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Cover Page Footnote
I would like to acknowledge my advisor Annie Selden for her help, as well as the online proof working group. Finally, I'd like to acknowledge my family, particularly my wife, who help me concentrate on my research.

Does Content Matter in an Introduction-to-Proof Course?

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Abstract

Introduction-to-proof courses are becoming more prevalent in mathematics departments as more recognize the need to support students while they transition from courses focused on computation (such as calculus) to proof-intensive courses (such as real analysis). In such introduction courses, there are some common proving techniques to teach (induction, contradiction, and contraposition to name a few), but the content varies from institution to institution. This note adds to the discussion on content in such courses, by analyzing two prior studies, one using a coding scheme designed to illuminate step-by-step justifications in a proof, and the other focused on interviews with course instructors. Our analysis of the literature shows that there may be reason to believe that content-based introduction-to-proof courses inadvertently overemphasize specific mathematical-area reasoning, which may not translate effectively to subsequent proof-based courses in different content areas. Simply put, while some mathematicians may be convinced of this, a real analysis, number theory, or abstract algebra course may not be the most effective introduction-to-proof course for students to transition to other proof-based courses.

1. Introduction

Introduction-to-proof courses (also known as transition-to-proof courses or bridge courses) have gained popularity in the last twenty years due to the difficulties that many mathematics and mathematics education students have with proving [9]. In some instances, these courses allow students to focus directly on proving techniques such as induction, contradiction, and contraposition. In others, the emphasis on proof techniques may be more latent. In short, while all introduction-to-proof courses share the common goal of introducing students to mathematical proof, there is considerable variation in the mathematical content used in these courses. For example, some courses emphasize truth tables (in a limited sense) and logical reasoning explicitly [6], while others tend to focus on proving techniques (e.g., [12]). Some universities use a content course, such as linear algebra, abstract algebra, or real analysis as an introduction-to-proof course [11]. Scouring the mathematics education literature, one can detect no clear sense of content for such courses.

There has been much research on introduction-to-proof courses. For instance, in one study, Alcock [2] interviewed five mathematicians experienced in teaching an introduction-to-proof course. She identified four modes of thinking (instantiation, creative thinking, critical thinking, and structural thinking) considered important by the mathematicians for successful proving. Her conclusion was that "it certainly seems reasonable to claim that collaborative classroom environments, in which students investigate, refine, and prove mathematical conjectures" [2, page 94] foster the flexible use of all four modes. Although researchers (e.g., [2, 8]) have discussed pedagogical strategies, implications, and suggestions for teaching these four modes of thinking, they did not specify which mathematical content would be useful in a transition-to-proof course for developing these four modes of thinking, nor did they address the question of whether transition-to-proof courses adequately prepare students for more advanced mathematics courses.

Nonetheless there is precedent for a thorough examination of mathematical content in proving. Dawkins and Karunakaran [4] claimed that previous proof literature made general declarations about proving without taking into consideration the content that the data were generated with. The authors analyzed three proving situations, two in geometry and another in real analysis, and concluded that many of the nuances of proving that are content-specific are often lost: "content-general models such as logic may be applied to a broad range of students' proving activity [in mathematics education research], but may also be misleading or dishonest to the underlying reasoning process" [4, page 73]. While Dawkins and Karunakaran addressed content-specificity in the research of proving, In this note we raise related but different questions regarding educational practice:

- Does specific content matter in an introduction-to-proof course?

- If it does, how can one improve an introduction-to-proof course by implementing both larger ideas, such as proving techniques, and content-specific reasoning?

Two courses that are commonly taken after an introduction-to-proof course in the United States are abstract algebra and real analysis. Common proving techniques permeate both courses. However, content such as continuity in real analysis consistently require students to consider mixed quantifiers (which is somewhat problematic [5]), and in algebra, some content such as identity and inverse in groups drive students to prove uniqueness of an element. While mixed quantifiers and uniqueness occur in theorems from both courses, a dissection of each content course with respect to proving may give suggestions for content to focus on in introduction-to-proof courses. In the next few paragraphs, two studies are discussed in detail to strengthen the previous conjecture.

2. A "chunk-by-chunk" examination of proofs from multiple mathematical topics

In a previous study [10], 43 student-constructed proofs (created in an inquiry-based learning course and verified by the professor of the course as valid) were dissected into "chunks," defined as "meaningful units" in the proof (in the sense of Miller [7]). A coding scheme with thirteen codes (influenced by Baker [3]) was used on the chunks to investigate how often certain codes appeared. These codes were used to illuminate step-by-step justifications for why the chunk appeared. Examples of these codes include "definition of (DEF)," defined as "a chunk in a proof that calls on the definition of a mathematical term" [10, page 22]. An example of the code DEF is: "consider the line 'Since $x \in A$ or $x \in B$, then $x \in A \cup B$.' The conclusion [chunk] 'then $x \in A \cup B$' implicitly calls on the definition of union" [10, page 22]. The rest of the codes, along with explanations, are located in Appendix A.

The conclusion of that study was that around 30% of the chunks in the final proofs were derived from a use of a definition (DEF) in the course, and that percentage, combined with assumptions (A, 25%) were over half of all proofs created in the course. The author re-investigated the data from the study, organizing the chunks and percentages based upon content area in the course. The four distinct content areas in the course were elementary set and function theory, real analysis (continuous functions), abstract algebra (semigroup theory), and topology (Hausdorff and regular topological spaces). The results are given in Table 1. An examination of the differences in percentages for various content yields two interesting large disparities: the use of high-school level algebra (ALG, 17.1%) in real analysis was much larger than any of the other content, and there was a lack of definition usage (DEF, 6.7%) in real analysis compared to the other content.

3. Interviewing instructors of introduction-to-proof courses

In another study [11], seven mathematicians in different research fields answered questions about introduction-to-proof courses. A1 and A2 were mathematicians from a small mathematics department (faculty size: eight) located in the Midwest. Professors B1 (a mathematician researching in mathematics education), B2, and B3 were from a large mathematics department (faculty size: 138) in the Midwest. Professors C1 and C2 were from a medium-sized mathematics department (faculty size: 32) in the south central U.S.

All of the participants acknowledged there were fundamental differences in reasoning between abstract algebra and real analysis. Below are samples of the overall responses:

B1: "Well you have different definitions that are fundamental... In elementary analysis one needs to know, not only epsilon-delta definitions but all these tricks: given this epsilon we're gonna construct a delta using some sort of magic that comes from experience and you don't need that technique in an algebra class."

B3: "When I teach analysis I try to highlight certain things that are sort of very analysis-y. Like proving equality by proving two inequalities, which is not something you typically find in an algebra course."

C2: "I think the use of quantifiers is more difficult in analysis than in algebra... but I think that might possibly be offset a little by the fact that the analysis content area relates so solidly to the calculus that they have studied, so they have a good deal of computational experience."

All of the professors stated that teaching a content course (such as an upper-level undergraduate real analysis or abstract algebra course) could be possible as an introduction-to-proof course with the caveat that the amount of content was drastically reduced and time was devoted to explicitly discussing proving techniques.

A1: "Yes, if the credits and time are increased enough to allow sufficient time to develop the habits of mind... I think [habits of mind do] need to [be] explicitly addressed, not just implicitly."

C1: "Yes. I absolutely believe so. But, I mean, you have to do it... so I can't teach a ring theory course [as a transition-to-proof course] with the idea that I have to cover a bunch of material."

Finally, the interviewees claimed there were skills that students need to learn in order to be successful in proving. Those skills include sense making, learning proving techniques, precision, reading/validating proofs, and flexibility in the proving process. Most (six out of the seven mathematicians) agreed that learning to use definitions in an introduction-to-proof course is a required skill for students to be successful in subsequent proof courses. One mathematician, C1, stated: "I tell [students] over and over and over again, 'Definitions tell you how to write proofs.' ... you look at the definition and that will tell you where to start." This resonates with previous research [1] and corroborates results in the previous section.

4. Discussion

Does content matter in an introduction-to-proof course? It appears the mathematician interviewees had mixed feelings about this question. Participants agreed that there are differences between reasoning and proof in abstract algebra and real analysis. However, the same participants never acknowledged those differences explicitly when discussing content in introduction-to-proof courses, contradicting their previous statements. In fact, all participants stated that one content course can satisfy the need for an introduction course. How can a student be introduced to rigorous proving and reasoning in one content area, say real analysis, and be expected to transfer that reasoning to algebra, especially when the reasoning is acknowledged as different? It seems difficult for an undergraduate student to absorb new proving techniques and deeper content, and then to be expected to transfer those techniques to new content.

The chunking study (Section 2) allowed a conclusion that, according to step-by-step justifications, there are differences between real analysis and the rest of the other content. Also, other research has shown that undergraduate students enter proving with mostly step-by-step, or local, proof comprehension [13]. This then generates a (common) hypothetical scenario: Suppose an institution has a real analysis course as their introduction to proving. How can a student transition to proving in abstract algebra where the content-specific reasoning is vastly different?

The author acknowledges limitations with both studies. The qualitative interview study used a small sample size, and while the participants were varied in areas of research, the results may not reflect the entire mathematics community. Although there were 43 proofs consisting of 673 chunks, the chunking study also involved a small sample size; content was limited to one topic per mathematical area. For example, the study data contained only a small number of continuity proofs in the real analysis portion, and there could be other topics, such as integration using Riemann sums, that might yield different chunk percentages. In fact, there was a small percentage of "use of definition" chunks in the real analysis proofs; however, all but one of the proofs concerned proving some function was continuous, thus satisfying the definition of a continuous function. This small percentage is due to the scope of the coding. The purpose of the codes was for local proof comprehension [13], and satisfying the full definition of continuity may require putting together pieces of the definition such as finding a correct δ.

The aim of this note is not to be declarative but rather to be stimulating; both studies above, coupled with that of Dawkins and Karunakaran [4], give us reason to believe that there may be content-specific reasoning that needs to be addressed in an introduction-to-proof course. Different content areas seem to require different reasoning, so using a single content area as an introduction-to-proof course may not transition students fully to proving. The challenge then is both to design larger studies that explicate those differences and to experiment with new approaches to teaching introduction-to-proof courses. One suggestion is to include many areas of content, with emphasis on proving, and to engage students in reflecting on and improving their own proving process. Perhaps then, an introduction-to-proof course can both be more encompassing in terms of content and improve the effectiveness of the students' transition to their subsequent upper-level mathematics courses.

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A. Appendix: Codes for the chunks of student-constructed proofs

The codes for the chunks of student-constructed proofs from [10] are listed below, with some minor revisions made for clarity.

(A) Assumption

Choice: When a symbol is chosen to represent an object (often fixed, but arbitrary) about which something will be proved — but not the assumption of additional properties given in a hypothesis.

Hypothesis: When the hypothesis of a theorem or argument is assumed (often stating properties of an object in the proof).

Example: For the theorem "For all $n \in \mathbb{N}$, if $n > 5$ then $n^2 > 25$."

A (Choice): "Let $n \in \mathbb{N}$." (fixed but arbitrary)

A (Hypothesis): "Suppose $n > 5$."

(ALG) Algebra

Any high school or lower algebra done in the proof.

Example: "… $|x + 4| - 4 \leq |x| + |4| - 4 = |x|$ …"

ALG: "$|x + 4| - 4 \leq |x| + |4| - 4$"

(C) Conclusion statement

This statement summarizes the conclusion of a theorem or an argument.

Example: "… So $x \in B$. Therefore $A \subseteq B$."

C: "Therefore $A \subseteq B$."

(CONT) Contradiction statement

This chunk is the conclusion of a contradiction proof or argument.

Example: For a proof: "… We found $x \in A$, which is a contradiction."

CONT: "which is a contradiction".

(D) Delimiter

A delimiter is a word or group of words signifying the beginning or end of a sub-argument. Common delimiters include "now," "next," "firstly," "Lastly," "Case 1," "In both cases," "Part," " $\Longrightarrow$," "($\subseteq$)," "base case" (in an induction proof), and "by induction" (in an induction proof).

Example: For a proof: "... In both cases, we conclude that $x \in B$."
D: "In both cases".

(DEF) Definition of

When the proof-writer uses a definition of a mathematical object.

Example: "... so $x \in A$... thus $x \in B$... Then $x \in A \cap B$... "
DEF: "Then $x \in A \cap B$".

(ER) Exterior reference

When a property depends on another theorem proved previously.

Example: In a proof "... Now, by Theorem 6, $x \in A$... "
ER: "by Theorem 6"

(FL) Formal logic

Any logic that is not common sense will be considered as "Formal Logic".

Example: In a proof "... If $x \in A$ and $x \notin B$, and $A, B \subseteq X$, then $x \notin (X - A) \cup B$... "
FL: "then $x \notin (X - A) \cup B$"

(II) Informal inference

"Informal inference" refers to an inference depending on common sense or basic logic.

Example: In a proof "...then $x \in B$. So $x \in A$ or $x \in B$ ..." and there is no mention of $x \in A$ earlier in the proof.
II: "So $x \in A$ or $x \in B$"

(IR) Interior reference

"Interior reference" refers to when a chunk of a proof calls on anything stated earlier in the proof.

Example: In a proof: "... Let $x \in A$ ... $A \subseteq B$... Since $x \in A$, $x \in B$... "

IR: "Since $x \in A$"

(REL) Relabeling

When an object is given a new (usually shorter) label.

Example: In a proof: "... Thus $e_a = e_b$ is the identity. Set $e = e_a = e_b$... "

REL: "Set $e = e_a = e_b$"

(SI) Statement of intent

A small statement in a proof that indicates what is intended in the rest of the argument.

Example: In a proof: "... We want to show that $x \in A$... "

SI: "We want to show that $x \in A$"

(SIM) Similarity in Proof

When a section of a proof can be repeated with the same arguments for another part of a proof, we call it "Similarity in proof".

Example: In a proof: "... Therefore A is a left ideal. Similarly, A is a right ideal... "

SIM: "Similarly, A is a right ideal"

Table 1: Numbers of different types of proof chunks in the four content areas of the course.

	Chunks	A	ALG	C	CONT	D	DEF	ER	II	IR	FL	REL	SIM	SI
Set and Function	266	70	0	27	0	24.5	92.5	1	12	33	3	1	1	1
% of total		26.3	0.0	10.2	0.0	9.2	34.8	0.4	4.5	12.4	1.1	0.4	0.4	0.4
Real Analysis	116	35	20.5	7	0	5	8	8.5	13.5	17.5	0	1	0	0
% of total		29.2	17.1	5.8	0.0	4.2	6.7	7.1	11.3	14.6	0.0	0.8	0.0	0.0
Abstract Algebra	149	32	2	10.5	1	1	51	4	8	33	0.5	1	2	3
% of total		21.3	1.3	7.0	0.7	0.7	34.0	2.7	5.3	22.0	0.3	0.7	1.3	2.0
Topology	142	29	0	11.5	3	1	51.5	3	10	24	9	0	0	0
% of total		20.4	0.0	8.1	2.1	0.7	36.3	2.1	7.0	16.9	6.3	0.0	0.0	0.0