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WHEN IS $|C(X \times Y)| = |C(X)|\,|C(Y)|$?

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Abstract. Sufficient conditions on the Tychonoff spaces X and Y are found that imply that the equation in the title holds. Sufficient conditions on the Tychonoff space X are found that ensure that the equation holds for every Tychonoff space Y. A series of examples (some using rather sophisticated cardinal arithmetic) are given that witness that these results cannot be generalized much.

1. Introduction

Throughout, all topological spaces considered will be assumed to be Tychonoff spaces (i.e., subspaces of compact Hausdorff spaces) unless the contrary is stated explicitly. The cardinal number of a set S will be denoted by $|S|$. The set of continuous real-valued functions on a space X is denoted by $C(X)$. In this paper, we try to determine for which pairs (X, Y) it is true that:

$$|C(X \times Y)| = |C(X)|\,|C(Y)|.$$

Such a pair will be said to be *functionally conservative*. If $f \in C(X)$ and $g \in C(Y)$, then $fg \in C(X \times Y)$ is defined by letting $(fg)(x, y) = f(x)g(y)$ if $(x, y) \in X \times Y$. So it is always true that $|C(X \times Y)| \geq |C(X)|\,|C(Y)|$. Our problem is to determine when this inequality can be reversed.

We begin by recalling the definitions of some familiar cardinal functions. The symbols used in the literature are not standardized. We follow [Hod] most, but not all of the time.

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In Section 2, we recall the definitions of some cardinal functions and some facts from [CH]. We use them to obtain some sufficient conditions for the existence of functionally conservative pairs of topological spaces. Section 3 is devoted to showing how certain kinds of singular cardinals (e.g., beth cardinals) can be used to create pairs of spaces that are not functionally conservative. These involve the use of cardinals that are "large" and not particularly familiar. In Section 4, topological techniques are used to create pairs of spaces each of cardinality 2^ω that fail to be functionally conservative. Use is made of the Alexandroff double (see [E] or [Wa]) to create larger closed C-embedded subspaces in a product of two spaces than exist in either factor. In Section 5, the techniques developed in Section 4 are used to construct infinite families of spaces such that, for $n < \omega$ specified in advance, any n-tuple $(X_0, X_1, \cdots, X_{n-1})$ satisfies $|C(\prod_{i=0}^{n-1} X_i)| > |C(\prod_{i\in F} X_i)|$, for every proper subset F of $\{0, 1, \cdots, n-1\}$. It then follows that for each $n < \omega$ there are spaces Z such that $|C(Z^{n+1})| > |C(Z^n)|$. Our problem is studied in the context of linearly ordered spaces and products of discrete spaces in Section 6. Use is made of pseudocompactness numbers and of functions on product spaces determined by countably many coordinates.

A (Tychonoff) space X is said to be *functionally conservative* if (X, Y) is a functionally conservative whenever Y is a Tychonoff space. Section 7 is devoted to studying this class of spaces. Separable spaces and σ-compact spaces of weight no larger than $\mathbf{c}$ are functionally conservative. The class of functionally conservative spaces is countably productive but is not productive. If X is functionally conservative, then so is its Stone-Čech compactification. The converse holds if X is locally compact and σ-compact, but does not hold in general. Other ways of creating new functionally conservative spaces from old ones are studied and some unsolved problems are posed.

2. Background and some positive results

Definitions 2.1. Suppose X is a topological space and $\kappa \geq \omega$ is a cardinal number.

(a) Let $w(X) = \min\{|\mathcal{B}| : \mathcal{B}$ is a base for the topology of $X\} + \omega$.

(b) Let $d(X) = \min\{|S| : S$ is a dense subspace of $X\} + \omega$.

(c) A family of pairwise disjoint nonempty open subsets is called a *cellular family*, and the *cellularity* cX of X is defined to be $\sup\{|K| : K$ is a cellular family in $X\} + \omega$. If $c(X) = \omega$, then X is called a *c.c.c.-space*, or is said to satisfy the *countable chain condition*. If $X \times Y$ is a c.c.c.-space for every c.c.c.-space Y, then X is said to be *productively c.c.c.*; see p. 35 of [CN2].

(d) The *Lindelöf number* $\ell(X)$ of X is defined to be $\min\{\kappa :$ every open cover of X has a subcover of cardinality $\kappa\} + \omega$. The *weak Lindelöf number* $wl(X)$ of X is defined to be $\min\{\kappa :$ every open cover of X has a subfamily of cardinality κ whose union is dense in $X\} + \omega$. If $\ell(X) = \omega$ (resp. $wl(X) = \omega$), then X is called a *Lindelöf* (resp. *weak Lindelöf*) *space*. It is noted in 3P of [PW] that every c.c.c.-space is weakly Lindelöf.

(e) The *extent* $e(X)$ is defined to be $\sup\{|D| : D$ is closed and discrete$\} + \omega$.

(f) Let $e^{\#}(X) = \min\{\kappa : D$ closed and discrete $\Rightarrow |D| < \kappa\}$, and let $\ell^{\#}(X) = \min\{\kappa :$ every open cover of X has a subcover of cardinality $\leq \mu \Rightarrow \mu < \kappa\}$.

(g) Let $2^{<\kappa}$ denote $\sup\{2^\alpha : \alpha < \kappa\}$.

(h) Denote the set of bounded elements of $C(X)$ by $C^*(X)$. If $C(X) = C^*(X)$, then X is said to be *pseudocompact*.

(i) If the map $f \to f\,|_S$ from $C(X)$ (resp. $C^*(X)$) into $C(S)$ (resp. $C^*(S)$) is a surjection, then $S \subset X$ is said to be *C-embedded* (resp. *C^*-embedded*) in X. The essentially unique compact space in which X is dense and C^*-embedded is denoted by βX. (See Chapter 6 of [GJ]).

(In [CH], our $d(X)$ is denoted by $\delta(X)$, our $c(X)$ by $o(X)$, and our $wl(X)$ by $wc(X)$.)

Remark. The convention that each of these cardinal functions take on values no smaller than ω is not always followed by Engelking in [E]. It turns out, however, that if one uses the definitions given above, the pertinent theorems in [E] remain correct unless they involve finite topological products of finite topological spaces. The reader will be reminded of this where appropriate.

Most parts of the following proposition are known, and are needed below.

Proposition 2.2. *Suppose X is a (Tychonoff) space.*

(a) $|C(X)| = |C(X)|^\omega = |C^*(X)|$.
(b) $|C(X)| = \big(w(\beta X)\big)^\omega$.
(c) $|C(X)| \leq \big(w(X)\big)^{wl(X)} \leq 2^{d(X)}$.
(d) *If X is weakly Lindelöf, in particular if X is a c.c.c.-space, then* $|C(X)| = \big(w(X)\big)^\omega$.
(e) *If X is metrizable, then* $|C(X)| = 2^{d(X)} = 2^{w(X)}$.
(f) *For any space Y,* $|C(X \times Y)| \leq \min\{|C(X)|^{d(Y)}, |C(Y)|^{d(X)}\}$.
(g) *For any space Y, if $Z = X \oplus Y$ is the free union of X and Y, then*

$$|C(Z \times Z)| = \max\{|C(X \times X)|, |C(X \times Y)|, |C(Y \times Y)|\}.$$

Proof. (a) The first equality is shown in 10.10 in [Hod]. (See also [Kr] for a more general result.) By corresponding to each element f of $C(X)$ the sequence $\{(-n) \vee (f \wedge n) : n < \omega\}$ in $C^*(X)$, we get $|C(X)| \leq |C^*(X)|^\omega$. Because $C^*(X)$ and $C(\beta X)$ are isomorphic, this yields

$$|C(X)| = |C(\beta X)|^\omega = |C(\beta X)| = |C^*(X)|.$$

Proofs of (b), (c), (d), and (e) may be found in Sections 2, 6, and 7 of [CH].

(f) By symmetry, it suffices to show that $|C(X \times Y)| \leq |C(X)|^{d(Y)}$. If K is a dense subspace of Y of cardinality $d(Y)$, then any element of $C(X \times Y)$ is determined by its values on $X \times K$, and there are at most $|C(X)|^{|K|}$ such functions. So (f) holds.

(g) Because Z is the free union of X and Y, we have

$$\begin{aligned}|C(Z \times Z)| &= |C(X \times X)|\,|C(X \times Y)|\,|C(Y \times Y)| \\ &= \max\{|C(X \times X)|, |C(X \times Y)|, |C(Y \times Y)|\}.\end{aligned}$$

□

The next theorem summarizes our main positive results. For any ordinals α, β the sets $[\alpha, \beta)$ and $[\alpha, \beta]$ are assumed to carry the interval topology.

Theorem 2.3. *(X, Y) is functionally conservative provided that any of the following hold.*

(a) *X is separable.*
(b) *$X \times Y$ is pseudocompact; in particular if one of X and Y is compact and the other is pseudocompact.*
(c) *$X \times Y$ is metrizable.*
(d) *$X \times Y$ is weakly Lindelöf, in particular if X is σ-compact and Y is Lindelöf, or X is productively c.c.c. and Y is a c.c.c. space.*

Proof. (a) By 2.2(a,f), $|C(X \times Y)| \leq |C(Y)|^\omega = |C(Y)| \leq |C(X \times Y)|$, while $|C(X)|\,|C(Y)| = |C(Y)|$ because $|C(X)| = 2^\omega \leq |C(Y)|$, so (X, Y) is functionally conservative.

(b) By a well-known theorem of I. Glicksberg, $X \times Y$ is pseudocompact if and only if there is a homeomorphism of $\beta(X \times Y)$ onto $\beta X \times \beta Y$ keeping $X \times Y$ pointwise fixed. (See 3.12.20 in [E].) So, if $X \times Y$ is pseudocompact, then $\left|w\big(\beta(X \times Y)\big)\right| = |w(\beta X \times \beta Y)| = |w(\beta X)|\,|w(\beta Y)|$. So (b) follows immediately from 2.2(b) followed by 2.2(a).

(c) follows immediately from 2.2(e) and the fact that the weight of a product of two spaces is the product of their weights.

(d) Because a continuous image of a weakly Lindelöf space is weakly Lindelöf, if $X \times Y$ weakly Lindelöf, then both X and Y are weakly Lindelöf. So (d) follows from 2.2(d) and the fact that the weight of a product of two spaces is the product of their weights. □

Examining Example 4.7 below will show the reader that it will be difficult to improve the result of 2.3(d).

To improve on the result of 2.3(c), we introduce the following definition.

Definition 2.4. A Tychonoff space Z such that $|C(Z)| = 2^{d(Z)}$ is said to be *functionally metrizable*.

Because $d(X \times Y) = d(X)d(Y)$, it is clear that if X and Y are functionally metrizable, then (X, Y) is functionally conservative.

By 6.1 of [CH], every metrizable space is functionally metrizable, and the separable space $\beta\omega$ witnesses the fact that the converse need not hold. The next result is a consequence of 2.2(f).

Theorem 2.5. *If X is functionally metrizable, and either*

(a) $|C(Y)| = 2^\beta$ *for some* $\beta \geq \omega$, *or*

(b) $|C(Y)| \leq |C(X)|$,

then (X, Y) is functionally conservative.

Proof. (a) By the definition of functionally metrizable and 2.2(f), $2^{d(X)}2^\beta = |C(X)|\,|C(Y)| \leq |C(X \times Y)| \leq |C(Y)|^{d(X)} = 2^{\beta d(X)}$. So the pair (X, Y) is functionally conservative since $|C(X)|\,|C(Y)| = 2^{d(X)}2^\beta = \max\{2^{d(X)}, 2^\beta\} = 2^{\beta d(X)}$.

(b) By assumption and 2.2(f), $2^{d(X)} = |C(X)| = |C(X)|\,|C(Y)| \leq |C(X \times Y)| \leq |C(Y)|^{d(X)} \leq |C(X)|^{d(X)} = 2^{d(X)}$. So (X, Y) is functionally conservative. □

Recall that a collection of nonempty subsets of a topological space X is called *discrete* if each point of the space has a neighborhood meeting at most one of its members. X is said to be *collectionwise normal* if for each (closed) discrete collection $\{A_\xi : \xi \in \Lambda\}$ of closed subsets of X, there is a discrete collection $\{U_\xi : \xi \in \Lambda\}$ of open subsets of X such that $A_\xi \subset U_\xi$ for each $\xi \in \Lambda$. X is called *paracompact* if each of its open covers has an open locally finite refinement. It is well-known that paracompact spaces are collectionwise normal and collectionwise normal spaces are normal. Neither of these last two implication can be reversed; see Chapter 5 of [E].

Proposition 2.6. *If X is a paracompact space with an open cover $\mathcal{U}$ of infinite cardinality λ that has no subcover of cardinality less than λ, then there is a closed discrete subspace D of cardinality λ.*

Proof. Let $\mathcal{U} = \{U_i : i \in I\}$ with $|I| = \lambda$ and let $\mathcal{V} = \{V_j : j \in J\}$ be a locally finite open refinement of $\mathcal{U}$. From the minimality condition on $\mathcal{U}$ we may assume that $|J| = \lambda$. For $j \in J$, choose $x_j \in V_j$. The locally finite cover $\mathcal{V}$ is point finite, so the map that sends $V_j \in \mathcal{V}$ to x_j is finite to one. Hence with $D = \{x_j : j \in J\}$, we have $|D| = \lambda$. Each point of X has a neighborhood meeting only finitely many V_j, and hence has a neighborhood containing at most one of the points x_j. This shows that D is closed and discrete. □

Let $p(X)$ denote the least cardinal number κ such that no discrete collection of nonempty open subsets of X has cardinality κ. Note that a Tychonoff space is pseudocompact if and only if $p(X) = \omega$. The lemma that follows is well-known. A proof is included because we cannot cite an explicit source for it.

Lemma 2.7.

(i) *$p(X)$ is the least infinite cardinal κ such that every locally finite family of nonempty open sets of X has cardinality $< \kappa$.*

(ii) *If each family of κ nonempty open subsets of X fails to be locally finite, then $p(X) \leq \kappa$.*

Proof. To see (i), it is enough to show that if X has a faithfully indexed locally finite family $\mathcal{U} = \{U_\xi : \xi < \kappa\}$ of nonempty open sets, then it has such a discrete family. Choose $x_\xi \in U_\xi$. Because $\mathcal{U}$ is point-finite, the map $\xi \to x_\xi$ is finite to one; then since $\kappa \geq \omega$, we may assume, passing to a subfamily if necessary, that this map is one-one. Since $\mathcal{U}$ is locally finite, every subset of $\{x_\xi : \xi < \kappa\}$ is closed, so there is an open set V_ξ such that $x_\xi \in V_\xi \subset clV_\xi \subset U_\xi$ and $x_\eta \notin V_\xi$ if $\eta \neq \xi$. Now define $W_\xi = V_\xi \setminus \cup\{clV_\eta : \eta \neq \xi\}$. Because $\{V_\xi : \xi < \kappa\}$ is locally finite, W_ξ is an open neighborhood of x_ξ. Finally, for each ξ, choose a neighborhood Y_ξ of x_ξ such that $clY_\xi \subset W_\xi$. Then $\mathcal{Y} = \{Y_\xi : \xi < \kappa\}$ is a faithfully indexed discrete family of open subsets of X. In fact, for any $x \in X$, if $O_x = X \setminus \cup\{clY_\xi : x \notin clY_\xi\}$, then O_x is an open neighborhood of x meeting at most one member of $\mathcal{Y}$.

(ii) This follows immediately from (i). □

The cardinal $p(X)$ is often called the *pseudocompactness number of* X, and if $p(X) \leq \aleph$, then X is said to be *pseudo-$\aleph$-compact.* For background discussion, see [A], Chapter 9 of [CN2], p.135 of [I], and [NU]. As will be noted again in

Definition 3.3(a) below, the *cofinality* $cf(\alpha)$ of cardinal α is the least cardinal number ξ such that $[0, \alpha)$ has a cofinal subset of power ξ.

Theorem 2.8. *For every Tychonoff space X:*

(i) $2^{<p(X)} \leq |C(X)|$, *and*
(ii) $2^{p(X)} \leq |C(X)|^{cf(p(X))}$.

Proof. (i) By the definition of $p(X)$, if $\omega \leq \kappa < p(X)$, then there is a discrete collection of nonempty open subsets of X of cardinality κ, so $2^\kappa \leq |C(X)|$ by 3L in [GJ]. Hence $2^{<p(x)} = \sup\{2^\kappa : \kappa < p(X)\} \leq |C(X)|$.

(ii) Suppose $\{\kappa_\xi\}_{\xi < cf(p(X))}$ is an increasing sequence of infinite cardinals whose supremum is $p(X)$. Then $2^{p(X)} = 2^{\sup\{\kappa_\xi : \xi < cf(p(X)\}} = 2^{\Sigma\{k_\xi : \xi < cf(p(X)\}} = \prod\{2^{\kappa_\xi} : \xi < cf(p(X))\} \leq |C(X)|^{cf(p(X))}$ by (i). □

Corollary 2.9. *If $cf\big(p(X)\big) = \omega$ or if X is pseudocompact, then $2^{p(X)} \leq |C(X)|$.*

It is clear from the definitions of these concepts that $p(X) = e^{\#}(X)$ if X is collectionwise normal. We can do a bit better than just combining this latter with 2.8 if we recall from 10.2 of [Hod] that if D is an infinite (closed) discrete subspace of a normal space X, then $2^{|D|} \leq |C(X)|$. This coupled with 2.2(c) and 2.6 yields the following corollaries.

Corollary 2.10. *If X is normal, then $2^{<e^{\#}(X)} \leq |C(X)| \leq 2^{d(X)}$.*

Corollary 2.11. *If X is infinite and paracompact, then $e^{\#}(X) = \ell^{\#}(X)$. (See 2.1(g).)*

Corollary 2.12. *If $2^{<p(X)} = 2^{d(X)}$, then X is functionally metrizable.*

Corollary 2.13. *If X and Y are infinite, paracompact spaces such that $2^{<\ell^{\#}(X)} = 2^{d(X)}$ and $2^{<\ell^{\#}(Y)} = 2^{d(Y)}$, then (X, Y) is functionally conservative.*

Proof. By corollaries 2.10, 2.11, and 2.5, $|C(X)| = 2^{d(X)}$ and $|C(Y)| = 2^{d(Y)}$. The conclusion follows from the fact that $|d(X \times Y)| = |d(X)|\,|d(Y)|$. □

Remark. The space of countable ordinals witnesses the fact that the hypothesis that X is paracompact in Proposition 2.6 may not be replaced by the assumption that X is hereditary normal, and Example 4.7 below witnesses that "$2^{<\ell^{\#}(X)}$" cannot be replaced by "$2^{\ell^{\#}(X)}$" in 2.13.

We conclude this section with the following analogues of Corollary 2.9.

Proposition 2.14. *If $2^{<p(X)} = 2^{d(X)}$ or $cf\big(p(X)\big)$ is countable while $2^{p(X)} = 2^{d(X)}$, and if $2^{<p(Y)} = 2^{d(Y)}$ or $cf\big(p(Y)\big)$ is countable while $2^{p(Y)} = 2^{d(Y)}$, then (X, Y) is functionally conservative.*

In the next two sections, ways to create pairs that are not functionally conservative will be described.

3. Using singular cardinals to create pairs that are not functionally conservative; paracompact spaces

In this section, we determine when a pair of spaces, one entry of which is infinite and discrete, is functionally conservative.

Definition 3.1. An ordered pair (m,t) of infinite cardinal numbers such that

$$m^t > m^\omega = m \geq 2^t$$

is called a *bad cardinal pair*.

Remarks. (i) It is clear that if $m = 2^\alpha$ for some infinite cardinal α, then there is no cardinal t such that (m,t) is a bad cardinal pair.

(ii) It is an exercise to verify that if (m,t) is a bad cardinal pair, then so is (m^ω, t).

Theorem 3.2. *Suppose X is a Tychonoff space, D is an infinite discrete space, and α is an infinite cardinal.*

(a) *If $m = |C(X)|$, and $t = |D|$, then (X,D) fails to be functionally conservative if and only if (m,t) is a bad cardinal pair.*

(b) *If X is weakly Lindelöf; in particular, if X is a c.c.c.-space, and if $\big(w(X), |D|\big)$ is a bad cardinal pair, then (X,D) fails to be functionally conservative.*

(c) *If (m,t) is a bad cardinal pair, and $t = |D|$, then (X,D) is not functionally conservative if either*

  (i) *X is the one-point compactification of an (uncountable) discrete space K of cardinal m, or*

  (ii) *X is the product of m separable metric spaces each of cardinality at least 2, or*

  (iii) *X is the first countable subspace obtained from the space $[0,m]$ (with the interval topology) by deleting all limit ordinals of uncountable cofinality.*

(d) *If $|C(X)| = 2^\alpha$ for some cardinal α, then (X,D) is functionally conservative.*

Proof. (a) Because $C(X \times D)$ is the free union of $|D|$ copies of $C(X)$, $|C(X \times D)| = m^t$. Clearly $|C(D)| = (2^\omega)^t = 2^t$, so it follows that

$$(\uparrow) \qquad |C(X \times D)| = m^t \geq |C(X)|\,|C(D)| = m|C(D)| = m2^t = \max\{m, 2^t\}.$$

In particular, if (X, D) fails to be functionally conservative, then $m^t > m$, and since $m^t > 2^t$, it follows that $m > 2^t$. Because m is the cardinal number of $C(X)$, $m^\omega = m$ by 2.2(a), so (m, t) is a bad cardinal pair.

If conversely, (m, t) is a bad cardinal pair, then by ($\uparrow$) and 2.2(a), $m^t = C(X \times D) > m2^t = |C(X)|\,|C(D)|$. So (X, D) is not functionally conservative.

(b) Remark (ii) following 3.1 coupled with 2.2(d) shows that $\big(|C(X)|, |D|\big)$ is a bad cardinal pair. So the conclusion follows from (a).

(c) The space of (i) is compact and the space of (ii) is c.c.c. by 11.3 of [Hod], so both are weakly Lindelöf. In case (i), the weight of X is m because the base for it is the family of singletons of K together with the sets consisting of the point at ∞ together with a cofinite subset of K. In case (ii), because by 4.1.16 of [E], a separable metric space has weight ω, the product space X has weight m by 2.3.13 of [E]. (See the remark following 2.1.) Thus the conclusion in these two cases follows from (b).

Clearly the space X of (iii) satisfies the first axiom of countability and has weight m because it has the set of open intervals as a base. Moreover, X is C^*-embedded in $Y = [0, m]$. To see this, note that for any uncountable $\alpha \in Y$ of uncountable cofinality, each $f \in C^*\big([0, \alpha) \cap X\big)$ has an extension to $C^*([0, \alpha])$ because $cf(\alpha) > \omega$. Inducting over these uncountable ordinals yields the desired result. Thus $\beta X = Y$, so $|C(X)| = |C(Y)| = \big(w(Y)\big)^\omega$ by 2.2(b) since the weight of the totally ordered space Y is m. So the conclusion of (iii) follows from Remark (ii) following 3.1 and (b).

(d) follows immediately from (a) and Remark (i) following Definition 3.1. □

Next, after giving some pertinent definitions, we turn our attention to the existence of bad cardinal pairs. For background, see Chapter 1 of [CN1] and [L]. We continue to identify cardinals with their initial ordinals. We assume the axiom of choice throughout, in which case the set of cardinals less than any given cardinal is well-ordered and every cardinal α has an immediate cardinal successor α^+.

Definitions 3.3. Suppose α is a cardinal number.

(a) The *cofinality* $cf(\alpha)$ of α is the least cardinal number ξ such that $[0, \alpha)$ has a cofinal subset of power ξ.

(b) α is called *singular* if it is the sum of fewer cardinals of smaller power; that is, if $cf(\alpha) < \alpha$. Otherwise it is called *regular*.

(c) For any ordinal ξ and infinite cardinal α, the *beth cardinal* $\beth_\xi(\alpha)$ is defined inductively as follows:

$$\beth_0(\alpha) = \alpha$$

$$\text{if } \xi \text{ is a nonzero limit ordinal, then } \beth_\xi(\alpha) = \sum_{\eta<\xi} \beth_\eta(\alpha), \quad \text{and}$$

$$\text{if } \xi \text{ is an ordinal, then } \beth_{\xi+1}(\alpha) = 2^{\beth_\xi(\alpha)}.$$

If $\alpha = \omega$, we abbreviate $\beth_\xi(\alpha)$ by $\beth_\xi$.

(d) The *continuum hypothesis CH* asserts that $2^\omega = \omega^+$, and the *generalized continuum hypothesis GCH* asserts that $2^\alpha = \alpha^+$ for every infinite cardinal α.

Theorem 3.4.

(a) *If λ is a cardinal of uncountable cofinality and $\lambda < \beth_\lambda$, then $(\beth_\lambda, \lambda)$ is a bad cardinal pair. In particular, $(\beth_{\omega_1}, \omega_1)$ is a bad cardinal pair.*

(b) *If (m, t) is a bad cardinal pair, then $t \geq \omega_1$. If in addition, $2^\alpha = \alpha^+$ whenever $\alpha < \beth_{\omega_1}$ (in particular if GCH holds), then $m \geq \beth_{\omega_1} = \aleph_{\omega_1}$.*

Proof. (a) It is noted in the proof of 1.25 in [CN1] that $\beth_\lambda^\omega = \beth_\lambda$. The map from λ into $\beth_\lambda$ given by $\xi \to \beth_\xi$ carries each cofinal subset of λ into a cofinal subset of $\beth_\lambda$, so using the familiar relation $\alpha < \alpha^{cf(\alpha)}$ (see 1.20 of [CN1]), we have $\beth_\lambda^\lambda \geq \beth_\lambda^{cf(\beth_\lambda)} > \beth_\lambda > \lambda$. Because $\lambda < \beth_\lambda = \sup\{\beth_\xi : \xi < \lambda\}$, there is an η such that $\lambda < \beth_\eta < \beth_\lambda$. So, $2^\lambda \leq \beth_{\eta+1} < \beth_\lambda$, and the proof of (a) is complete.

(b) That $t \geq \omega_1$ is obvious from 2.3(a). If we assume that $2^\alpha = \alpha^+$ whenever $\alpha < \beth_{\omega_1}$, it is clear from its definition that $\beth_{\omega_1} = \aleph_{\omega_1}$. To complete the proof, it is enough to show that

$$(✠) \qquad \aleph_{\omega_1} = \min\{\alpha : \alpha^{\omega_1} > \alpha^\omega \quad \text{and} \quad \alpha \geq 2^{\omega_1}\}.$$

To verify (✠), let $\alpha < \aleph_{\omega_1}$; then either α is a successor cardinal or $cf(\alpha) = \omega$. In the first case, $\alpha = \beta^+ = 2^\beta$ for some $\beta \geq \omega_1$, so $\alpha^{\omega_1} = 2^{\beta\omega_1} = 2^\beta = \alpha$, and hence $\alpha^{\omega_1} = \alpha^\omega = \alpha$. In the second case, when α is singular, $\alpha < \alpha^\omega \leq \alpha^{\omega_1} \leq 2^{\alpha\omega_1} = 2^\alpha = \alpha^+$. Therefore, $\alpha^\omega = \alpha^{\omega_1} = \alpha^+$. □

Remarks 3.5. It is clear that $\beth_{\omega_1}$ is a singular cardinal, but the answer to the question:

(∗) If (m, t) is a bad cardinal pair, must m be singular?

seems to depend on which set-theoretic assumptions are made. We elaborate briefly.

(a) Recall from A.4 of [CN2] that if $\kappa \geq \omega$, then $\kappa^{cf(\kappa)} > \kappa$. Moreover by A.8 of [CN2], GCH implies that $cf(\kappa)$ is the least cardinal λ such that $\kappa^\lambda > \kappa$. So GCH and $m^t > m \geq 2^t > t$ imply $t \geq cf(m)$ and hence that m is singular. So the answer to (∗) is yes if GCH holds.

(b) Abbreviate $\beth_{\omega_1}$ by κ and consider the assumption:

$$(\nabla) \qquad \kappa^{\omega_1} \geq (\kappa^+)^+.$$

It follows from the discussion on p. 181 of [L] that (∇) is consistent with ZFC. Then (∇) implies that (κ^+, ω_1) is a bad cardinal pair. For $(\kappa^+)^{\omega_1} \geq (\kappa^+)^+ > \kappa^+ = \max\{\kappa^+, \kappa^\omega\} = (\kappa^+)^\omega$, and it is clear that $(\kappa^+)^{\omega_1} \geq \kappa^{\omega_1} 2^{\omega_1}$. Because successor cardinals are regular, if (∇) holds, then question $(*)$ has a negative answer.

(c) Since the hypothesis of 3.4(a) requires that $\lambda < \beth_\lambda$, it seems natural to ask if, in ZFC, there are cardinals λ such that $\lambda = \beth_\lambda$. Examining 3.27 and 4.24 in [L] will convince the reader that indeed there are in ZFC cardinals λ such that $\lambda = \beth_\lambda$ and also cardinals μ such that $\mu = \aleph_\mu$; in each case the smallest such cardinal has countable cofinality.

The following proposition will be used below. It follows immediately from 3.2(c) and 3.4.

Proposition 3.6. *If D is a discrete space of cardinality ω_1, then (X, D) is not functionally conservative if either:*

(a) *X is the first countable space obtained from $[0, \beth_{\omega_1})$ by deleting all uncountable limit ordinals, or*

(b) *X is the compact c.c.c.-space $\{0,1\}^{\beth_{\omega_1}}$. (More generally, if X is the product of $\beth_{\omega_1}$ separable metric spaces each of cardinality at least 2.)*

Proposition 3.7. *There are locally compact spaces Z of arbitrarily large cardinality such that $|C(Z \times Z)| > |C(Z)|$.*

Proof. Suppose (m, t) is a bad cardinal pair. By the proof of 3.2(a), there is a compact space X and a discrete space D such that $|C(X \times D)| = m^t > 2^t = |C(X)|\,|C(D)|$, $|X| = m$, $|D| = t$, $|C(X)| = m$, and $|C(D)| = 2^t$. If $Z = X \oplus D$, then Z is locally compact, and by 2.2(g), $|C(Z \times Z)| = m^t > m = |C(Z)|$. By 3.4(a), there are bad cardinal pairs (m, t) with t arbitrarily large. □

Theorem 3.8. *Suppose (X, Y) is not functionally conservative. If Y contains a dense C^*-embedded functionally metrizable subspace Z with $d(Z) = t$, and $|C(X)| = m = m^\omega$, then (m, t) is a bad cardinal pair.*

Proof. By assumption, $|C(Y)| = |C(Z)| = 2^t$, and by 2.2(f), $|C(X \times Y)| \leq |C(X)|^{d(Y)} = m^t$. As (X, Y) is not functionally conservative, it follows that $m^t > \max\{m^\omega, 2^t\}$. Moreover, $m > 2^t$ since if $m \leq 2^t$, then $m^t \leq (2^t)^t = 2^t$. So the relation $|C(X)|\,|C(Y)| < |C(X \times Y)|$ becomes $m2^t = m^\omega 2^t < (m^\omega)^t = m^t$. □

Corollary 3.9. *If (X,Y) is not functionally conservative and Y contains a dense C^*-embedded functionally metrizable subspace Z, then:*

(a) $|C(X)| > |C(Y)|$, *and*
(b) $d(X) > d(Y)$.

Proof. (a) Let $d(Z) = t$. If $|C(X)| \leq |C(Y)|$ then $|C(X)|\,|C(Y)| = 2^t = |C(Y)| < |C(X \times Y)| \leq |C(X)|^{d(Y)} \leq (2^t)^t = 2^t$, a contradiction.

(b) Using 2.2(c), if (b) fails, then $|C(X \times Y)| \leq 2^{d(X \times Y)} = 2^{d(Y)} \leq 2^{d(Z)} = |C(Z)| = |C(Y)| = |C(X)|\,|C(Y)|$, contrary to assumption. □

An immediate consequence of 3.9 follows.

Corollary 3.10. *If Z is a space such that $|C(Z \times Z)| > |C(Z)|$, then Z cannot contain a dense C^*-embedded functionally metrizable subspace.*

Remarks 3.11. (a) Because $(\beth_{\omega_1}, \omega_1)$ is a bad cardinal pair, 3.2 and 3.4 show that 2.3(a) is a best possible result in the sense that a pair of spaces can fail to be functionally conservative if either of the members has density larger than ω. Nor is it enough in 2.3(b), 2.3(c), or 2.3(d) to assume that only one of the members is either pseudocompact, metrizable, or weakly Lindelöf; this follows from 3.2 and 3.4.

(b) In 2.2(f), density may not be replaced by cellularity. To see this, let $X = \{0,1\}^{\beth_{\omega_1}}$, and let D denote a discrete space of cardinality ω_1. As in the proof of 3.4(b), X is compact, satisfies the countable chain condition, and has weight $\beth_{\omega_1}$. Then $|C(X)| = \beth_{\omega_1}$ and we have $|C(X \times D)| = \beth_{\omega_1}^{\omega_1} > \beth_{\omega_1} = |C(X)|^{c(X)} = |C(X)|^{\omega} = |C(X)|\,|C(D)|$.

(c) It cannot be shown in ZFC that if $|C(Y)| = 2^{\omega}$, then (X,Y) is a functionally conservative pair for every compact space X. To see this, assume $2^{\omega_1} = 2^{\omega}$, suppose (m, ω_1) is a bad cardinal pair, let Y denote the discrete space of power ω_1 and let X denote the one-point compactification of a discrete space of power m. Now, $|C(Y)| = 2^{\omega} = 2^{\omega_1}$ by assumption, and by 3.2(b) $|C(X \times Y)| > |C(X)|\,|C(Y)|$.

Another method for creating pairs of spaces that fail to be functionally conservative follows.

Theorem 3.12. *If κ is a singular cardinal of uncountable cofinality such that $2^{<\kappa} = \kappa$ (in particular if κ is a cardinal of the form $\beth_\alpha$ where α is an ordinal of uncountable cofinality), then there is a paracompact space X of cardinality κ and character $cf(\kappa)$ such that $|C(X)| = 2^{<\kappa} < |C(X \times X)| = 2^{\kappa}$.*

Proof. Let $\lambda = cf(\kappa)$ and let $\{\kappa(\alpha) : \alpha < \lambda\}$ denote a strictly increasing sequence of regular cardinals such that $\lambda < \kappa(\alpha) < \kappa$ and $\kappa = \sup\{\kappa(\alpha) : \alpha < \lambda\}$. Define $X = \kappa \cup \{\infty\}$, and define a topology on X by letting each subset of κ be open and by letting V be a neighborhood of ∞ if $\infty \in V$ and there is an $\beta < \lambda$ such that V contains $\{\alpha \in \kappa : \alpha > \kappa(\beta)\}$. It is clear that $|X| = \kappa$ and that the character of X is $cf(\kappa)$. Since X is a Hausdorff space with exactly one nonisolated point, it is paracompact. (Indeed, each open cover of such a space has a disjoint open refinement.) Because κ has uncountable cofinality, each $f \in C(X)$ is constant on some set $\{\alpha \in \kappa : \alpha > \kappa(\beta)\}$, so $|C(X)| = 2^{<\kappa}$. Now, $\big\{\{(\alpha+1, \tau)\} : \alpha < \lambda, \kappa(\alpha) < \tau \leq \kappa(\alpha+1)\big\}$ is a discrete collection of open subsets of $X \times X$ of cardinality κ, so $|C(X \times X)| = 2^{\kappa}$. □

Theorem 3.13. *If X has an uncountable discrete family of nonempty open subsets (that is, if $p(X) \geq \omega_2$), then there is a paracompact space Y such that (X, Y) is not functionally conservative.*

Proof. Let $\mathcal{A} = \{U(\xi + 1) : \xi < \omega_1\}$ denote a discrete family of nonempty open subsets of X of cardinality ω_1. Choose $\kappa(0) > |C(X)|$, let $\kappa = \beth_{\omega_1}\big(\kappa(0)\big)$, and let $Y = \kappa \cup \{\infty\}$. Define a topology on Y by letting each subset of Y be open, and by letting neighborhoods of ∞ consist of ∞ together with subsets of κ that contain $\big\{\xi \in \kappa : \xi > \beth_\eta\big(\kappa(0)\big)\big\}$ for some $\eta < \omega_1$. Arguing as in Theorem 3.12 shows that Y is paracompact. Moreover, $\big\{U(\xi + 1) \times \{\tau\} : \xi < \omega_1, \beth_\xi\big(\kappa(0)\big) < \tau \leq \beth_{\xi+1}\big(\kappa(0)\big)\big\}$ is a discrete collection of nonempty open subsets of $X \times Y$ of cardinality κ. Hence $|C(X \times Y)| \geq 2^{\kappa}$. Clearly $|C(X)| \leq |C(Y)|$, so $|C(X)|\,|C(Y)| = |C(Y)|$. Clearly $\kappa = 2^{<\kappa}$ is uncountable, so we may argue as in Theorem 3.12 that $|C(Y)| \leq \kappa < 2^{\kappa}$. Hence (X, Y) is not functionally conservative. □

Corollary 3.14. *If X is paracompact and (X, Y) is functionally conservative for every paracompact space Y, then X is a Lindelöf space. If, in addition, X is metrizable, then X is separable.*

Proof. By Proposition 2.6, if the paracompact space X fails to be Lindelöf, it has an uncountable closed discrete subspace, and since paracompact spaces are collectionwise normal, it also has an uncountable discrete family of nonempty open subsets. By Theorem 3.13, because (X, Y) is functionally conservative for every paracompact space Y, every discrete family of nonempty open subsets of X is countable. Hence X is a Lindelöf space, and the second assertion is clear. □

Examples 3.15. The requirement in 3.14 that X be paracompact may not be dropped. For, if X is separable, then (X, Y) is functionally conservative for every Tychonoff space Y by 2.3(a). Also, every countably compact Lindelöf space is compact as noted in Chapter 8 of [GJ]. Thus for any point $x \in \beta\omega \setminus \omega$, the space $\beta\omega \setminus \{x\}$ is not Lindelöf, even though $(\beta\omega \setminus \{x\}, Y)$ is functionally conservative for all Y.

Theorem 3.16. *If X is a Tychonoff space that contains a cellular family $\mathcal{A}$, where $\lambda = |\mathcal{A}|$ is an uncountable regular cardinal, and each point of X has a neighborhood meeting fewer than λ members of $\mathcal{A}$, then there is a paracompact space Y such that (X, Y) is not a functionally conservative pair.*

Proof. Let $\{\kappa_\xi : \xi < \lambda\}$ denote a strictly increasing sequence of cardinals such that $\kappa_0 > |C(X)|$ and $\kappa = \sup\{\kappa_\xi : \xi < \lambda\} = 2^{<\kappa}$. (To achieve this, we could take, for example, $\kappa_\xi = 2^{\sup\{\kappa_\eta : \eta < \xi\}}$ whenever $0 \leq \xi \leq \lambda$.) Note that $\kappa = \cup\{\kappa_\xi : \xi < \lambda\}$, and let $Y = \kappa \cup \{\infty\}$. Make Y into a topological space by letting each subset of κ be open, and by letting neighborhoods of ∞ contain ∞ together with the complement in κ of κ_ξ for some $\xi < \lambda$. As in Theorem 3.13, Y is paracompact and $|C(Y)| = 2^{<\kappa}$. It will be shown next that $X \times Y$ contains a discrete cellular family $\mathcal{M}$ of cardinality κ, and hence $|C(X \times Y)| \geq 2^\kappa$. To see this, write $\mathcal{A} = \{U_\xi : \xi < \lambda\}$, and let $\mathcal{M} = \{U_{\xi+1} \times \{t\} : \xi < \lambda, t \in \kappa_{\xi+1} \setminus \kappa_\xi\}$. Clearly $\mathcal{M}$ is a cellular family. To see that this family is discrete, suppose $(x, y) \in X \times Y$. If $y \in \kappa$, then $X \times \{y\}$ is open and meets at most one member of $\mathcal{M}$. If $y = \infty$, choose an open neighborhood W of x that meets fewer than λ members of $\mathcal{A}$. There must be an $\xi_* < \lambda$ such that $W \cap U_\xi = \varnothing$ whenever $\xi \geq \xi_*$. Consider the neighborhood $W \times \big((\kappa \setminus \kappa_\xi) \cup \{\infty\}\big)$ of (x, ∞). If it meets some member $U_{\eta+1} \times \{t\}$ of $\mathcal{M}$, then $W \cap U_{\eta+1} \neq \varnothing$, so $\eta + 1 < \xi_*$. But $t \in \kappa_{\eta+1} \setminus \kappa_\eta \subset \kappa_{\xi_*}$; a contradiction. Hence $\mathcal{M}$ is discrete. Because $\{\kappa_\xi : \xi < \lambda\}$ is strictly increasing, $|\kappa_{\xi+1} \setminus \kappa_\xi| = |\kappa_{\xi_*}|$, so $|\mathcal{M}| = \kappa$. □

Corollary 3.17. *There is a paracompact space Y such that $\big([0, \omega_1), Y\big)$ fails to be functionally conservative. (Here $[0, \omega_1)$ denotes the space of countable ordinals in the interval topology.)*

Each of the examples given above of pairs of spaces that fail to be functionally conservative has at least one factor of "large" cardinality (usually at least $\beth_{\omega_1}$). In the next section, topological techniques are used to create pairs that fail to be functionally conservative where each factor has reasonable small cardinality; often 2^ω.

4. Using the Alexandroff double to create pairs that are not functionally conservative

By the *Alexandroff double (or Alexandroff duplicate)* $ad(X)$ of a topological space X is meant the free union $X \times \{0,1\} = X(0) \oplus X(1)$ topologized as follows:

Let $\mathcal{B}$ denote a base for the topology of X. Let $\{(x,1)\}$ be open in $ad(X)$ for each $x \in X$, and for each $(x,0) \in X(0)$ and open neighborhood $B_x \in \mathcal{B}$ of x, let $ad(x, B_x) = B_x \times \{0,1\} \setminus \{(x,1)\}$ denote a basic neighborhood of $(x,0)$. Then $ad(\mathcal{B}) = \big\{\{(x,1)\} : x \in X\big\} \cup \{ad(x,B_x) : x \in B_x \in \mathcal{B}\}$ is a base for a topology on $ad(X)$. Note that X and $X(0)$ are homeomorphic, $X(1)$ is a dense discrete subspace of $ad(X)$ each point of which is isolated, and that $ad(X)$ is a Hausdorff space if X is.

This construction appeared first in a paper of Alexandroff and Urysohn published (after Urysohn's death) in 1929 in the special case when X is the unit circle. It is used often, but few systematic developments of it have been written. We rely on one given by S. Watson in Section 3 of [Wa] while departing from his notation for technical reasons. See also 3.1.26 in [E].

It is shown in 3.1.2 of [Wa] that:

Lemma 4.1. *If X is a compact Hausdorff space, then so is $ad(X)$.*

Next, with the aid of the Alexandroff double a method is described for creating a large number of examples of pairs of spaces that fail to be functionally conservative. We begin by proving three lemmas stated using the following notation:

> K will denote a Tychonoff space, $ad(K) = K \times \{0,1\} = K(0) \oplus K(1)$ denotes its Alexandroff double, and we abbreviate $K(1)$ by D.

Lemma 4.2. *If M and L are disjoint subspaces of $K(0)$, $X = D \cup M$, and $Y = D \cup L$, then $|C(X \times Y)| = 2^{|K|}$.*

Proof. Let $\Delta = \{(d,d) \in X \times Y : d \in D\}$. Then Δ is an open discrete subspace of $X \times Y$ of cardinality $|D| = |K|$. It will be shown next that Δ is closed in $X \times Y$. For, let $(p,q) \in (X \times Y) \setminus \Delta$. If $p \in D$, then $\{p\} \times (Y \setminus \{p\})$ is open in $X \times Y$ and disjoint from Δ. If $q \in D$, then $(X \setminus \{q\}) \times \{q\}$ is open in $X \times Y$ and disjoint from Δ. If $(p,q) \in M \times L$, then $p \neq q$ since $M \cap L = \emptyset$ and because $ad(K)$ is Hausdorff, there are disjoint open sets U, W in $ad(K)$ such that $p \in U$ and $q \in W$. Then $(p,q) \in (U \times W) \cap (X \times Y) \subset (X \times Y) \setminus \Delta$. Thus Δ is a clopen discrete subset of $X \times Y$ of cardinality $|K|$, and so $|C(X \times Y)| \geq 2^{|K|}$. Since $D \times D$ is dense in $X \times Y$, it follows immediately that $|C(X \times Y)| \leq |C(D \times D)| = 2^{|D \times D|} = 2^{|K|}$. Hence $|C(X \times Y)| = 2^{|K|}$. □

Lemma 4.3. *If $2^{w(K)} = |K|$, then there are disjoint subsets M, L of $K(0)$ that meet each closed subspace F of $K(0)$ such that $|F| > w(K)$.*

PROOF. K has at most $2^{w(K)} = |K|$ closed sets F for which $|F| > w(K)$. Enumerate them as $(F_\alpha)_{\alpha<|K|}$. The sets M and L are constructed inductively.

Choose p_0, q_0 to be distinct points of F_0. Now assume $0 < \alpha < |K|$ and that for all $\beta < \alpha$, $(p_\beta)_{\beta<\alpha}$ and $(q_\beta)_{\beta<\alpha}$ have been chosen such that:

(a) $(p_\beta)_{\beta<\alpha} \cap (q_\beta)_{\beta<\alpha} = \emptyset$, and
(b) $\{p_\beta, q_\beta\} \subset F_\beta$.

Because $|(x_\beta)_{\beta<\alpha} \cup (y_\beta)_{\beta<\alpha}| < |K|$, there are distinct points p_α, q_α in $F_\alpha \setminus [(p_\beta)_{\beta<\alpha} \cup (q_\beta)_{\beta<\alpha}]$. This completes the induction, and $M = (p_\alpha)_{\alpha<|K|}$ and $L = (q_\alpha)_{\alpha<|K|}$ are as required. □

Lemma 4.4. *If K, M and L are chosen as in Lemma 4.3, $X = D \cup M$, and $Y = D \cup L$, then:*

(i) $w(X) = w(Y) = |K|$, *and*
(ii) $|C(X)| = |C(Y)| = 2^{w(K)} = |K|$.

PROOF. It is enough to establish this result for the space X. Because M is a subspace of $K(0)$, $w(M) \leq w(K)$, and because $\big\{\{d\}\big\}$ is a neighborhood base in X for each $d \in D$, it is clear that $w(X) = |D| + w(K) = |K| + w(K) = |K|$. It will be shown next that $\ell(X) \leq w(K)$.

Let $\mathcal{U}$ denote an open cover of X. Since $\ell(M) \leq w(M) \leq w(K)$, there is a subfamily $\mathcal{V}$ of $\mathcal{U}$ of cardinality no larger than $w(K)$ whose union V contains M. Because M meets each closed subset of $K(0)$ whose cardinality exceeds $w(K)$, it follows that $|K(0) \setminus V| \leq w(K)$. Now, each element of $\mathcal{U}$ contains a member of the basis $ad(\mathcal{B})$ for $ad(X)$. So by selecting from $\mathcal{U}$ one member that contains each element of $K(0) \setminus V$ and one member that contains $(x, 1)$ for each $(x, 0) \in K(0) \setminus V$, a subcover of $\mathcal{U}$ of cardinality no larger than $w(K)$ is obtained. Thus $\ell(X) \leq w(K)$.

By 2.2(d), $|C(X)| \leq w(X)^{wl(X)} \leq w(X)^{\ell(X)} \leq |K|^{\ell(X)} \leq (2^{w(X)})^{w(X)} = 2^{w(X)} = |K|$. Clearly $|C(X)| \geq |K|$ as X has $|K|$ isolated points. Hence $|C(X)| = |K|$. □

From these three lemmas we obtain:

Theorem 4.5. *Suppose that K is a Tychonoff space such that:*

(i) $|K| = 2^{w(K)}$, *and*
(ii) *if $F \subset K$ is closed and $|F| > w(K)$ then $|F| = 2^{w(K)}$.*

Then there are Tychonoff spaces X, Y with $|C(X)| = |C(Y)| = 2^{w(K)} = |K|$ and $|C(X \times Y)| = 2^{|K|}$.

(Note that (i) implies (ii) if GCH holds.)

Proof. Apply Lemmas 4.2, 4.3, and 4.4. □

Corollary 4.6. *If GCH holds, then for any infinite cardinal t, there are spaces X, Y for which $|C(X)| = |C(Y)| = 2^t$ and $|C(X \times Y)| = 2^{2^t}$.*

Proof. Letting $K = \{0,1\}^t$, this is immediate from Theorem 4.5 and the fact that $w(K) = t$, while $|K| = 2^t$. □

Next, some more explicit examples of pairs of spaces of "small" cardinality that are not functionally conservative are given. In what follows, 2^ω is abbreviated by $\mathfrak{c}$.

Example 4.7. A pair (X, Y) of first countable Lindelöf spaces such that: $|C(X)| = |C(Y)| = \mathbf{c}$, while $|C(X \times Y)| = 2^{\mathbf{c}}$.

Let $K = [0,1]$ and construct spaces X, Y using 4.5. Then $w(K) = \omega$ and $|K| = 2^\omega$. Clearly $X = D \cup M$ and $Y = D \cup L$ are first countable since $[0,1]$ is first countable. By the proof of 4.4, $\ell(X) \leq w([0,1])$, so X (and Y) are Lindelöf. By 4.5, the various cardinalities are as indicated.

This example should be contrasted with 2.3(d).

Example 4.8. A pair (X, Y) of countably compact spaces such that $|C(X)| = |C(Y)| = 2^{\mathbf{c}}$, while $|C(X \times Y)| = 2^{2^{\mathbf{c}}}$.

Proof. It is well-known that every infinite closed subspace of ω^* has cardinality $2^{\mathfrak{c}} = |\omega^*|$, and that $w(\omega^*) = \mathbf{c}$. (See 9H.2 and 6S3 in [GJ].) So, if we let $K = \omega^*$, Theorem 4.5 supplies us with a pair of Tychonoff spaces satisfying the advertised cardinality conditions. It remains only to prove that these spaces are countably compact. We do this for $X = D \cup M$.

Note first that M is countably compact because by 4.3, every infinite closed subset of $K(0)$ meets M. If $\mathcal{U}_X$ is a countable open cover of X, there is a countable family $\mathcal{U}$ of open subsets of $ad(K)$ whose union contains M such that $\mathcal{U}_X = \{U \cap X : U \in \mathcal{U}\}$. Then $\{U \cap K(0) : U \in \mathcal{U}\}$ is a countable family of open subsets of $X(0)$ containing the countably compact subspace M, so there is a finite set F and a subfamily $\mathcal{U}_F = \{U_i : i \in F\}$ of $\mathcal{U}$ whose union V contains M. So, $K(0) \setminus V$ is a closed subset of $K(0)$ disjoint from M, and hence is a finite set G. For each $(x_j, 0) \in G$, there is $U_j \in \mathcal{U}$ containing it. If $\mathcal{U}_G = \{U_j : j \in G\}$, then by definition of the topology $ad(\mathcal{B})$, $\mathcal{U}_F \cup \mathcal{U}_G$ covers M and all but finitely many points of $D = K(1)$. So, only finitely many more elements of $\mathcal{U}$ are needed to provide for a finite subcover of $\mathcal{U}_X$, and (i) holds. □

Example 4.9. Yet another way of creating pairs of spaces of small cardinality that are not functionally conservative. Let X_{ω_1} denote the one-point Lindelöfication of $D(\omega_1)$; that is, let $X_{\omega_1} = D(\omega_1) \cup \{\infty\}$, where each point of $D(\omega_1)$ is isolated and neighborhoods of ∞ consist of ∞ together with a co-countable subset of $D(\omega_1)$. Because the interval $[0, \omega_1)$ has no countable cofinal subset, $|C(X_{\omega_1})| = 2^\omega = \big|C\big([0, \omega_1)\big)\big|$. Clearly also $\{(\alpha, \alpha + 1) : \alpha < \omega_1\}$ is a clopen discrete subspace of $X_{\omega_1} \times [0, \omega_1)$, so $\big|C\big(X_{\omega_1} \times [0, \omega_1)\big)\big| \geq 2^{\omega_1}$. We conclude that $\big(X_{\omega_1}, [0, \omega_1)\big)$ is not a functionally conservative pair if $2^\omega < 2^{\omega_1}$.

Remark 4.10. By replacing the spaces X, Y by $Z = X \oplus Y$ in 4.5, 4.6, 4.7, and 4.8, we can produce a single space Z with the appropriate properties such that (Z, Z) is not functionally conservative.

Remark 4.11. It is not difficult to see that 4.4 can be reworked to find three pairwise disjoint subsets M_i $(1 \leq i \leq 3)$ of $K(0)$, each meeting each uncountable closed subset of $K(0)$. Then, writing $Y_j = D \cup (\cup\{M_i : 1 \leq i \leq 3, j \neq i\})$ it follows much as in the verification of Example 4.5 that $|C(Y_j \times Y_k)| = 2^\omega$ for $1 \leq j, k \leq 3$, while $|C(Y_1 \times Y_2 \times Y_3)| = 2^{2^\omega}$; further, the free union $Z = \oplus_{1 \leq i \leq 3} Y_i$ satisfies $|C(Z^2)| = 2^\omega < 2^{2^\omega} = |C(Z^3)|$. In the next section we extend the construction of the present section, proving these remarks and a bit more.

5. Spaces Z such that $|C(Z^n)| < |C(Z^{n+1})|$

In this section for X a set and λ a (finite or infinite) cardinal we write $[X]^\lambda = \{A \subseteq X : |A| = \lambda\}$.

As usual, we equate each ordinal with its set of ordinal predecessors. In particular for $n < \omega$ we write $n = \{0, 1, \ldots, n-1\}$.

Given a family $\{X_i : i \in I\}$ of sets and $\emptyset \neq F \subseteq I$, we write $X_F = \Pi_{i \in F} X_i$.

In light of the results of Section 4, it is reasonable to inquire whether for $n > 1$ one can find spaces $Y_0, Y_1, \ldots, Y_{n-1}$ such that every proper subset F of $\{0, 1, \ldots, n-1\}$ satisfies

$$|C(\Pi_{i=0}^{n-1} Y_i)| > |C(Y_F)|.$$

In Theorem 5.6 for fixed $n > 1$ and for $\kappa < \mathfrak{c}$ we find κ-many spaces, every n of which satisfy this inequality. Indeed we arrange that $|C(Y_F)| = c$ for each $F \in [\kappa]^{n-1}$, while $|C(Y_F)| = 2^{\mathfrak{c}}$ for each $F \in [\kappa]^n$.

Definition 5.1. An uncountable, closed subset B of $[0,1]^H$ is *diagonal-like* if for each $i \in H$ and $x \in [0,1]_i$ the set $\pi_i^{-1}(\{x\}) \cap B$ is countable.

Lemma 5.2. *Suppose $0 < n < \omega$ and let A be an uncountable, closed subset of $[0,1]^n$. Either A is diagonal-like in $[0,1]^n$ or there is a nonempty subset F of n and $p \in [0,1]^F$ such that A contains an uncountable closed subset of the form $\{p\} \times B$ with B diagonal-like in $[0,1]^{n \setminus F}$.*

Proof. Among all subsets F of n such that some $p \in [0,1]^F$ has $|\pi_F^{-1}(\{p\}) \cap A| = \mathbf{c}$, fix F of maximal cardinality. Then the set

$$B = \{x \in [0,1]^{n \setminus F} : (p,x) \in A\}$$

is as required. □

Theorem 5.3. *Suppose $\kappa < \mathbf{c}$. Then there is a family $\{M(\eta) : \eta < \kappa\}$ of pairwise disjoint subsets of $[0,1]$ such that for each $k < \omega$ each set $\big(M(\eta)\big)^k$ meets each uncountable, closed diagonal-like subset of $[0,1]^k$.*

Proof. Each space $[0,1]^k$ with $k < \omega$ has $\mathbf{c}$-many uncountable closed subsets, hence only $\mathfrak{c}$-many diagonal-like uncountable closed subsets (each of cardinality $\mathbf{c}$). Let $\{B_\xi : \xi < \mathfrak{c}\}$ be a listing of all sets which, for some $k < \omega$, are diagonal-like, closed and uncountable in $[0,1]^k$.

Let $B_0 \subseteq [0,1]^{k(0)}$. Since $|\pi_i[B_0]| = \mathbf{c}$ for $i < k(0)$ and $\kappa < \mathbf{c}$, there is a κ-sequence $\{x(0,\eta) : \eta < \kappa\}$ in B_0 such that if η and η' are distinct predecessors of κ and if i and j are not necessarily distinct predecessors of $k(0)$ then $x(0,\eta)_i \neq x(0,\eta')_j$.

Now let $\xi < \mathbf{c}$ and suppose that points $x(\zeta,\eta)$ in $B(\zeta) \subseteq [0,1]^{k(\zeta)}$ have been chosen for all $\zeta < \xi$, $\eta < \kappa$, and define $S(\xi) = \{x(\zeta,\eta)_i : \zeta < \xi, \eta < \kappa, i < k(\xi)\}$. Since $|S(\xi)| \leq |\xi|{\cdot}\kappa{\cdot}\omega < \mathbf{c}$ and $B_\xi \subseteq [0,1]^{k(\xi)}$ is diagonal-like, there is a κ-sequence $\{x(\xi,\eta) : \eta < \kappa\}$ in B_ξ such that

(i) $\{x(\xi,\eta)_i : i < k(\xi)\} \cap S(\xi) = \emptyset$, and
(ii) if η and η' are distinct predecessors of κ and if i and j are not necessarily distinct predecessors of $k(\xi)$ then $x(\xi,\eta)_i \neq x(\xi,\eta')_j$.

It is then clear that the sets

$$M(\eta) = \{x(\xi,\eta)_i : \xi < \mathbf{c}, i < \kappa(\xi)\}$$

are as required. Indeed $M(\eta) \cap M(\eta') = \emptyset$ when $\eta \neq \eta'$, and for each set $B(\xi) \subseteq [0,1]^{k(\xi)}$ $(\xi < c)$ and $\eta < \kappa$ we have $x(\xi,\eta) \in B_\xi$ and $x(\xi,\eta)_i \in M(\eta)$, so that $x(\xi,\eta) \in M(\eta)^{k(\xi)} \cap B_\xi$. □

[The restriction to diagonal-like subsets in the above is necessary, in order that disjoint sets $M(\eta)$ as asserted can be found. For example, consider the non-diagonal-like subset $B = \{\frac{1}{2}\} \times [0,1]$ of $[0,1]^2$; clearly every $M \subseteq [0,1]$ such that $M^2 \times B \neq \emptyset$ must satisfy $\frac{1}{2} \in M$.]

Theorem 5.4. *Let M be a subset of $[0,1]$ such that, for $0 < k < \omega$, M^k meets each closed, uncountable, diagonal-like subset of $[0,1]^k$, and let $X = D \cup M$ as in 4.5 (with $D = K(1)$, $K = [0,1]$). Then $|C(X^n)| = \mathbf{c}$ for each integer $n > 0$.*

Proof. We proceed by induction on n. Since by Lemma 5.2 M meets each uncountable, closed subset of $[0,1]$, we have $|C(X)| = \mathbf{c}$ by 4.5.

Suppose the statement true for $n = k$, and for $i < k+1$ write X^{k+1} in the form

$$X^{k+1} = (D \cup M)^{k+1} = \widehat{X_i} \times (D_i \cup M_i) = (\widehat{X_i} \times D_i) \cup (\widehat{X_i} \times M_i)$$

with $\widehat{X_i} = \Pi\{X_j : 0 \leq j < k+1, j \neq i\}$. Since $\widehat{X_i}$ is homeomorphic to X^k and M_i is separable we have

$$|C(\widehat{X_i} \times M_i)| \leq |C(\widehat{X_i})|^\omega = c^\omega = c$$

by the inductive hypothesis and Proposition 2.2(d). It then follows, since $X^{k+1} \setminus D^{k+1} = \cup_{i=0}^{k}(\widehat{X_i} \times M_i)$, that $|C(X^{k+1} \setminus D^{k+1})| = c$. To prove that $|C(X^{k+1})| = c$ it is therefore enough to show that the map from $C(X^{k+1})$ onto $C(X^{k+1} \setminus D^{k+1})$ given by $f \to f \mid (X^{k+1} \setminus D^{k+1})$ is at most $\mathfrak{c}$-to-one, and for that it suffices to show that at most $\mathbf{c}$-many functions $f \in C(X^{k+1})$ satisfy $f \equiv 0$ on $X^{k+1} \setminus D^{k+1}$.

We claim: If $f \in C(X^{k+1})$ with $\mathrm{coz}(f) \subseteq D^{k+1}$, then $|\mathrm{coz}(f)| \leq \omega$. If the claim fails then, replacing if necessary f by $-f$, we have for some integer $m > 0$ that the set $A = \{x \in D^{k+1} : f(x) \geq \frac{1}{m}\}$ satisfies $|A| > \omega$. Identifying $A \subseteq D^{k+1} = K(1)^{k+1}$ with its natural copy (also denoted A) in $[0,1]^{k+1}$, and denoting by $\overline{A}$ the closure of A in $[0,1]^{k+1}$, we have from Lemma 5.2 that either $\overline{A}$ is diagonal-like in $[0,1]^{k+1}$ or there is a non-empty subset F of $k+1 = \{0,1,\cdots,k\}$ and $p \in [0,1]^F$ and an uncountable, closed, diagonal-like subset B of $[0,1]^{(k+1)\setminus F}$ such that $\overline{A} \supseteq \{p\} \times B$. Then, choosing $x \in M^{(k+1)\setminus F} \cap B$ we have $(p,x) \in X^{k+1} \setminus D^{k+1}$ and $(p,x) \in \overline{A}$, so that $f(p,x) \geq \frac{1}{m}$, contrary to the condition $\mathrm{coz}(f) \subseteq D^{k+1}$. The claim is established.

Since D^{k+1} has just $\mathbf{c}$-many countable subsets, and each of these can be $\mathrm{coz}(f)$ for at most $|\mathbb{R}|^\omega$-many (that is, $\mathbf{c}$-many) continuous functions f, we have $|C(X^{k+1})| \leq |C(X^{k+1} \setminus D^{k+1})| \cdot \mathbf{c} = \mathbf{c}^2 = \mathbf{c}$, as desired. □

It follows from Theorems 5.3 and 5.4 that with $X_\eta = D \cup M(\eta)$ $(\eta < \kappa)$ as in 5.4, each integer $n > 0$ satisfies $\left|C\big((X_\eta)^n\big)\right| = \mathbf{c}$. Now for fixed $n < \omega$ we want to use the spaces X_η $(\eta < \kappa)$ to build new spaces $Y_\xi (\xi < \kappa)$ with the properties suggested in Remark 4.10. The following combinatorial lemma will be useful.

Lemma 5.5. *Suppose $\kappa \geq \omega$ and $0 < n < \omega$. Then there is a faithfully indexed family $\{S_\xi : \xi < \kappa\}$ of subsets of κ such that*

(i) $F \in [\kappa]^n \Rightarrow \cap_{\xi \in F} S_\xi \neq \emptyset$, *and*
(ii) $F \in [\kappa]^{n+1} \Rightarrow \cap_{\xi \in F} S_\xi = \emptyset$.

Proof. We proceed by induction. For $n = 1$, choose any family of κ-many pairwise disjoint, nonempty subsets of κ.

Suppose the lemma holds for $n = k < \omega$. Write κ in the form $\kappa = X_0 \cup X_1$ with $|X_i| = \kappa$ and $X_0 \cap X_1 = \emptyset$, and (using the inductive hypothesis) choose a faithfully indexed family $\{T_\xi : \xi < \kappa\}$ of subsets of X_0 such that

$$F \in [X_0]^k \Rightarrow \cap_{\xi \in F} T_\xi \neq \emptyset,$$

and

$$F \in [X_0]^{k+1} \Rightarrow \cap_{\xi \in F} T_\xi = \emptyset.$$

Let $f\colon [X_0]^{k+1} \to X_1$ be a one-to-one function, and for $\xi \in X_0$ define

$$S_\xi = T_\xi \cup \{f[F] : F \in [X_0]^{k+1}, \xi \in F\}.$$

The family $\{S_\xi : \xi \in X_0\}$ is faithfully indexed, since $S_\xi \cap X_0 = T_\xi$.

For $F \in [X_0]^{k+1}$ we have $f[F] \in \cap_{\xi \in F} S_\xi$, so (i) holds for $n = k + 1$. To prove (ii) for $n = k + 1$ it is enough to note that since the function f is one-to-one, each point $p \in X_1 \cap (\cup_{\xi<\kappa} S_\xi)$ satisfies $p \in S_\xi$ for exactly $(k + 1)$-many $\xi < \kappa$, namely for $\xi \in F \in [X_0]^{k+1}$ with $p = f[F]$; thus for $F \in [X_0]^{k+2}$ we have $\cap_{\xi \in F} S_\xi = \cap_{\xi \in F} T_\xi = \emptyset$. □

Theorem 5.6. *Suppose $\omega \leq \kappa < \mathbf{c}$ and $0 < n < \omega$. There is a set $\{Y_\xi : \xi < \kappa\}$ of spaces such that*

(i) $F \in [\kappa]^n \Rightarrow |C(Y_F)| = \mathbf{c}$, *and*
(ii) $F \in [\kappa]^{n+1} \Rightarrow |C(Y_F)| = 2^{\mathbf{c}}$.

Proof. Using $\{M(\eta) : \eta < \kappa\}$ as in Theorem 5.3, define $X(\eta) = D \cup M(\eta)$ $(\eta < \kappa)$. Then with $\{S_\xi : \xi < \kappa\}$ defined as in Lemma 5.5, set

$$Y_\xi = \cup\{X(\eta) : \eta \in S_\xi\}.$$

We verify (i) and (ii).

(i) Given $F \in [\kappa]^n$ there is $\eta \in \cap_{\xi \in F} S_\xi$. For this η we have $X(\eta) \subseteq \cap_{\xi \in F} Y_\xi$, so $\big(X(\eta)\big)^F$ is a (dense) subspace of $\Pi_{\xi \in F} Y_\xi = Y_F$. From Theorem 6.4 we have $\big|C\big(X(\eta)^F\big)\big| = \mathbf{c}$, from which (i) follows.

(ii) Let $f \in [\kappa]^{n+1}$, let $\overline{\Delta}$ be the diagonal in $(D \cup [0,1])^F$, and let $\Delta = \overline{\Delta} \cap Y_F$. Since $\overline{\Delta}$ is closed in $(D \cup [0,1])^F$ the set Δ is closed in Y_F. Thus Δ is a closed, discrete subset of Y_F of cardinality $\mathbf{c}$, and $|C(Y_F)| \geq |C(\Delta)| = 2^{\mathfrak{c}}$ follows. □

Remark 5.7. A more delicate selection of the points $x(\xi, \eta) \in B_\xi$ in Theorem 5.3 allows the construction of $\mathbf{c}$-many sets $\{M(\eta) : \eta < \mathbf{c}\}$ rather than of (only) κ-many for fixed $\kappa < \mathbf{c}$. This yields a family $\{Y_\xi : \xi < \mathbf{c}\}$ with properties (i) and (ii) of Theorem 5.6. We leave the details to the interested reader.

Theorem 5.8. *Let $0 < n < \omega$. There are spaces Z such that $|C(Z^k)| = \mathbf{c}$ for $1 \leq k \leq n$, and $|C(Z^k)| = 2^{\mathbf{c}}$ for $n < k < \omega$.*

Proof. Choose $\{Y_\xi : \xi < \kappa\}$ as in Theorem 5.6, fix arbitrary $F \in [\kappa]^{n+1}$, and let Z be the free union of the spaces Y_ξ for $\xi \in F$. It is enough to show that $|C(Z^n)| = \mathbf{c}$ and $|C(Z^{n+1})| = 2^{\mathbf{c}}$. The space Z^n is the free union of $(n+1)^n$-many spaces of the form $Y_{\xi_0} \times Y_{\xi_1} \times \cdots \times Y_{\xi_{n-1}}$ (repetitions allowed). With $\eta \in \cap_{i<n} S_{\xi_i}$ this space contains densely the space $\big(X(\eta)\big)^n$. Since $\big|C\big(X(\eta)\big)^n\big| = \mathbf{c}$ by Theorem 5.4 we have $|C(Z^n)| \leq \mathbf{c} \cdot (n+1)^n = \mathbf{c}$. Since Z^{n+1} contains an open-and-closed copy of the space $Y_F = \Pi_{\xi \in F} Y_\xi$, we have from Theorem 5.6(ii) that $|C(Z^{n+1})| \geq |C(Y_F)| = 2^{\mathbf{c}}$. □

6. Products involving spaces of ordinals

For any cardinal $\alpha > 0$, $D(\alpha)$ will denote the interval $[0, \alpha)$ with the discrete topology, while $[0, \alpha)$ will denote this set with the interval topology. Let $\log(\alpha)$ denote $\min\{\beta : 2^\beta \geq \alpha\}$. (Thus, $\log(\mathfrak{c}) = \omega = \log(\omega_1)$.) Finally, for any space, ${}^\beta\alpha$ will abbreviate the space $D(\alpha)^\beta$.

Theorem 6.1. *For any cardinal α and Tychonoff space X, $|C({}^\omega\alpha \times X)| \geq |C(X)|^\alpha$.*

Proof. Because ${}^\omega\alpha = D(\alpha)^\omega = D(\alpha) \times D(\alpha)^\omega$, it follows that $D(\alpha)^\omega$ is the free union of α copies of itself; say $D(\alpha)^\omega = \bigcup\{D^\omega_\xi : \xi < \alpha\}$, where each $D_\xi = D(\alpha)$. For each $\xi < \alpha$, choose $d_\xi \in D^\omega_\xi$. Then $T = \{d_\xi : \xi < \alpha\} \times X$ is C-embedded in $D(\alpha)^\omega \times X$. For, if $f \in C(T)$, the function F such that $F(d, x) = f(d_\xi, x)$ whenever $d \in D_\xi$ and $x \in X$ is a continuous extension of f over $D(\alpha)^\omega \times X$. Hence $|C({}^\omega\alpha \times X)| \geq |C(T)| = |C(X)|^\alpha$. □

The following lemma may be found in 7.5 of [vDZ].

Lemma 6.2. *If α is an infinite cardinal, then:*

(i) $w\big([0,\alpha)\big) = \alpha$,
(ii) $wl\big([0,\alpha)\big) = \ell\big([0,\alpha)\big) = \omega + cf(\alpha)$,
(iii) $\big|C\big([0,\alpha)\big)\big| = \alpha^\omega$.

Use will also be made of the Hewitt-Pondiczery-Marzewski theorem; namely:

Theorem 6.3 ([E, 2.3.15])**.** *If α is an infinite cardinal, and $(X_i)_{i\in I}$ is a family of no more than 2^α spaces with $d(X_i) \le \alpha$, then $d(\Pi_{i\in I} X_i) \le \alpha$.*

See the remark following 2.1.

The next example supplies an alternate way of using a bad cardinal pair to produce a pair of spaces that fails to be functionally conservative.

Example 6.4. If (m,t) is a bad cardinal pair, then $\big({}^\omega t,\ [0,m)\big)$ fails to be a functionally conservative pair.

Proof. By Theorems 6.1 and 6.2, $\big|C\big({}^\omega t \times [0,m)\big)\big| \ge \big|C\big([0,m)\big)\big|^t = m^t$. By 6.3 and 2.2, $d({}^\omega t) = t$. So $|C({}^\omega t)| \le 2^t$. By 2.2 and 6.3, $\big|C\big([0,m)\big)\big| = m$. Because $m^t > m \ge 2^t$, the conclusion follows. □

We also use 6.3 to show:

Lemma 6.5. *If $\omega \le \gamma \le 2^\alpha$, then ${}^\gamma\alpha$ is functionally metrizable.*

Proof. By 2.2 and 6.3, $|C({}^\gamma\alpha)| \le 2^{d({}^\gamma\alpha)} = 2^\alpha$. As in the proof of 6.1, ${}^\gamma\alpha$ has a C-embedded discrete subspace D of cardinality α, so $|C({}^\gamma\alpha)| \ge 2^\alpha$ and the conclusion follows. □

Theorem 6.6. *Suppose Y is a Tychonoff space. If $\omega \le \gamma \le 2^\alpha$ and either $\log|C(Y)| \le \alpha$ or $d(Y) \le \alpha$, then $({}^\gamma\alpha, Y)$ is functionally conservative.*

Proof. Under either assumption on Y, 2.2 implies that $|C(Y)| \le 2^\alpha = |C({}^\gamma\alpha)|$. So this pair is functionally conservative by 2.5. □

Stronger results may be obtained if GCH is assumed. First, we prove:

Theorem 6.7. *For all $\alpha \ge \omega$ the following assertions are equivalent:*

(a) *If $\omega \le \lambda < cf(\alpha)$, then $\alpha^\lambda = \alpha^\omega 2^\lambda$.*
(b) *If Y is a Tychonoff space such that $\omega \le d(Y) < cf(\alpha)$, then*

$$\alpha^\omega |C(Y)| \le \big|C\big([0,\alpha)\times Y\big)\big| \le \alpha^\omega 2^{d(Y)}.$$

Proof. Suppose (a) holds and Y satisfies $\omega \le d(Y) = \lambda < cf(\alpha)$. Then by 6.2 and the hypothesis,

$$\begin{aligned}\alpha^\omega |C(Y)| &= \big|C\big([0,\alpha)\big)\big|\,|C(Y)| \le \big|C\big([0,\alpha)\times Y\big)\big| \le \big|C\big([0,\alpha)\times D(\lambda)\big|\\ &= (\alpha^\omega)^\lambda = \alpha^\lambda = \alpha^\omega 2^\lambda = \alpha^\omega 2^{d(Y)}.\end{aligned}$$

So (b) holds.

Conversely, if (b) holds, suppose $\omega \le \lambda < cf(\alpha)$, and let $Y = D(\lambda)$. Then (b) implies $\alpha^\omega 2^\lambda \le \big|C\big([0,\alpha)\big)\times D(\lambda)\big| \le \alpha^\omega 2^\lambda$. Because $\big|C\big([0,\alpha)\big)\times D(\lambda)\big| = (\alpha^\omega)^\lambda = \alpha^\lambda$, (a) holds. □

Corollary 6.8. *Suppose GCH holds. If Y is a Tychonoff space for which $\omega \le d(Y) < cf(\alpha)$, then $\big([0,\alpha), Y\big)$ is functionally conservative.*

Proof. GCH implies $\alpha^\omega = \alpha$ and $2^\lambda \le \alpha = \alpha^\lambda$ whenever $\omega \le \lambda < cf(\alpha)$, so the conclusion follows from 6.7. □

We close this section by exhibiting some additional kinds of functionally conservative pairs. First, we recall a definition and a result from [CN2]. Let Γ denote an infinite set, and suppose that $X_\Gamma = \prod_{\gamma\in\Gamma} X_\gamma$ is a topological product of Tychonoff spaces indexed by Γ. If $\aleph$ is a cardinal and $\aleph \le |\Gamma|$, then an element $f \in C(X_\Gamma)$ is said to be *determined by* $<\aleph$ *coordinates* if there is a subset $\Gamma(f)$ of Γ of cardinality no greater than $\aleph$ such that $f = f_0 \circ \pi_{\Gamma,\Gamma(f)}$ for some $f_0 \in C(X_{\Gamma(f)})$, where $\pi_{\Gamma,\Gamma(f)}$ denotes the projection of X_Γ onto $X_{\Gamma(f)}$. It is shown in Lemma 10.2 of [CN2], citing earlier results, that for any infinite set Γ and regular cardinal $\aleph$, every $f \in C(X_\Gamma)$ is determined by $<\aleph$ coordinates if X_Γ is pseudo-$\aleph$-compact.

Lemma 6.9. *If X is compact and Y is pseudo-$\aleph$-compact, then $X\times Y$ is pseudo-$\aleph$-compact.*

Proof. Suppose there were a faithfully indexed locally finite family $\{S_\xi : \xi < \aleph\}$ of nonempty open subsets of $X \times Y$. Because X is compact, the projection map $p_Y : X\times Y \to Y$ is perfect as well as open by 3.7.1 in [E]. (Recall that a continuous surjection is *perfect* if it is closed and inverse images of points are compact.) By 3.10.11 in [E], $\{p_Y(S_\xi) : \xi < \aleph\}$ is locally finite as well as being open, and it follows from 2.7(ii) that Y fails to be pseudo-$\aleph$-compact. □

This enables us to prove:

Theorem 6.10. *If $\{X_\alpha : \alpha < \omega_2\}$ is a family of separable metric spaces each having at least two points, and $\omega_1 \le p(Y) \le \omega_2$, then $(\prod\{X_\alpha : \alpha < \omega_2\}, Y)$ is a functionally conservative pair.*

Proof. Suppose first that $p(Y) = \omega_1$. Let $X = \prod\{X_\alpha : \alpha < \omega_2\}$. As noted in the proof of 3.2(c)(ii), $w(X) = \omega_2$, so $|C(X)| = \omega_2^\omega$. If $f \in C(X \times Y)$, then by 2.3(a) and 6.9 and [10.2] of [CN2], f depends on only countably many coordinates; that is, there is a countable subset $\Gamma(f)$ of ω_2 such that f coincides with any function that agrees with it on $X_{\Gamma(f)\times Y}$. It follows from 6.3 and the above that

$$\begin{aligned}|C(X \times Y)| &\le \sum\{|C(X_A \times Y)| : |A| = \omega\} = \sum\{|C(X_A)|\,|C(Y)| : |A| = \omega\}\\ &\le \omega_2 \mathbf{c}|C(Y)| \le \omega_2^\omega |C(Y)| = |C(X)|\,|C(Y)|.\end{aligned}$$

This completes the proof in case $p(X) = \omega_1$. If $p(X) = \omega_2$, then 6.9 implies that each $f \in C(X \times Y)$ depends on $< \omega_2$ coordinates, and the rest of the proof is similar. □

Remark 6.11. It seems natural to ask if the conclusion of 6.10 holds when $p(X) > \omega_2$; say if $p(X) = \omega_{n+1}$ for some n such that $1 < n < \omega$. In a model for set theory in which $\omega_n \le \mathbf{c}$, in which case 2.3(a) and 6.9 imply that (X_A, Y) is functionally conservative, then the proof of 6.10 will yield an affirmative answer in this case. So, a negative answer cannot be established in ZFC.

7. Spaces X such that (X, Y) is functionally conservative for every Y

Definition 7.1. If (X, Y) is functionally conservative for every Y in a class $\mathcal{P}$ of Tychonoff spaces, then X is said to be *functionally conservative for* $\mathcal{P}$. If (X, Y) is functionally conservative for every Tychonoff space Y, then X is said to be *functionally conservative.*

The purpose of this final section is to describe what we know about spaces functionally conservative for various classes of space. Many of our earlier results may be rephrased in terms of this new terminology; e.g., 2.3, 2.5, 3.13, 3.14, and 3.15. This task is largely left to the reader, but the following immediate consequences of 2.3(a), 3.13, and 3.15 are worth restating explicitly.

Theorem 7.2.

(a) *Every separable space is functionally conservative.*

(b) *If a space is functionally conservative for paracompact spaces, then each of its discrete families of nonempty open sets is countable.*

(c) *If a paracompact space is functionally conservative for paracompact spaces, then it is a Lindelöf space.*

(However, not all Lindelöf spaces are functionally conservative for Lindelöf spaces; see 4.7.)

Let $\mathcal{Z}(X)$ denote the family of zerosets of X, let $\mathcal{CZ}(X)$ denote its family of cozerosets, let $\mathcal{RCZ}(X)$ denote the family of closures of its cozerosets, and let $\mathcal{Z}^{\#}(X)$ the family of closures of interiors of its zerosets. Because the map $Z \to X \setminus Z$ is a bijection of $\mathcal{Z}(X)$ onto $\mathcal{CZ}(X)$, these families have the same cardinality. Similarly, $|\mathcal{Z}^{\#}(X)| = |\mathcal{RCZ}(X)|$. The main results of [vDZ] are that if X is an infinite Tychonoff space, then $|C(X)| = |\mathcal{CZ}(X)| = |\mathcal{RCZ}(X)|$. (These authors use the notation $\mathcal{R}(X)$ for our $\mathcal{RCZ}(X)$.) Recall from Section 10 of [We] that a subspace S of X is said to be *z-embedded in* X if the map $Z \to Z \cap S$ is a surjection of $\mathcal{Z}(X)$ onto $\mathcal{Z}(S)$, and from [HVW] that a dense subspace S is *$z^{\#}$-embedded in* X if this map is a surjection of $\mathcal{Z}^{\#}(X)$ onto $\mathcal{Z}^{\#}(S)$. If S is C^*-embedded in X, then it is z-embedded in X, and the latter implies that it is $z^{\#}$-embedded in X if S is dense in X. (Neither of these implications can be reversed.) It is shown in [We] that every cozeroset, every C^*-embedded subspace, and every Lindelöf subspace of a Tychonoff space is z-embedded, and in [HVW] that a dense weakly Lindelöf subspace of a Tychonoff space S is $z^{\#}$-embedded in S. The next theorem will be used below to construct new functionally conservative spaces from old ones. We will make use of the following lemma pointed out to us by A. Hager.

Lemma 7.3 (Hager)**.** *If S is (dense and) $z^{\#}$-embedded in X, then* $|C(S)| = |C(X)|$.

Proof. Because for any $f \in C(X)$, the map $cl\big(intZ(f)\big) \to cl\big(\mathrm{coz}(f)\big)$ is a bijection of $\mathcal{Z}^{\#}(X)$ onto $\mathcal{RCZ}(X)$, by the result in [vDZ] cited above $|C(S)| = |\mathcal{RCZ}(S)| = |\mathcal{Z}^{\#}(S)| \leq |\mathcal{Z}^{\#}(X)| = |\mathcal{RCZ}(X)| = |C(X)|$. So $|C(S)| \leq |C(X)|$, and the reverse inequality holds because S is dense in X. □

Theorem 7.4. *If X contains a dense subspace $D = \bigcup_{n<\omega} F_n$, where each F_n is functionally conservative and z-embedded in X, then X is functionally conservative.*

Proof. Given Y and $f \in C(X \times Y)$ we associate the sequence $\{f_n\}$, where $f_n = f \mid (F_n \times Y) \in C(F_n \times Y)$ for each $n < \omega$. Because D is dense, it is routine to verify that the map $f \to \{f_n\}$ from $C(X \times Y)$ into $\prod_n C(F_n \times Y)$ is an injection, and since each F_n is z-embedded, $|C(F_n)| \le |C(X)|$. Because each F_n is functionally conservative, it follows that

$$|C(X \times Y)| \le \Big|\prod_n C(F_n \times Y)\Big| = \prod_n |C(F_n \times Y)| = \prod_n |C(F_n)|\,|C(Y)|$$
$$\le \prod_n |C(X)|\,|C(Y)| = |C(X)|^\omega |C(Y)|^\omega = |C(X)|\,|C(Y)|.$$

□

Corollary 7.5.

(a) *If X is functionally conservative, and $X \subset T \subset \beta X$, then T is functionally conservative.*

(b) *A space containing a dense functionally conservative weakly Lindelöf subspace or a dense functionally conservative cozeroset is functionally conservative.*

(c) *A countable product of nonempty functionally conservative spaces is functionally conservative. That is, the property of being functionally conservative is countably productive.*

Proof. Parts (a) and (b) follow immediately from the theorem and the remarks preceding 7.3.

(c) Let $\{X_n\}_{n<\omega}$ denote a sequence of functionally conservative spaces and let X denote their product. Pick $(x_0, x_1, \dots, x_n, \dots)$ in X, and let $F_n = X_0 \times \cdots \times X_n \times \{x_{n+1}\} \times \{x_{n+2}\} \times \cdots$ if $n < \omega$. A routine induction shows that any finite product of functionally conservative spaces is functionally conservative, so each F_n is functionally conservative. Moreover $\{F_n\}$ is an increasing sequence of C^*-embedded subspaces whose union is dense in X. Thus the conclusion follows from the theorem. □

We turn next to try to determine when various kinds of subspaces of functionally conservative spaces are functionally conservative. The following technical lemma will prove useful for what follows.

Lemma 7.6. *If (X, Y) is a functionally conservative pair and K is a subspace of X such that $|C(X)| \le |C(K)|$ (e.g., if K is dense in X), and $K \times Y$ is z-embedded in $X \times Y$, then (K, Y) is a functionally conservative pair.*

Proof. By the remarks preceding 7.3 and the hypothesis, $|C(K \times Y)| \le |C(X \times Y)| = |C(X)|\,|C(Y)| = |C(K)|\,|C(Y)|$. □

Theorem 7.7. *If X is a (Tychonoff) space, then the following are equivalent:*

(a) *X is functionally conservative.*
(b) *One of its dense cozerosets is functionally conservative.*
(c) *Each of its dense cozerosets is functionally conservative.*

Proof. If a dense cozeroset K of a space X is functionally conservative, then so is X by 7.5(b). So (b) implies (a).

If $K = \mathrm{coz}(f)$ is a cozeroset of a functionally conservative space X, then for any space Y, $K \times Y$ is the cozeroset of the function $F \in C(X \times Y)$ such that $F(x, y) = f(x)$ for all $(x, y) \in X \times Y$, and is dense in $X \times Y$ if K is dense in X. So, K is functionally conservative by 7.6. Hence (a) implies (c), and obviously, (c) implies (b). □

The next corollary follows immediately from the theorem and the fact that every locally compact and σ-compact space is a dense cozeroset in each of its compactifications.

Corollary 7.8. *If X is locally compact and σ-compact, then the following are equivalent:*

(a) *X is functionally conservative.*
(b) *Some compactification of X is functionally conservative.*
(c) *Every compactification of X is functionally conservative.*

Next, we show that every compact space of weight no larger than 2^ω is functionally conservative. This will enable us to show that the converse of 7.5(a) need not hold (that is, to exhibit a space X that is not functionally conservative such that βX is functionally conservative) and that the hypothesis that X is locally compact and σ-compact in Theorem 7.8 may not be omitted. To do so, use will be made of the following theorem of M. Starbird given in Theorem 3 of [S1] in different wording. See also, [S2] and [Hos].

Theorem 7.9. *If X and Y are Tychonoff spaces, and K is a compact subspace of X, then $K \times Y$ is C^*-embedded in $X \times Y$.*

Theorem 7.10. *If K is a σ-compact space of weight no larger than 2^ω, or equivalently if $|C(K)| = 2^\omega$, then K is functionally conservative.*

Proof. The equivalence of the two properties in the hypothesis follows from 2.2(b), (c), (d).

Assume first that K is compact. By 3.2.5 in [E], the compact space K of weight 2^ω embeds in $[0, 1]^{2^\omega}$, and hence is C^*-embedded in $[0, 1]^{2^\omega}$. This latter space is separable by 6.3, and hence is functionally conservative by 2.3. So, by 7.9, for any space $Y, K \times Y$ is C^*-embedded in $[0,1]^{2^\omega} \times Y$, and the conclusion follows from 7.6 in case K is compact. The general case then follows from 7.4. □

In the special case when K is the one-point compactification of a discrete space of cardinality no larger than 2^ω, this result was communicated orally by S. Watson to M. Henriksen using a different argument.

If X is a topological space, λ is an infinite cardinal, and $p \in X^\lambda$, we call the subspace of points in X^λ that differ from p in at most finitely (resp. at most countably) many coordinates the σ- (resp. Σ-) *product of X^λ based at p.*

Corollary 7.11. *Let X be a compact space such that $|C(X)| = 2^\omega$, let $\omega \leq \lambda \leq 2^\omega$, let $S = S(p)$ be the σ-product in X^λ based at $p \in X^\lambda$, and let $S \subseteq Z \subseteq X^\lambda$. Then Z is functionally conservative.*

Proof. For each $n < \omega$, the set σ_n of elements in the σ-product that differ from p at no more than n coordinates is compact, and the σ-product is the union of the σ_n for $n < \omega$. So the σ-product is functionally conservative by 7.10. The σ-product is a dense Lindelöf subspace of X^λ and hence z-embedded in X^λ, so the conclusion follows from 7.4. □

We note that it follows in particular, given X, λ and p as in 7.11, that the Σ-product in X^λ based at p is functionally conservative.

Examples and Comments 7.12. (1) There are functionally conservative spaces that are neither separable nor compact. Such a space may be obtained by taking the topological sum of a space satisfying the hypotheses of Theorem 7.9 that fails to be separable (e.g., the one point compactification of a discrete space of power $\leq 2^\omega$ or $\beta\omega \setminus \omega$) and a separable space that fails to be compact. This also supplies us with nonseparable compact functionally conservative spaces of cardinality 2^c.

(2) By 3.17, $[0, \omega_1)$ fails to be functionally conservative for paracompact spaces, while $[0, \omega_1] = \beta\big([0, \omega_1)\big)$ is functionally conservative by 7.9. Thus, the converse of 7.5(a) need not hold, and the hypothesis that X is σ-compact in Theorem 7.8 may not be omitted. Nor is it true that X Lindelöf and $|C(X)| = 2^\omega$ imply X is functionally conservative as is witnessed by Examples 4.7 and 4.9.

(3) In Remark 3.11(c) above, it is shown that if $2^{\omega_1} = 2^\omega$, then the discrete space $D(\omega_1)$ of power ω_1 is not functionally conservative for compact spaces. Under this same set-theoretic assumption (consistent with Martin's axiom and

not CH), it follows that $\beta\big(D(\omega_1)\big)$ is functionally conservative. We do not know if this result holds in all models of ZFC.

Definition 7.13. If there is a continuous surjection of a Tychonoff space T onto a dense subspace of a Tychonoff space X such that $|C(X)| = |C(T)|$, then X is called a *functionally tight image of* T.

Proposition 7.14. *A Tychonoff functionally tight image X of a functionally conservative space T is functionally conservative.*

Proof. If $\varphi\colon T \to X$ is a continuous surjection, then for any space Y, the map $\Phi\colon T \times Y \to X \times Y$ given by $\Phi(x, y) = \big(\varphi(x), y\big)$ is also a continuous surjection. So, $|C(X \times Y)| \leq |C(T \times Y)| = |C(T)|\,|C(Y)| = |C(X)|\,|C(Y)|$ because X is a functionally tight image of T and T is functionally conservative. □

Two kinds of topological conditions that guarantee that a space has a functionally tight image are considered next. To describe them, we borrow from [DHH], [HVW], and [V]. A Tychonoff space X is called a *quasi-F space* if each of its dense cozerosets is C^*-embedded, and is called *basically disconnected* each of its cozerosets has an open closure. It is well known that every basically disconnected space is a quasi-F-space, but not conversely, and that open subspaces of basically disconnected spaces are basically disconnected. (See [Ko].) It is noted in 1.1 of [HVW] and in [V] that if X is Tychonoff, there is a unique quasi-F space $QF(X)$ (called the *quasi-F-cover of* X) (resp. a unique minimal basically disconnected cover $BD(X)$ called the basically disconnected cover) and an irreducible perfect continuous surjection of $QF(X)$ (resp. $BD(X)$) onto X that is minimal among all quasi-F spaces (resp. basically disconnected spaces) that map irreducibly, perfectly, and continuously onto X. It is shown in Sections 1-3 of [DHH] that if X is compact, then so is $QF(X)$, and that $C\big(QF(X)\big)$ consists of certain kinds of equivalence classes of sequences of elements of $C(X)$. So $\big|C\big(QF(X)\big)\big| = |C(X)|^\omega = |C(X)|$, and it is noted in [V] that if X is compact, then $C\big(BD(X)\big)$ is the σ-completion of $C(X)$, whence $\big|C\big(QF(X)\big)\big| = |C(X)|^\omega = |C(X)|$. Hence by Theorem 7.13, we have:

Proposition 7.15. *If X is compact and its quasi-F cover or its basically disconnected cover is functionally conservative then X is functionally conservative.*

By 3.6(b), $\{0,1\}^{\beth_{\omega_1}}$ is not functionally conservative. So functional conservatism is not a productive property. (We leave it to the reader to ponder the political implications of this result.) Despite this, functional conservatism shares something in common with productive properties. By 7.5(c) and 3.13, finite products of functionally conservative spaces are pseudo-ω_1-compact. Hence we have:

Theorem 7.16. *Every continuous real-valued function on a product of functionally conservative Tychonoff spaces is determined by countably many coordinates. Hence every product of functionally conservative spaces is pseudo-ω_1-compact.*

We conclude with some open questions about functionally conservative spaces.

Question 7A. Is there a functionally conservative space X such that $|C(X)| > 2^\omega$?

Question 7B. By Theorem 7.10, $[0, \omega_1]$ is functionally conservative. If (m, t) is a bad cardinal pair, in particular if $m = \beth_{\omega_1}$, then $[0, m]$ is not functionally conservative by 3.3 and 3.4. What is the least cardinal α such that $[0, \alpha]$ is not functionally conservative? Indeed, what can be said about the set α of cardinals such that $[0, \alpha]$ is not functionally conservative?

Question 7C. Just before 7.16, we noted that $\{0, 1\}^{\beth_{\omega_1}}$ is not functionally conservative, thereby showing that the class of functionally conservative spaces is not productive. By 7.5(c), the product of countably many functionally conservative spaces is functionally conservative. Thus with λ denoting the least cardinal for which some product of λ-many functionally conservative spaces fails to be functionally conservative, we have $\omega < \lambda \leq \beth_{\omega_1}$. What is the value of λ? Is $\lambda = \mathbf{c}$? Is there a functionally conservative space X such that X^λ is not functionally conservative?

Question 7D. Suppose X is compact and functionally conservative. Must $QF(X)$ or $BD(X)$ be functionally conservative?

Question 7E. Is the functional conservatism of $\beta\big(D(\omega_1)\big)$ independent of ZFC?

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