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In 1940, Helmer showed [1, Theorem 9] that in the ring $R$ of entire functions, every finitely generated ideal is principal. That is, if $f, g$ are entire functions without zeros in common, there exist $s, t$ in $R$ such that

$$sf + tg = 1. \quad (1)$$

He asked if this theorem is true in the ring $R^*$ of all entire functions of finite order. A negative answer to this question existed already in 1936 in a paper of Whittaker [7, p. 256]. In particular if the zeros $(a_n)$, $(b_n)$ of $f, g$ respectively are not sufficiently separated as $n \to \infty$, the equation (1) cannot hold with $s, t$ in $R^*$. In 1940, making use of results of [7], Mursi showed [6] that if there is an $k > \max (\text{ord } f, \text{ord } g)$ such that the circles $S(a_n, |a_n|^{-k})$ with center $a_n$ and radius $|a_n|^{-k}$ intersect none of the corresponding circles $S(b_m, |b_m|^{-k})$, then (1) holds with both ord $s$ and ord $t$ no greater than $\max (\text{ord } f, \text{ord } g)$.

In an earlier paper [2], the author showed that if $M$ is any maximal ideal of $R$, the residue class field $R/M$ is isomorphic with the complex field $K$. In this paper, under some restrictions, this theorem is extended to the ring $R_\lambda$ of all entire functions of order no greater than $\lambda$, and hence to $R^*$. 

**Definition.** Let $i_n(f, n)$ be the number of zeros of $f$ contained in $S(a_n, |a_n|^{-h})$, where a zero of multiplicity $m$ is counted $m$ times.

**Theorem.** Let $M'$ be a maximal ideal of $R_\lambda$ containing a function $f$ such that $i_n(f, n)$ is bounded for some $n$. Then $R_\lambda/M'$ is isomorphic with $K$. 

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The proof proceeds as in [2, Theorem 5]. It is easily seen that $R/M'$ has degree of transcendency $c$ (where $c$ is the cardinal number of the continuum) over the rational field. By a well-known theorem of Steinitz, it is only necessary to show that $R/M'$ is algebraically closed.

Since a maximal ideal of any integral domain is prime, there is a $g$ in $M'$ such that $i_k(g, n) = 1$, for all $n$. In particular, all the zeros $(b_n)$ of $g$ are simple. Let $\Phi(z, X) = X^n + f_1(z) X^{n-1} + ... + f_m(z)$ be a polynomial with coefficients in $R/M'$, of degree $m > 0$. For each $n$, $\Phi(b_n, X)$ is a polynomial with coefficients in $K$, which has $m$ complex roots. Choose any such and call $c_n$. It is well known that $|c_n| < 1 + \max (|f_1(b_n)|, ... , |f_m(b_n)|)$. Since the order of the $f_i$, and the exponent of convergence of $(b_n)$ do not exceed $\lambda$, it follows from a theorem of Macintyre and Wilson [5, Theorem 4] (also obtained independently by Leont'ev [4]) that there is a $t$ in $R$ such that $i(b_n) = c_n$. So $\Phi(z, t(z))$ is in $M'$, whence the theorem.

**Remarks:**
1. The author does not know if there is a maximal ideal in $R$ that fails to satisfy the hypothesis of the theorem.

2. There exist prime ideals of $R_4$ and $R^*$ that fail to satisfy this hypothesis. For, the set $B$ of elements of $f$ of $R_4$ (or $R^*$) with $i_n(f, n)$ bounded for some $h$, is closed under multiplication. Hence, one can construct, with the aid of Zorn's lemma, prime ideals not intersecting $B$. See also [3].

3. If in the theorem $R_4$ is replaced by $R^*$, the constant $h$ in the definition of $i_n(f, n)$ can be replaced by a positive, increasing function of $|z|$ such that $\lim \sup \frac{\log h(|a_n|)}{\log |a_n|}$ is finite. See [5, Theorem 5].

**References**

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