Local Connectedness in the Stone-Cech Compactification

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BY MELVIN HENRIKSEN AND J. R. ISBELL

Introduction

This is a study of when and where the Stone-Čech compactification of a completely regular space may be locally connected. As to when, Banaschewski [1] has given strong necessary conditions for $\beta X$ to be locally connected, and Wallace [19] has given necessary and sufficient conditions in case $X$ is normal. We show below that Banaschewski's necessary conditions are also sufficient and may be restated as follows: $\beta X$ is locally connected if and only if $X$ is locally connected and pseudo-compact (Corollary 2.5). Moreover, the requirement that $\beta X$ be locally connected is so strong that it implies that every completely regular space containing $X$ as a dense subspace is locally connected (Corollary 2.6).

As to where $\beta X$ is locally connected, we note first (1.15) that the completion $\langle aX \rangle$ of $X$ in its finest uniformity is a subspace of $\beta X$. Then $\beta X$ is never locally connected at any point not in $\langle aX \rangle$ (Theorem 2.2) and is locally connected at a point of $X$ if and only if $X$ is locally connected there (Corollary 1.5). In the remaining case, we have only that if $X$ is locally connected, then $\beta X$ is locally connected at every point of $\langle aX \rangle$ (Theorem 2.1).

These results, together with some lemmas, are given in the first two sections. Two lemmas worthy of independent mention are Lemma 1.4: An open subset $U$ of $\beta X$ is connected if and only if $U \cap X$ is connected, and Lemma 1.14: $X$ is locally connected if and only if every normal covering has a normal refinement consisting of connected sets. (The first of these was obtained by Wallace in [19] for normal spaces.)

In our last section we discuss Wallace's conditions which are stated in terms of Property $S$, a name which is given in the literature to three related but different concepts. We show that Property $S$ in the sense of Wallace is equivalent to local connectedness and countable compactness; then our Corollary 2.5 appears as a direct generalization of Wallace's result.

1. The lemmas

In this paper, we are concerned almost exclusively with subspaces of compact Hausdorff spaces. These are the completely regular spaces, and throughout the paper "space" will abbreviate "completely regular space" unless an exception is made explicitly.

For any space $X$, let $C(X)$ denote the set of all continuous real-valued functions on $X$, and let $C^*(X)$ denote the set of all bounded functions in $C(X)$.

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1 This paper was written while the first author was an Alfred P. Sloan fellow, and the second author was a National Science Foundation fellow.
1.1. Every completely regular regular space is a subspace of an essentially unique compact space $\beta X$ such that every $f \in C^*(X)$ has a (unique) extension $\tilde{f} \in C(\beta X) = C^*(\beta X)$. The space $\beta X$, which is usually called the Stone-Čech compactification of $X$, is unique in the sense that if $Y$ is any compact space containing $X$ as a dense subspace and such that every $f \in C^*(X)$ has a continuous extension over $Y$, then there is a homeomorphism of $\beta X$ onto $Y$ keeping $X$ pointwise fixed [3, 16].

If $A \subseteq X$, we use $A^\beta$ to denote the closure of $A$ in $\beta X$. Two subsets $A, B$ of $X$ are said to be completely separated if there is an $f \in C^*(X)$ such that $f[A] = 0$ and $f[B] = 1$. Two subsets of $X$ have disjoint closures in $\beta X$ if and only if they are completely separated [3].

1.2. We denote by $vX$ the subspace of $\beta X$ consisting of all $p \in \beta X$ over which every $f \in C(X)$ has a continuous real-valued extension. If $X = vX$, then $X$ is called a $Q$-space. The space $vX$ is unique in the sense that if $X$ is dense in a $Q$-space $Y$ such that every $f \in C(X)$ has a (unique) extension $\tilde{f}$ over $\beta X$ into the one point compactification $R \cup \{\infty\}$ of the real line $R$. A point $q$ of $\beta X$ fails to be in $vX$ if and only if there is an $f \in C(X)$ such that $\tilde{f}(q) = \infty$ [5].

1.3. In [1], Banaschewski showed that if $\beta X$ is locally connected (i.e., every point has a base of connected open neighborhoods), then (i) $X$ is locally connected, and (ii) $X$ cannot have an infinite family of open subsets whose closures are pairwise disjoint and have a closed union. In [6], it was noted that (ii) is equivalent to $X$ being pseudo-compact (i.e., every $f \in C(X)$ is bounded). Equivalently, $vX = \beta X$.

Below (Corollary 1.5 and Lemma 1.6), we improve Banaschewski’s result by making it local in character. In particular, we show that $\beta X$ cannot be locally connected at any point $x$ of $X$ unless $X$ is locally connected there, and that $\beta X$ fails to be locally connected at any point not in $vX$. Moreover, the converse of Banaschewski’s theorem is true. Indeed if $X$ is locally connected and pseudo-compact, and $X$ is dense in a completely regular space $Y$, then $Y$ is locally connected (Theorem 2.4).

The following lemma was obtained by Wallace for normal spaces [19].

1.4. Lemma. An open subset $U$ of $\beta X$ is connected if and only if $U \cap X$ is connected.

Proof. If $U = U_1 \cup U_2$, where $U_1$ and $U_2$ are disjoint open subsets of $\beta X$, then $U \cap X = (U_1 \cap X) \cup (U_2 \cap X)$, so $U \cap X$ is disconnected if $U$ is disconnected.

If $U \cap X$ is disconnected, then there exist nonempty disjoint open subsets $V_1, V_2$ of $X$ such that $U \cap X = V_1 \cup V_2$. Since $X$ is dense in $\beta X$, $(U \cap X)^\beta = V_1^\beta \cup V_2^\beta$ contains $U$. If $V_1^\beta \cap V_2^\beta$ is empty, then

$$U = (V_1^\beta \cap U) \cup (V_2^\beta \cap U)$$
is disconnected, and we are done. On the other hand, if there is a \( p \notin V^2 \cap V^3 \),
construct a \( \varphi \in C(\beta X) \) such that \( \varphi(p) = 0 \), and \( \varphi(\beta X - U) = 1 \). Define a
function \( f \) on \( X \) by letting \( f(x) = \varphi(x) \) except where \( x \in V_2 \) and \( \varphi(x) < \frac{1}{2} \),
and by letting \( f(x) = \frac{1}{2} \) otherwise. It is easily verified that \( f \in C^*(X) \).
The continuous extension (1.1) \( \hat{f} \) of \( f \) over \( \beta X \) coincides with \( \varphi \) on \( V^2 \), so
\( \hat{f}(p) = 0 \). But \( \hat{f} \geq \frac{1}{2} \) on \( V^3 \). This contradiction shows that \( U \) is discon-
nected, and completes the proof of the lemma.
Clearly \( \Omega_x \) is a base of open neighborhoods of \( x \in X \) if and only if \( \Omega_x \cap X \)
is a base of open neighborhoods of \( x \) in \( X \). We may conclude:

1.5. Corollary. For each point \( x \) of \( X \), \( \beta X \) is locally connected at \( x \) if and only if \( X \) is locally connected at \( x \).

1.6. Lemma. \( \beta X \) is never locally connected at any point not in \( \nu X \).

Proof. If \( p \in \beta X - \nu X \), there is an \( f \in C(X) \) such that \( \hat{f}(p) = \infty \) (1.2).
For \( i = 0, 1, 2, 3 \), let \( Z_i \) be the set of all \( x \) in \( X \) such that \( n \leq f(x) \leq n + 1 \)
for some integer \( n \equiv i \) (mod 4). The four sets \( Z_i \) cover \( X \), so \( p \) is in one of
their closures in \( \beta X \), say \( p \in Z^3_1 \). Then \( p \) is not in \( Z^3_2 \), since \( Z_1 \) and \( Z_3 \) are
obviously completely separated (1.1). Hence there is a neighborhood \( U \) of \( p \)
disjoint from \( Z^3_2 \). If \( \beta X \) is locally connected at \( p \), then \( U \) contains a
connected open neighborhood \( U' \) of \( p \). By Lemma 1.4, \( U' \cap X \) is connected. So,
by the construction above, there is an integer \( n \) such that \( 4n \leq f(x) \leq 4n + 3 \)
for all \( x \in U' \cap X \), contrary to the fact that \( \hat{f}(p) = \infty \).

1.7. Corollary (Banaschewski). \( \beta X \) is not locally connected unless \( X \) is
locally connected and pseudo-compact.

For the remainder of the paper, we shall need some elementary facts about
normal coverings and uniformities in the sense of Tukey [18].

1.8. An open covering \( v \) of a space is said to be a refinement of a covering \( u \)
if every member of \( v \) is a subset of some member of \( u \). The open covering
\( v = \{ V_\beta \} \) is said to be a star-refinement of the open covering \( u \) if every \( V_\beta \)
is contained in some member \( U \) of \( u \) in such a way that \( U \) contains every member of \( v \)
that meets \( V_\beta \). An open covering \( u \) is normal if there is an infinite
sequence \( \{ u^n \} \) of open coverings beginning with \( u^1 = u \), such that \( u^{n+1} \) is
a star-refinement of \( u^n \). A binary open covering \( \{ U, V \} \) is normal if and
only if \( X - U \) and \( X - V \) are completely separated [18, V. 9.3].

Some insight into this concept may be gained by the following remark
which is given in [9, Corollary 2.2].

An open covering \( u \) of a space \( X \) is normal if and only if there exists a metrizable
space \( Y \), an open covering \( v \) of \( Y \), and a continuous function \( f \) on \( X \) onto \( Y \) such
that \( f^{-1}(v) \) is a refinement of \( u \).

1.9. We presuppose a familiarity with Tukey's development of uniform
spaces, but we will repeat some known facts about uniformities, primarily
those described with the aid of nonstandard terminology. The open cover-
ings of a uniform space that are members of its uniformity are called large coverings. Every large covering is normal. A filter $\mathcal{F}$ on a uniform space $\mu X$ is called a Cauchy filter if every large covering contains a member of $\mathcal{F}$. A uniform space $\mu X$ is complete if every Cauchy filter on $\mu X$ converges. Every uniform space $\mu X$ is a dense subspace of a unique complete uniform space $\langle \mu X \rangle$, called the completion of $\mu X$, such that every Cauchy filter on $X$ converges to a point in $\langle \mu X \rangle$. A uniform space is called precompact if its completion is compact. There is a finest uniformity on a space $X$ compatible with its topology. It consists of all normal (open) coverings of $X$. The associated uniform space is denoted by $aX$.

The next two lemmas are due essentially to Tukey and Doss.

1.10. Lemma. For every point $x$ of a space $X$ and every open neighborhood $U$ of $x$, there is a closed neighborhood $V$ of $x$ such that $\{U, X - V\}$ is a normal covering.

Proof. There is an $f \in \mathcal{C}(X)$ such that $f(x) = 0$ and $f[X - U] = 1$. Let $V = \{x \in X : f(x) \leq \frac{1}{2}\}$. Then it is easily seen that $X - U$ and $V$ are completely separated. So, as noted in 1.8, $\{U, X - V\}$ is normal.

1.11. Lemma. The space $X$ is pseudo-compact if and only if every normal (open) covering of $X$ has a finite normal subcovering.

Proof. In [4], Doss has shown that $X$ is precompact in all its uniformities if and only if $X$ is pseudo-compact. Tukey [10, p. 60] has shown that a uniform space is precompact if and only if every large covering has a finite large subcovering. (Tukey uses "largely compact" for our "precompact"). But then $X$ is precompact in all its uniformities if and only if $aX$ is precompact, so we have the lemma.

The next lemma is due to A. H. Stone. Although a weaker statement is made in [15, p. 979], the following is actually proved therein.

1.12. Lemma (A. H. Stone). Every normal covering has a normal refinement that can be written as the union of countably many collections $\{V_n\}$, $n = 1, 2, \cdots$, such that for each fixed $n$, the $V_n$'s have pairwise disjoint closures.

1.13. Recall that a space $X$ is locally connected (connected im kleinen) at a point $x$ if every neighborhood of $x$ contains a connected open neighborhood (connected neighborhood). Locally, im kleinen connectedness is a weaker property; but in the large the two are equivalent [10, p. 94]. Therefore, to show that a space is locally connected, it suffices to show that it is connected im kleinen at each of its points.

A space is locally connected if and only if components of open sets are open. The union of a family of connected sets that meet a given connected set is connected [21, p. 10, p. 45].

1.14. Lemma. A space $X$ is locally connected if and only if every normal covering has a normal refinement consisting of connected sets.
Proof. Let $U$ be any open neighborhood of a point $x$ of $X$. By Lemma 1.10, there is a closed neighborhood $V$ of $x$ such that $\{U, X - V\}$ is a normal covering of $X$. Hence the sufficiency follows.

To prove the necessity, we will show that if $\{U, x\} = u$ is a normal covering of a locally connected space $X$, then the covering $v$ consisting of all of the components of the elements of $u$ is normal.

Since $u$ is normal, there is a sequence of (normal) coverings $\{u^n\}$ with $u^1 = u$ and such that $u^{n+1}$ is a star-refinement of $u^n$. If $v^n$ denotes the set of all components of elements of $u^n$, then since $X$ is locally connected, $v^n$ is an open covering (1.13). Moreover, if $V \in v^{n+1}$, then $V$ is a component of some $U \in u^{n+1}$, and therefore $V$ is a subset of some $U' \in u^n$ which contains every member of $u^{n+1}$ meeting $U$. A fortiori, $U'$ contains all the elements of $v^{n+1}$ that meet $V$. But, as noted in 1.13, this latter is a connected set, and thus is a subset of a component of $U'$. Therefore $v^{n+1}$ is a star-refinement of $v^n$, and hence $v'$ is normal.

Next, we will make some remarks comparing $\mathfrak{v}X$ with $\langle aX \rangle$.

1.15. The completion $\langle aX \rangle$ of $X$ in its finest uniform structure is a subspace of $\mathfrak{v}X$.

Proof. As was shown by Tukey [18, VI. 5.5], every $f \in C(X)$ is uniformly continuous on $aX$ and hence has a continuous extension over $\langle aX \rangle$, which in turn has an extension over $\mathfrak{v}(aX)$. Thus $X$ is dense in the $Q$-space $\mathfrak{v}(aX)$, and every $f \in C(X)$ is extensible over it. From (1.2), there is a homeomorphism of $\mathfrak{v}(aX)$ upon $\mathfrak{v}X$ keeping $X$ pointwise fixed, which serves to embed $\langle aX \rangle$ in $\mathfrak{v}X$.

1.16. Actually, under very weak hypotheses on $X$, we may identify $\mathfrak{v}X$ with $\langle aX \rangle$. More precisely, Shirota showed in [13] that if $X$ has a base of open sets whose cardinal number is not strongly inaccessible from $\aleph_0$ in the sense of Tarski and Ulam, then $X$ is a $Q$-space if (and only if) it admits a uniformity in which it is complete. Actually this hypothesis may be weakened a bit further, but we shall not dwell on the matter since we do not use Shirota’s theorem explicitly in the sequel. Moreover, Tarski [17] has shown that it is consistent with the axioms of set theory to reject the existence of strongly inaccessible cardinals.

To see that $\mathfrak{v}X$ and $\langle aX \rangle$ coincide under the hypothesis stated above, it suffices to note that Shirota’s theorem yields that $\langle aX \rangle$ is a $Q$-space, and that $\langle aX \rangle$ contains $X$ as a dense subspace so that every $f \in C(X)$ has a continuous extension over $\langle aX \rangle$ (1.2).

The next lemma, which we will need explicitly below, is also due to Shirota [13].

1.17. Lemma (Shirota). $\mathfrak{v}X$ is the completion of $X$ relative to the uniformity defined by all countable normal coverings.

Our last lemma, which is due to Morita, gives us a way of passing from
normal coverings of $X$ to normal coverings of $\langle aX \rangle$. For any subset $A$ of $X$, we let $\overline{A}$ denote the closure of $X$ in $\langle aX \rangle$, and we let $A^*$ denote the interior (in $\langle aX \rangle$) of $\overline{A}$. The proof of this lemma may be obtained by reading in the order given [11, Theorem 3], [12, Lemma 1], [11, Lemma 9] and by recalling that every large covering is normal (1.9).

1.18. **Lemma (Morita).** For any normal covering $\{ U_\alpha \}$ of $X$, $\{ U_\alpha^* \}$ is a normal covering of $\langle aX \rangle$.

### 2. The theorems

2.1. **Theorem.** If $X$ is locally connected, then $\langle aX \rangle$ is locally connected, and $\beta X$ is locally connected at each point of $\langle aX \rangle$.

**Proof.** Let $u^1 = \{ U_\alpha \}$ be a normal covering of $\langle aX \rangle$, and let $u^n$ be a descending sequence of star-refinements. Let $v^n$ denote the restriction of $u^n$ to $X$. Obviously $v^{n+1}$ is a star-refinement of $u^n$, so $v^n$ is normal—as is $v^\infty$. By Lemma 1.14, $v^\infty$ has a normal refinement $w = \{ W_\beta \}$ consisting of connected open sets. By Lemma 1.18, $\{ W_\beta^* \}$ is a normal covering of $\langle aX \rangle$. Each $W_\beta$ is contained in a member $U_\alpha$ of $u^\infty$ which contains every member of $u^n$ meeting $W_\beta$; a fortiori $U_\alpha$ contains $\overline{W_\beta}$, and hence $W_\beta^*$. Finally, each $W_\beta \in w$ is connected and dense in $W_\beta^*$, so $W_\beta^*$ is connected. By Lemma 1.14 again, $\langle aX \rangle$ is locally connected. The second part of the theorem follows from the above, Corollary 1.5, and the fact that $\beta \langle aX \rangle = \beta X$ (1.15).

2.2. **Theorem.** $\beta X$ is not locally connected at any point not in $\langle aX \rangle$.

**Proof.** By Lemma 1.6, we need only consider points of $vX$. Suppose that $p$ is in $vX$, but not in $\langle aX \rangle$. Let $\mathcal{F}$ denote the filter of all $U \cap X$, where $U$ is a neighborhood in $\beta X$ of $p$. Since $p$ is not in $\langle aX \rangle$, $\mathcal{F}$ is not a Cauchy filter on $\langle aX \rangle$, so there is a normal covering $\{ U_\alpha \}$ of $X = \langle aX \rangle$ which is normal in $\beta X$. By Lemma 1.12 (and the definition of normal covering) we may replace $\{ U_\alpha \}$ by a normal star-refinement $\{ V_\beta \}$, where for each fixed $n$, the $V_\beta$’s have pairwise disjoint closures. Since $\mathcal{F}$ is a filter containing no $U_\alpha$, it contains no $V_\beta$. However, if for $n = 1, 2, \ldots$, we put $V_n = \bigcup_\beta V_\beta$, then $\{ V_n \}$ is a countable normal covering. But by Lemma 1.17, $vX$ is the completion in the uniformity on $X$ defined by all countable normal coverings, so $\mathcal{F}$ is a Cauchy filter relative to this uniformity, whence $\mathcal{F}$ must contain some $V_0$. This means that for this $n$, $V_n = U_\alpha \cap X$ for some neighborhood $U$ of $p$. Since $X$ is in $\beta X$, $V_n \cap X$ contains $U$, and hence is a neighborhood of $p$. If $\beta X$ is locally connected at $p$, then there is a connected open neighborhood $U'$ contained in $V_n$. By Lemma 1.4, $U' \cap X$ is connected and hence is contained in one of the sets $\overline{V}_\gamma$. But $U' \cap X$ is in $\mathcal{F}$, and by the above no $\overline{V}_\gamma$ can be in $\mathcal{F}$. Hence $\beta X$ cannot be locally connected at $p$.

From Theorem 2.1, Theorem 2.2, Corollary 1.5, and the fact that $\beta \langle aX \rangle = \beta X$, we obtain the following.
2.3. **Corollary.** $X$ is locally connected if and only if $(aX)$ is locally connected.

Note that in the corollary above, we cannot replace $(aX)$ by $\nu X$ without some cardinality restriction on $X$ as in 1.16. For if there exists a discrete space $X$ that is not a $Q$-space, then $X$ is locally connected, but $\nu X$ is not locally connected.

2.4. **Theorem.** If $X$ is locally connected and pseudo-compact, then any (completely regular) space $Y$ containing $X$ as a dense subspace is locally connected.

**Proof.** Let $X$ be dense in $Y$. For any $y \in Y$, and any open neighborhood $U$ of $y$, by Lemma 1.10, there is a closed neighborhood $V$ of $y$ such that \{U, Y - V\} is a normal covering of $Y$. Then \{U \cap X, X - V\} forms an open covering of $X$ which is clearly normal. Since $X$ is locally connected, by Lemma 1.14, this open covering has a normal refinement consisting of connected (open) subsets of $X$. Since $X$ is pseudo-compact, the latter has a finite subfamily \{F_i\} that covers $X$. Let $G$ denote the closed subset of $Y$ which consists of the union of the closures in $Y$ of all those $F_i$ such that $y$ is a limit point of $F_i$. None of these $F_i$ can be contained in $X \cap U$; hence they are all in $U \cap X$, so $G$ is a subset of the closure in $Y$ of $U$. Now, since $Y$ is regular, the closed neighborhoods of $y$ form a basis at $y$. Moreover, $Y - G$ is a subset of the union of the closures in $Y$ of all those $F_i$ of which $y$ is not a limit point, so $G$ is a neighborhood of $y$. Finally, $G$ is a union of connected sets having a point in common, and hence is connected (1.13). Hence, $Y$ is connected im kleinen at each of its points, so $Y$ is locally connected (1.13).

The next two corollaries follow from Theorem 2.4 and Corollary 1.5.

2.5. **Corollary.** $\beta X$ is locally connected if and only if $X$ is locally connected and pseudo-compact.

2.6. **Corollary.** $\beta X$ is locally connected if and only if every (completely regular) space $Y$ containing $X$ as a dense subspace is locally connected.

We conclude this section by remarking that under the added assumption that $Y$ is compact, Corollary 2.6 can be obtained more simply. For, Whyburn [20] has shown that every closed continuous image of a locally connected space is locally connected, and by a theorem of Čech [3], every compact space containing $X$ as a dense subspace is a continuous (closed) image of $\beta X$. (We are indebted to E. Michael for the reference to Whyburn’s paper.)

3. **Property S**

The term *Property S* has two definitions in the literature which are not seriously liable to be confused. In each case, the idea is that a set having Property S should be locally connected and “smooth”. The original formulation of Sierpiński [14] is metric: for every real $\varepsilon > 0$, the space is a union of
finely many connected sets of diameter less than \( \varepsilon \). This definition is used e.g. in Bing’s solution of the convex metric problem [2], and in a textbook [7]. However, in the theory of generalized manifolds [21], it seems to be convenient to use a related property that is topological and relative; a subspace \( Y \) of a regular space \( X \) has Property \( S \) if every open covering of \( X \) can be refined on \( Y \) by a finite family of connected sets.

Wallace has introduced a third property of the same name, and has given some applications of it in the theory of extension spaces [19]. He says that a topological space \( X \) has Property \( S \) provided every finite open covering of \( X \) has a finite refinement consisting of connected sets. We shall show below that this use of the terminology is unnecessary, at least for regular spaces.

3.1. Theorem. The following properties of a regular space \( X \) are equivalent:

(a) \( X \) has Property \( S \) in the sense of Wallace.

(b) Every finite open covering has a finite refinement consisting of connected open sets.

(c) \( X \) is locally connected and countably compact.

Proof. (a) implies (c). Suppose that \( X \) has Property \( S \). For any point \( x \) of \( X \), let \( U \) denote an arbitrary open neighborhood of \( x \) and let \( V \) be any closed neighborhood of \( x \) contained in \( U \). Then \( \{U, X \setminus V\} \) has a finite refinement \( \{F_i\} \) consisting of connected sets. By the argument given in the proof of Theorem 2.4, the union of the closures of those \( F_i \) of which \( x \) is a limit point is a connected neighborhood of \( x \) contained in \( U \). Thus \( X \) is connected im kleinen at each of its points, and hence is locally connected (1.13).

Suppose next that \( X \) is not countably compact, and let \( D = \{d_i\} \) denote a countably infinite closed discrete subset of \( X \). Since \( X \) is regular, a simple induction yields a sequence \( \{U_i\} \) of pairwise disjoint open sets such that each \( U_i \) is a neighborhood of \( d_i \). Clearly, \( \{X \setminus D, \bigcup_i U_i\} \) has no finite refinement consisting of connected sets.

(c) implies (b). Suppose that \( X \) is locally connected, so that components of open sets are open (1.13). We shall assume that (b) does not hold and construct an infinite closed discrete subset of \( X \). Let \( \{V_j\} \) denote a finite open covering of \( X \) that has no finite connected open refinement.

Consider the open components \( \{C_{ja}\} \) of the sets \( \{V_j\} \). Successively for each \( j \), delete those \( C_{ja} \) which are contained in the union of (1) all \( V_k \) for \( k > j \), and (2) all \( C_{ka} \), \( k < j \), such that \( C_{ka} \) was not deleted at the \( k \)th step. The remaining \( C_{ja} \) still form an open covering refining \( \{V_j\} \), so there are infinitely many of them. Hence there are infinitely many of them in some one set \( V_j \). For this \( j \), each \( C_{ja} \) contains a point \( p_a \) not in any other undeleted component \( C_{jb} \). But, the infinite set \( \{p_a\} \) has no limit point in any of the open sets \( C_{ib} \) which form a covering of \( X \). Hence \( \{p_a\} \) is closed and discrete, so \( X \) is not countably compact.
Clearly (b) implies (a).

In [19], Wallace showed that if $X$ is a normal space, then $X$ has Property $S$ if and only if $\beta X$ has Property $S$, and noted that for compact spaces Property $S$ is equivalent to local connectedness. Since countably compact (completely regular) spaces are pseudo-compact, Wallace's characterization follows from our Corollary 2.5 and Theorem 3.1.

References