Some Properties of Positive Derivations on f-Rings

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1. INTRODUCTION

Throughout A denotes an f-ring; that is, a lattice-ordered ring that is a subdirect union of totally ordered rings. We let \( \mathcal{D}(A) \) denote the set of derivations \( D : A \rightarrow A \) such that \( a > 0 \) implies \( Da > 0 \), and we call such derivations positive. In [CDK], P. Coleville, G. Davis, and K. Keimel initiated a study of positive derivations on f-rings. Their main results are (i) \( D \in \mathcal{D}(A) \) and A archimedean imply \( D = 0 \), and (ii) if A has an identity element 1 and \( a \) is the supremum of a set of integral multiples of 1, then \( Da = 0 \). Their proof of (i) relies heavily on the theory of positive orthomorphisms on archimedean f-rings and gives no insight into the general case. Below, in Theorem 4 and its corollary, we give a direct proof of (i), and in Theorem 10, we generalize (ii). Throughout, we improve on results in [CDK], and we study a variety of topics not considered therein.

2. THE RESULTS

In the sequel, A will always denote an f-ring, and 
\( A^+ = \{a \in A : a > 0\} \) its positive cone. If \( a \in A \), let \( a^+ = a \vee 0 \), \( a^- = (-a) \vee 0 \), and \( |a| = a \vee (-a) \). Then \( a = a^+ - a^- \), \( |a| = a^+ + a^- \), and \( a^+ a^- = a^- a^+ = a^+ a^- = 0 \). A subset I of A that is a ring ideal and such that \( |b| \leq |a| \), and \( a \in I \) imply \( b \in I \) is called an \( \ell \)-ideal. The \( \ell \)-ideals are the kernels of homomorphisms that preserve lattice as well as ring operations [BKW, Chap. 8].

A derivation on A is a linear map \( D : A \rightarrow A \) such that if \( a, b \in A \), then \( D(ab) = aDb + (Da)b \). A derivation D is called positive if \( D(A^+) \subseteq A^+ \). The family of all positive derivations on A will be denoted by \( \mathcal{D}(A) \).
In any f-ring \( \text{rad} \ A \), the set of all nilpotent elements of \( A \), coincides with the intersection of all the prime \( \mathcal{L} \)-ideals of \( A \), and hence is an \( \mathcal{L} \)-ideal [BKW, 9.2.6]. If \( \text{rad} \ A = \{0\} \), then \( A \) is said to be reduced. In [CDKJ], it is shown that if \( A \) is commutative and \( a^n = 0 \), then \( [Da]^{2n-1} = 0 \). We improve this result next. We begin by observing that if \( a, b, c \in A^+ \) then

\[(1) \quad ab = 0 \quad \text{implies} \quad aDb = (Da)b = 0.\]

1. **Proposition.** Suppose \( a \in A \) and \( D \in \mathcal{D}(A) \). Then \( a^n = 0 \) implies \( (Da)^n = 0 \). In particular, \( D[\text{rad} \ A] \subseteq \text{rad} \ A \).

**Proof.** Since \( a^n = 0 \) if and only if \( |a|^n = 0 \), we may assume \( a \in A^+ \) and \( n > 1 \). By (1), \( a^{n-1}Da = 0 \). So \( a^{n-2}(aDa) = 0 \). Using (1) again yields \( O = a^{n-2}D(aDa) = a^{n-1}D^2a + a^{n-2}(Da)^2 \). Since \( a \in A^+ \), \( a^{n-2}(Da)^2 = 0 \). Continuing this process yields \( (Da)^n = 0 \) and hence that \( D[\text{rad} \ A] \subseteq \text{rad} \ A \).

The next example will show that the index of nilpotency of \( Da \) need not be less than that of \( a \). We note first that if \( D \in \mathcal{D}(A) \) and \( I \) is an \( \mathcal{L} \)-ideal of \( A \) such that \( D(I) \subseteq I \), then \( D_I \in \mathcal{D}(A/I) \), where

\[(2) \quad D_I(a+I) = Da+I, \]

2. **Example.** Let \( R \) denote all rational functions with real coefficients of negative degree. If \( r(x) = \frac{p(x)}{q(x)} \in R \), we may assume that \( q(x) = x^m + a_1x^{m-1} + \ldots \) has leading coefficient 1, and we let \( r(x) \) be positive if the leading coefficient of \( p(x) \) is positive. With this order, \( R \) is a totally ordered ring. If \( r(x) \in R \), let \( Dr(x) = -r'(x) \) be the negative of the usual derivative. Then \( D \in \mathcal{D}(R) \), as is \( (xD): R \rightarrow R \), where \( (xD)r(x) = xDr(x) = -xr'(x) \).

If \( n \) is a positive integer, let \( I_n \) denote the set of all \( r(x) \) in \( R \) of degree \( \leq n \). Clearly \( I_n \) is an \( \mathcal{L} \)-ideal of \( R \), and \( (xD)(I_n) \subseteq I_{n+1} \). If \( R_n = R/I_n \), and \( (xD)_n(r(x)+I_n) = xDr(x) + I_n \), then \( (xD)_n \in \mathcal{D}(R_n) \), and \( (xD)_n(1+I_n) = \frac{1}{x} + I_n \) is nilpotent of index \( n \).

If \( G \) is an abelian \( \ell \)-group, and \( T: G \rightarrow G \) is an order preserving endomorphism of \( G \) such that \( x \wedge y = 0 \) implies \( x \wedge Ty = 0 \) for \( x, y \in G^+ \), then \( T \) is called a positive orthomorphism of \( G \). If \( A \) is reduced, then \( x \wedge y = 0 \) if and only if \( xy = 0 \) [BKW, 9.3.1].
So each positive derivation on an f-ring is an orthomorphism by (1). The next result appears implicitly in [CDKJ]. We include a proof for the sake of completeness.

3. PROPOSITION. If $P$ is a minimal prime $\mathcal{L}$-ideal of $A$, and $D \in \mathcal{D}(A)$, then $D(P) < P$. In particular, $D_P \in \mathcal{D}(A/P)$.

PROOF. As is noted in [BDKW, 9.3.2 and 12.1.1], if $A$ is reduced, then each positive orthomorphism of $A(\mathcal{L})$ maps a minimal prime subgroup into itself, and $P$ is a minimal prime $\mathcal{L}$-ideal of $A$ if and only if it is a minimal prime subgroup. So $D(P) < P$ if $A$ is reduced. In the general case, if we let $I = \text{rad } A$ in (2), we obtain $D(P) < P$.

We do not know if $D(P) < P$ for any prime $\mathcal{L}$-ideal of $P$.

Recall that $A$ is said to be archimedean if $a \in A^+$ and $(na : n = 1, 2, \ldots)$ bounded above imply $a = 0$. The next theorem is the key to an alternate proof of the fact that a reduced archimedean f-ring admits no nontrivial derivations [CDKJ].

4. THEOREM. Suppose $A$ is reduced, $D \in \mathcal{D}(A)$, $a \in A^+$, and $n$ is a positive integer. Then

(a) $n(a \land a^2)Da \leq (a \lor a^2)Da$,
(b) $nDa(a \land a^2) \leq Da(a \lor a^2)$, and
(c) $nD(a^2) \leq (a^2)Da + (Da)a^2 \lor Da$.

PROOF. Since $A$ is reduced, $(0)$ is an intersection of minimal prime ideals and $A$ is a subdirect sum of totally ordered rings $A/P$ such that $P$ is a minimal prime $\mathcal{L}$-ideal. Thus, by Proposition 3, it suffices to verify these identities in case $A$ is totally ordered and has no proper divisors of $0$ [BDKW, 9.2.5].

Let $x = (na-a^2)Da$. Then $x \in A^+$. We consider two cases:

(i) Suppose $x = 0$. Then $Da = 0$ or $na \leq a^2$. In either case we obtain

$$nDa \leq a^2Da \quad \text{and} \quad n(Da)a \leq (Da)a^2.$$  

(ii) Suppose $x > 0$. Then $Da > 0$ and $a^2 < na$. Hence $aDa + (Da)a \leq nDa$. Since $A$ is totally ordered, $aDa \leq (Da)a$ or $(Da)a \leq a(Da)$.

Suppose the former holds. Then

$$2aDa \leq nDa \quad \text{and hence} \quad (na-2a^2)Da \geq 0.$$
But $Da > 0$, so $2a^2 \leq na$. By induction, we get $2^k a^2 \leq na$ for $k = 0, 1, 2, \ldots$. If we choose $k$ so large that $n^2 \leq 2^k$, we get

$$n^2 a \leq a.$$  

If, instead, $(Da)a < aDa$, an obvious modification of this latter argument also yields (4). Pre or post multiplying by $Da$ yields

$$na^2 Da \leq aDa \quad \text{and} \quad n(Da)a^2 \leq (Da)a.$$  

Since either (3) or (5) must hold in $A/P$ for any minimal prime ideal $P$, the conclusions of (a) and (b) hold.

By (4), if $x > 0$, then $nD(a^2) \leq D(a)$. If $x = 0$, then adding the inequalities in (3) yields $nD(a^2) \leq (a^2 Da + (Da)a^2)$. Hence (c) holds as well.

5. COROLLARY. [CDKJ] If $A$ is archimedean and $D \in D(A)$, then $D(A) \subseteq \text{rad } A$ and $D(A^2) = 0$.

PROOF. By (c) of the last theorem and Proposition 3, if $a \in A$, then $D(a^2) \in \text{rad } A$. Since $aDa \leq D(a^2)$, $(Da)^2 \leq D(aDa) \leq D^2(a^2) \in D(\text{rad } A) \subseteq \text{rad } A$ by Proposition 1. Since each element of $\text{rad } A$ is nilpotent, so is $Da$.

If $a, b \in A$, then $D(ab) = aDb + (Da)b = 0$, since $(\text{rad } A)A = A(\text{rad } A) = 0$ in an archimedean f-ring [BKW, 12.3.11]. Hence $D(A^2) = 0$.

6. PROPOSITION. Suppose $e^2 = e \in A$ and $D \in D(A)$.

(a) $(De)^2 = e(De)e = (De)(De) = 0$.

(b) If $A$ is reduced or has an identity element or $e$ is in the center of $A$, then $De = 0$.

PROOF. Since $e^2 = e$, we have

$$eDe + (De)e = De.$$  

Multiplying (6) on the left by $e$ yields

$$e(De)e = 0.$$  

Applying $D$ to (7), we obtain

$$eD(De)e + (De)^2 e = 0 = e(De)^2 + D(eDe)e.$$
Hence

\[(8) \quad e(De)^2 = (De)^2 e = 0.\]

Multiplying both sides of (6) on the left by \((De)\) and using (8) yields

\[(9) \quad (De)e(De) = (De)^2.\]

By (7), (8), and (9), we obtain

\[CeDe - (De)e^2 + (De)e(De) = 0.\]

Hence \((De)^2 = (De)e(De) = 0\), which together with (7), completes the proof of (a).

Clearly \(De = 0\) if \(\text{rad } A = \{0\}\). If \(eDe = (De)e\), then by (6) and (7), \(De = 2eDe = 0\). If \(A\) has an identity element, then each of its idempotents is in the center of \(A\) by [BKX, 9.4.20]. This completes the proof of (b).

The next example shows that the hypotheses of (b) above cannot be omitted.

7. EXAMPLE. A totally ordered ring with an idempotent \(e\) and a positive derivation \(D\) such that \(De \neq 0\).

Let \(S\) denote the algebra over the real field \(\mathbb{R}\) (with the usual order) with basis \(\{e, z\}\), where \(e^2 = e\), \(ez = z^2 = 0\), and \(ze = z\). If \(x = ae + bz \in S\), let \(x > 0\) if \(a > 0\) or \(a = 0\) and \(b > 0\). If we let \(Dx = zx - xz = az\), then \(D \in D(S)\), and \(De = z \neq 0\).

If \(D \in D(A)\), let \(\ker D = \{a \in A : Da = 0\}\). If \(G\) is an abelian \(\mathcal{L}\)-group and \(H \subset G\), let \(H^\perp = \{g \in G : [g] \wedge [h] = 0\text{ for all } h \in H\}\), and let \(H^{\perp\perp} = (H^\perp)^\perp\). Note that \(H^\perp\) is an \(\mathcal{L}\)-subgroup of \(G\) (that is, \(H\) is a subgroup and \(|a| \leq |b|\), and \(b \in H^\perp\) implies \(a \in H\)). A band in \(G\) is an \(\mathcal{L}\)-subgroup \(H\) of \(G\) such that if \(K \subset H\) and \(\text{sup } K \subset G\), then \(\text{sup } K \subset H\). If \(H\) is a subset of \(G\), the intersection \(B(H)\) of all the bands in \(G\) containing \(H\) is also a band. Moreover, \(B(H) \subset H^{\perp\perp}\). See [LZ, Theorem 19.2]. An element \(e\) of \(G\) such that \((e)^\perp = 0\) is called a weak order unit of \(G\). An element \(e\) of an \(f\)-ring \(A\) such that \(ex = 0\) or \(xe = 0\) implies \(x = 0\) is called regular. Note that if \(e \in A\) is regular, then \(e\) is a weak order unit, and the converse holds if \(A\) is reduced.
The following lemma will be useful in what follows.

8. LEMMA. Suppose $A$ is an f-ring and $D \in \mathcal{D}(A)$.
   
   (a) $x Dx \wedge (Dx)x \geq 0$ for every $x \in A$.
   
   (b) If $A$ is reduced, then $D$ is an $\ell$-endomorphism.
   
   (c) If $A$ has an identity element $1$, and $n$ is a positive integer, then $nDx \leq x Dx \wedge (Dx)x$ for every $x \in A^+$ and $D(I) \subseteq I$ for every $\ell$-ideal $I$ of $A$.

PROOF. (a) holds since this inequality holds whenever $A$ is totally ordered.

(b) holds since if $A$ is reduced, then $D$ is a positive orthomorphism and hence an $\ell$-endomorphism [BKW, 12.1].

(c) by Proposition 6(b), $1 \in \ker D$, and by (a) $(x-nI)D(x-nI) \geq 0$.
Hence $nDx \leq xDx$. Similarly, $nDx \leq (Dx)x$. Hence $x \in I$ implies $Dx \in I$ since $I$ is an $\ell$-ideal.

Next, we provide some examples to show that the hypotheses of (b) and (c) above cannot be omitted.

9. EXAMPLES. (i) Let $E$ denote the direct sum of two copies of the real line $\mathbb{R}$ with trivial multiplication, and let $(r,s) \geq 0$ mean $r \geq s \geq 0$. As is noted in [CG], $5B$, the map $D: E \rightarrow E$ such that $D(r,s) = (r,0)$ is a positive endomorphism that is not an $\ell$-homomorphism.
To see the latter, note that $(1,2)^+ = (2,2)$. So $D((1,2)^+) = (2,0) \neq (1,0) = D(1,2)^+$.

(ii) Let $R$ and $(xD)$ be as in Example 2, and let $y = \frac{1}{x^2}$. Then $n(1)Dy = n_x$, while $y(xD)y = x^{-2}$, so the conclusion of (c) fails.

The next theorem summarizes most of what we know about kernels of positive derivations.

10. THEOREM. Suppose $D \in \mathcal{D}(A)$, $x \in A$, and $n$ is a positive integer.

   (a) If $e$ is regular, and $ex \in \ker D$, then $x \in \ker D$.

   (b) If $A$ is reduced then:

   (i) $x \in \ker D$ implies $(x)^+ \in \ker D$.

   (ii) $x^n \in \ker D$ implies $x \in \ker D$.

   (iii) $\ker D$ is a band,
(iv) \( D^n = 0 \) implies \( D = 0 \), and
(v) \( e^2 = e \in A \) implies \( e \in \ker D \).

(c) If \( A \) has an identity element and \( U(A) \) is the smallest band containing the units of \( A \), then \( U(A) \subseteq \ker D \). In particular, \( \text{rad} \ A \subseteq \ker D \). Also, if \( x^2 \leq x \), then \( x \in \ker D \).

**Proof.** (a) By (1), \( D(ex) = 0 \) implies \( eDx = 0 \), which, in turn implies \( Dx = 0 \).

(b) (i) By Lemma 8(b) and [EBKW, 3.2.2J], \( D(\{x\}^+) \subseteq D(\{x\}^+) \subseteq \{(Dx)\}^+ = \{0\} \) since \( x \in \ker D \) and \( A \) is reduced.

(ii) follows from (i) and the fact that \( \{x\}^+ \) is the intersection of all the minimal prime \( \ell \)-ideals that contains \( x \) [EBKW, 3.4.12J].

(iii) As was noted above, the smallest band containing \( \ker D \) is contained in \( \{(\ker D)^+\}^+ \) and the latter is contained in \( \ker D \) by (i).

(iv) Since \( x \) is a difference of positive elements, it suffices to show that \( Dx = 0 \) whenever \( x \in A^+ \). The proof will proceed by induction on \( n \). It is obvious when \( n = 1 \). Assume that \( D^n(A) = 0 \) implies \( D(A) = 0 \) whenever \( A \) is a reduced \( f \)-ring and \( n \geq 1 \) is an integer. If \( 0 = D^{n+1}(A) = D^n(D(A)) \), then \( D^n(D(A)^+) = 0 \) by (i). So \( D(D(A)^+) = 0 \) by the induction hypothesis. In particular, \( D^2(x^2) = 0 \). Since \( xDx \leq D(x^2) \), \( 0 = D(xDx) = xD^2x + (Dx)^2 \). So \( (Dx)^2 = 0 = Dx \) since \( A \) is reduced.

(v) is a restatement of Proposition 16(b).

(c) That \( U(A) \subseteq \ker D \) follows directly from (a) and (b) (iii) above. If \( x^n = 0 \), then \( (1-x)(1+x+\cdots+x^{n-1}) = 1 \), so \( 1-x \) is a unit and \( x = 1 - (1-x) \in U(A) \subseteq \ker D \). Finally, if \( x^2 \leq x \), then \( D(x^2) = xDx + (Dx)x \leq Dx \wedge (Dx)x \) by Lemma 8(c). Hence \( xDx = (Dx)x = 0 \). Thus \( Dx = 0 \). This completes the proof of Theorem 10.

**11. Examples and Remarks.** The assumption that \( A \) is reduced in Theorem 10(b) cannot be dropped. For example, if \( A = \mathcal{C} [0,1] \), the \( \ell \)-group of continuous real-valued functions on \( [0,1] \), with trivial multiplication for all \( f \in \mathcal{C} [0,1] \), we let \( Df = f \frac{1}{2} \), then \( D \in \mathcal{D}(A) \), and \( \ker D \) fails to be a band [DV, p. 123]. Also, the plane \( E^2 \) with the usual coordinatewise addition and trivial multiplication admits positive endomorphisms that are nilpotent. (For example, let \( T(a,b) = (0,a) \) for all \( (a,b) \in E^2 \).

Theorem 10(c) generalizes [ECDK, Theorem 7] where it is shown that \( \ker D \) contains the supremum of any set of elements bounded above by some integral multiple of the identity element.
As in [P], we let $I_0(A) = \{a \in A : n|a| \leq x$ for some $x \in A^+$ and $n = 1, 2, \ldots \}$. Clearly $I_0(A)$ is an $\mathcal{L}$-ideal and $I_0(A) = \{0\}$ if and only if $A$ is archimedian.

12. THEOREM. Suppose $D \in D(A)$.

(a) If $A$ is reduced, then $D(A^2) \subset I_0(A)$.

(b) If $A$ has an identity element, then $D(A) \subset I_0(A)$. If, moreover, $A$ is reduced and $I_0(A) \subset U(A)$, then $D = 0$.

PROOF. (a) follows immediately from Theorem 4 and the fact that $ab \leq (a + b)^2$ whenever $a, b \in A^+$.

(b) That $D(A) \subset I_0(A)$ is a restatement of Lemma 10(c). If $I_0(A) \subset U(A)$, then by Theorem 10(c), $D^2(A) \subset D(U(A)) = \{0\}$. Hence if $A$ is reduced, then $D = 0$ by Theorem 10(b).

13. EXAMPLES AND REMARKS.

(a) The reader may easily verify for the f-ring $R$ of Example 2, $I_0(R) = I_2$, while $(xO)(R) = R$. So the hypothesis in Theorem 12(b) that $A$ has an identity element may not be dropped if we wish to have $D(A) \subset I_0(A)$.

(b) Let $S$ denote the ring of all functions of the form

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

where $a_i$ is an integer and $r_i$ is a nonnegative rational number, ordered lexicographically, with the coefficient of the largest power of $x$ dominating. Then $I_0(S) = S$, and $U(A)$ is the set of constant polynomials. So, the condition of Theorem 12(b) fails. Despite this, $D \in D(S)$ implies $D = 0$.

For if $D \in D(S)$, then $D(x) = D((x^{1/2})^2 = 2x^{1/2}D((x^{1/4})^2) = 4x^{3/4}D((x^{1/8})^2) = \cdots = 2^n x^{1-1/2n}D(x^{1/2n})$. Hence $2^n D(x) = 0$ for $n = 0, 1, 2, \ldots$. Since the coefficients of any element of $S$ are integers, it follows that $D(x) = 0$. A similar argument will show that $x^r \in \ker D$ whenever $r$ is a nonnegative rational number. It follows that $D = 0$.

We do not, however, know of any such example that is an algebra over an ordered field. If $S^*$ is the result of allowing the coefficients of the elements of $S$ to be arbitrary rational numbers, and we let $D(x^r) = rx^r$ for any positive rational number $r$, then $D$ is a positive derivation. To see why, map $x^r$ to $e^{rx}$ and note that $S^*$ is isomorphic as an ordered ring to a subring of the ring of exponential polynomials, and the usual derivative on the latter maps the image of $S^*$ into itself.
Our last result applies more general theorems and techniques of Herstein [H1] [H2] to the context of positive derivations.

14. THEOREM. Suppose \( A \) is reduced and \( D \in \mathcal{D}(A) \).

(a) If \( D \neq 0 \), then the ring \( S \) generated by \( \{Da : a \in A\} \) contains a nonzero ideal of \( A \).

(b) If \( S \) is commutative, then \( S \) is contained in the center of \( A \).

(c) If \( z \in A \) commutes with every element of \( S \),
\[(az-za) \in \ker D \] for every \( a \in A \). If, in addition, \( A \) is totally ordered and \( D \neq 0 \), then \( z \) is in the center of \( A \).

PROOF. (a) It is shown in [H1] that the conclusion holds for any derivation on any ring if \( D^3 \neq 0 \). Since \( A \) is reduced, \( D^3 \neq 0 \) implies \( D \neq 0 \) by Theorem 10(b).

(b) Suppose \( a \in S \) and \( x \in A \). Then
\[0 = (Da)D(ax) - D(ax)(Da) = Da(aDx + (Da)x) - (aDx + (Da)x)Da = Da[(Da)x - x(Da)].\]

By [H3, Lemma 1.1.4], \( Da \) is in the center of \( A \).

(c) The second statement is shown in [H2], and the first follows immediately from the second and Theorem 10(b).

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