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An Algebraic Characterization of the Freudenthal Compactification for a Class of Rimcompact Spaces

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AN ALGEBRAIC CHARACTERIZATION
OF THE FREUDENTHAL
COMPACTIFICATION FOR A CLASS OF
RIMCOMPACT SPACES

by

MELVIN HENRIKSEN
1. Introduction

Throughout \( C(X) \) will denote the ring of all continuous real-valued functions on a Tychonoff space \( X \), and \( C^*(X) \) will denote the subring of bounded elements of \( C(X) \). The real line is denoted by \( R \), and \( N \) denotes the (discrete) subspace of positive integers. A subset \( S \) of \( X \) such that the map \( f \to f|_S \) is an epimorphism of \( C(X) \) (resp. \( C^*(X) \)) is said to be \( C\)-embedded (resp. \( C^*-\)embedded) in \( X \). As is well-known, every \( f \in C^*(X) \) has a unique continuous extension \( \hat{f} \) over its Stone-Cech compactification \( \beta X \) [GJ, Chapter 6]. That is, \( X \) is \( C^*-\)embedded in \( \beta X \).

In [NR], L. Nel and D. Riordan introduced the subset \( C^\#(X) \) of \( C(X) \) consisting of all \( f \) such that for every maximal ideal \( M \) of \( C(X) \), there is an \( r \in R \) such that \( (f-r) \in M \), and they noted that \( C^\#(X) \) is a subalgebra and sublattice of \( C(X) \) containing the constant functions. They show how \( C^\#(X) \) determines a compactification of \( X \) in a number of cases and leave the impression that it always does. In [Cl], E. Choo notes that this is true if \( X \) is locally compact and seems to conjecture that it need not be the case otherwise. In [SZ 1], O. Stefani and A. Zanardo show that every \( f \in C^\#(R^\omega) \) is a constant function, where \( R^\omega \) denotes a countably infinite product of copies of \( R \). In [SZ 2] they show that \( C^\#(X) \)
determines a compactification of $X$ in case $X$ is locally compact, pseudo compact, or zero-dimensional, and they describe the compactifications so determined when $X$ is realcompact \cite{GJ, Chapter 8}.

In this paper, I show that under certain restrictions on $X$, the ring $C\#(X)$ determines the Freudenthal compactification of $X$ \cite[pp. 109-120]{Il}, I observe that, at least in disguised form, $C\#(X)$ has been considered by a number of authors other than those named above, and some conditions are given that are either necessary or sufficient for $X$ to determine a compactification of $X$. In particular, it is shown that if $X$ is realcompact, and $C\#(X)$ determines a compactification of $X$, then $X$ is rimcompact and it determines the Freudenthal compactification $\phi X$ of $X$. There are realcompact rimcompact spaces $X$ for which $C\#(X)$ does not determine a compactification of $X$, but $C\#(X)$ does determine $\phi X$ if every point of $x$ has either a compact neighborhood, or a base of open and closed neighborhoods. Other sufficient conditions are given for $C\#(X)$ to determine $\phi X$. I close with some remarks and open problems.

2. Using $C\#(X)$ to Compactify $X$

We will make use of the following characterization of $C\#(X)$ due to a number of authors. Recall that $Z(f) = \{x \in X: f(x) = 0\}$ and $\nu X$ denotes the Hewitt real compactification of $X$.

2.1 Theorem. If $f \in C(X)$, then the following are equivalent.

(a) $f \in C\#(X)$. 
(b) \( f \in C^*(X) \) and \( f[D] \) is closed (and hence finite) for every \( C \)-embedded copy \( D \) of \( N \).

(c) \( f \in C^*(X) \) and \( f[Z] \) is closed for every zero-set \( Z \) in \( X \).

(d) \( f \in C^*(X) \) and for every \( r \in \mathbb{R} \), \( \text{Cl}_{\beta X} Z(f-r) = Z(\beta f-r) \).

(e) \( f \in C^*(X) \) and for every \( p \in \beta X \setminus \nu X \), there is a neighborhood of \( p \) in \( \beta X \) on which \( \beta f \) is constant.

The equivalence of (a) and (b) seems to appear first in [NR]. The equivalence of (a), (b), (c), (d) appears in [Cl], and that of (a), (b), (d), and (e) in [SZ 2]. Mappings that satisfy (d) are a special case of what are called WZ-maps by T. Isiwata, who showed that any map that sends zero-sets to closed sets in a WZ-map, and that a WZ-map on a normal space is closed [I 2], [W, p. 215]. More important for this paper is the following result. For any subset \( S \) of \( X \), let \( \text{Fr} S = \text{Cl}_S S \cap \text{Cl}(X \setminus S) \) denote the boundary (or frontier) of \( S \).

2.2 Theorem. If \( X \) is realcompact and \( f \in C^\#(X) \), then \( \text{Fr} Z(f-r) \) is compact for every \( r \in \mathbb{R} \), and \( f \) is a closed mapping.

By Theorem 2.1 (d,e) if \( r \in \mathbb{R} \), then either \( Z(f-r) \) is compact or \( \text{Fr} Z(\beta f-r) \subseteq X \). In the latter case, \( \text{Fr} Z(f-r) = \text{Fr} Z(\beta f-r) \). In either case \( \text{Fr} Z(f-r) \) is compact. In [I.2, 1.3], T. Isiwata shows that a WZ-map with this latter property is closed, so the theorem is proved.

Recall that a space \( X \) is called rimcompact if it has a base of open sets with compact boundaries. \( X \) is said to be zero-dimensional at \( x \) if \( x \) has a base of neighborhoods with
empty boundaries, and $X$ is called zero-dimensional if it is zero-dimensional at each of its points. It is shown in [M3] that every rimcompact space has a compactification $\phi X$ such that $\phi X \setminus X$ is zero-dimensional, and wherever $\gamma X$ is a compactification of $X$ with $\gamma X \setminus X$ zero-dimensional, there is a continuous map of $\phi X$ onto $\gamma X$ leaving $X$ pointwise fixed. $\phi X$ is called the Freudenthal compactification of $X$.

In [D], R. Dickman shows that if $X$ is rimcompact, then every $f \in C^*(X)$ such that $\text{Fr } Z(f-r)$ is compact for every $r \in \mathbb{R}$ has a (unique) extension in $C(\phi X)$. Hence the following is an immediate consequence of Theorem 2.2.

2.3 Corollary. If $X$ is rimcompact and realcompact, then every $f \in C^#(X)$ has a (unique) extension $\phi f \in C(\phi X)$.

Suppose $S$ is a subring of $C^*(X)$ that contains the constant functions and $\gamma X$ is a compactification of $X$ such that every $f \in S$ has an extension $\gamma f \in C(\gamma X)$ and $S^\gamma = \{ \gamma f: f \in S \}$ separates the points of $\gamma X$. (That is if $x_1, x_2 \in \gamma X$ and $x_1 \neq x_2$, there is an $f \in S$ such that $\gamma f(x_1) = 0$ and $\gamma f(x_2) = 1$). Then by the Stone-Weierstrass Theorem, $S^\gamma$ is dense in $C(\gamma X)$ in its uniform topology [GJ, 16.4], and we say that $S$ determines the compactification $\gamma X$ of $X$. Note that $S$ determines a compactification of $X$ if points can be separated from disjoint closed sets by functions in $S$.

If $\gamma_1 X$ and $\gamma_2 X$ are compactifications of $X$ for which there is a homeomorphism of $\gamma_1 X$ onto $\gamma_2 X$ keeping $X$ pointwise fixed, then we write $\gamma_1 X = \gamma_2 X$.

For any space $X$, let $C^#(\beta X) = \{ \beta f: f \in C^#(X) \}$ and note that $C^#(\beta X)$ and $C^#(X)$ are isomorphic. Similarly, if $X$ is
realcompact and rimcompact, then by Corollary 2.3, \( C^\#(X) \) is isomorphic to \( C^\#(\Phi X) = \{ \phi f : f \in C^\#(X) \} \).

A subring \( A \) of \( C^*(X) \) is called *algebraic* if it contains the constant functions and those members \( f \in C^*(X) \) such that \( f^2 \in A \). If, in addition, \( A \) is closed under uniform convergence, then \( A \) is called an *analytic* subring of \( C^*(X) \). The closure in the uniform topology of a subset \( B \) of \( C^*(X) \) will be denoted by \( uB \). It is noted in [GJ, 16.29], that if \( A \) is an algebraic subring of \( C^*(X) \), then \( uA \) is an analytic subring.

If \( B \subseteq C^*(X) \), then a *maximal stationary set* \( S \) of \( B \) is a subset of \( X \) maximal with respect to the property that every \( f \in B \) is constant on \( S \). In [GJ, 16.29-16.32], the following is established.

2.4 If \( X \) is compact and \( A \) is an algebraic subring of \( C^*(X) \), then every maximal stationary set of \( A \) is connected and \( uA = \{ f \in A : f \text{ is constant on every connected stationary set of } A \} \).

If \( X \) is rimcompact and realcompact, then, by the above \( C^\#(\Phi X) \) is an algebraic subring of \( C^*(\Phi X) \). Next, I make use of the above to establish:

2.5 Theorem. If \( X \) is a realcompact space and \( C^\#(X) \) determines a compactification \( \gamma X \) of \( X \), then \( X \) is rimcompact and \( \gamma X = \Phi X \).

Proof. Suppose \( x \in X \) and \( V \) is an open neighborhood of \( x \). By assumption there is an \( f \in C^\#(X) \) such that \( f(x) = 0 \) and \( f(X \setminus V) = 1 \). If \( g = (f - \frac{1}{2}) \vee 0 \), then, by Theorem 2.2 \( Z(g) \) is a neighborhood of \( x \) with compact boundary that is
contained in V. Hence X is rimcompact, and so \( A = C^\#(\mathcal{F}X) \) is an algebraic subring of \( C^*(\mathcal{F}X) \). Assume without loss of generality that X is not compact, let S denote a maximal stationary set of A, and suppose S has more than one point. Since A determines a compactification of X, it follows that \( S \subset \mathcal{F}X \setminus X \). Since the remainder of X in \( \mathcal{F}X \) is totally disconnected, S reduces to a point and Theorem 2.5 is established.

Next, I give an example to show that \( C^\#(X) \) need not determine a compactification of a realcompact and rimcompact space. For any space X, let \( R(X) \) denote the set of points of X which fail to have a compact neighborhood. Clearly \( R(X) \) is closed since \( X \setminus R(X) \) is open.

2.6 Example. A realcompact rimcompact space S for which \( R(X) \) is a compact connected maximal stationary set.

Let \( W^* \) denote the space of ordinals that do not exceed the first uncountable ordinal \( \omega_1 \), and let \( W = W^* \setminus \{\omega_1\} \). It is well known that \( W^* \) is compact and every \( f \in C(W) \) is eventually constant [GJ, 5.13]. Let \( X = [0,1] \times W^* \) with the topology obtained by adding to the product topology every subset of \([0,1] \times W\). Clearly X is rimcompact and \( R(X) = [0,1] \times \{\omega_1\} \). Moreover, X is the union of a realcompact discrete space and the compact space \( R(X) \), so X is realcompact [GJ, 8.16]. Suppose \( 0 < r < s < 1 \) and \( g \in C^*(X) \) is such that \( g(r,\omega) \neq g(s,\omega) \). Since \([0,1]\) is connected, since every \( f \in C(W) \) is eventually constant, and since \( W \) has no countable cofinal subset, there is an \( \alpha > \omega_1 \), and an increasing sequence \( \{x_n\} \) of real numbers between r and s such that \( g(x_n,\alpha) \neq g(x_m,\alpha) \) if \( n \neq m \). Thus g assumes infinitely many
values on a closed discrete subspace of \( X \) and hence cannot be in \( C^\#(X) \) by Theorem 2.1(b). So \( R(X) \) is a maximal stationary set of \( C^\#(X) \).

It is clear that \( C^\#(X) \) always contains both the subring \( C_\kappa(X) \) of all functions with compact support and the subring \( C_F(X) \) of functions with finite range. Clearly any point of \( X \setminus R(X) \) can be separated from any disjoint closed set by some element of \( C_\kappa(X) \), and if \( X \) is zero-dimensional at a point \( x \), then \( x \) can be separated from any disjoint closed set by some element of \( C_F(X) \). This together with 2.4 and Theorem 2.5 proves:

2.7 Theorem. If \( X \) is a rimcompact, realcompact space that is zero-dimensional at each point of \( R(X) \), then \( C^\#(X) \) determines \( \Phi X \); that is, \( \cup C^\#(\Phi X) = C(\Phi X) \).

Along these lines we have also:

2.8 Theorem. If \( X \) is a rimcompact and realcompact space such that \( c_{\Phi X}(\Phi X \setminus X) \) is zero-dimensional, then \( \cup C^\#(\Phi X) = C(\Phi X) \).

Proof. By the remarks proceeding the proof of Theorem 2.7, if \( S \) is a maximal stationary set for \( C^\#(\Phi X) \) with more than one point, then \( S \subset c_{\Phi X}(\Phi X \setminus X) \). Since the latter set is zero-dimensional, \( S \) reduces to a point and the conclusion follows.

In [II, Theorem 36, p. 114], it is shown that if \( \Phi X \setminus X \) is a Lindelöf space, then the Lebesgue dimension of \( \Phi X \setminus X \) is zero. In [P, Corollary 5.8] it is shown that if \( F \) is a closed subset of a normal space \( Y \), then the Lebesgue dimension...
of \( Y \) does not exceed the Lebesgue dimensions of \( A \) or \( (Y \setminus A) \).

It follows that if \( R(X) \) is compact and zero-dimensional, then 
\[
\partial_{\phi X}(\phi X \setminus X) = (\phi X \setminus X) \cup R(X)
\]
is zero-dimensional, for these two motions of dimensionality coincide at 0 if \( X \) is compact; see [P, pp. 156-157]. Note also that \( \phi X \setminus X \) is a Lindelöf space if and only if every compact subset of \( X \) is contained in a compact subset with a countable base of neighborhoods; in which case we will say that \( X \) is of countable type. [II, p. 119]. Thus we have established:

2.8 Corollary. If \( X \) is a rimcompact, realcompact space of countable type, and \( R(X) \) is compact and zero-dimensional, then
\[
\mathfrak{u} \subset \mathfrak{c}(\phi X) = \mathfrak{c}(\phi X)
\]

3. Remarks and Open Problems

A. In [N], the ring of all closed \( f \in C(X) \) is considered for \( X \) locally compact and weakly paracompact ( = metacompact). For \( X \) realcompact this latter ring coincides with \( C^\#(X) \) by Theorem 2.2. Recall also that W. Moran showed in [M3] that if every closed discrete subspace of a normal metacompact space \( X \) is realcompact, then so is \( X \). Also, examination of Example 3 of [N] shows that this latter need not hold if \( X \) fails to be normal.

B. In a private communication S. Willard notes that if \( f \in C^*(X) \) and \( f \) is a closed mapping, then \( Z(f) \) has a countable base of neighborhoods in \( X \). (I.e., \( Z(f) = \bigcap_{i=1}^\infty f^{-1}(-1/i,1/i) \)). It would be of great interest to characterize the zero-sets of elements of \( C^\#(X) \) at least in case \( X \) is rimcompact and realcompact. To determine which such spaces determine \( X \), it would probably be
enough to characterize zero-sets of restrictions to $X$ of $u \mathcal{C}^\#(\mathcal{F}X)$.

C. Willard notes also that if $S$ is a countable subset of $X$ and $\overline{S}_{\mathcal{F}X}$ is connected, then $S$ is a stationary set for $\mathcal{C}^\#(X)$. It follows from a theorem of McCartney [M1, Proposition 3.12] that if $Y = [0,1] \times (0,1] \cup Z$, where $Z = \{(q,0): 0 < q < 1 \text{ and } q \text{ is rational}\}$, then $\mathcal{F}Y = [0,1] \times [0,1]$. Hence, by the latter remark of Willard cited above, $Z$ is a stationary set for $\mathcal{C}^\#(Y)$, so $Y$ is a separable, metrizable rimcompact space such that $\mathcal{C}^\#(Y)$ does not determine a compactification of $Y$.

D. Suppose $X = [0,1] \times Q \cap [0,1]$, where the open sets of $X$ and those in the product topology together with any subset of $\{(a,b) \in X: b > 0\}$. Then $R(X) = \{(a,b) \in X: b = 0\}$ is compact and connected, $X$ is rimcompact, realcompact, and determines $\mathcal{F}X$. So the hypotheses of Theorem 2.7 or 2.8 are not necessary for $X$ to determine $\mathcal{F}X$.

References


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