
[7] A. Kokorin and V. Kopytov. *Fully Ordered Groups* (John Wiley and Sons, 1974).
[8] R. Mura and A. Rhemtulla. *Orderable Groups*. Lecture Notes in Pure and Appl. Math. **27** (Marcel Dekker, 1977).