Some Sufficient Conditions for the Jacobson Radical of a Commutative Ring with Identity to Contain a Prime Ideal

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THE JACOBSON RADICAL OF A COMMUTATIVE
RING WITH IDENTITY TO CONTAIN A PRIME IDEAL

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1. Introduction

Throughout, the word «ring» will abbreviate the phrase «commutative ring with identity element 1» unless the contrary is stated explicitly. An ideal I of a ring R is called pseudoprime if $ab = 0$ implies $a$ or $b$ is in I. This term was introduced by C. Kohls and L. Gillman who observed that if I contains a prime ideal, then I is pseudoprime, but, in general, the converse need not hold. In [9 p. 233], M. Larsen, W. Lewis, and R. Shores ask if whenever the Jacobson radical $J(R)$ of an arithmetical ring is pseudoprime, it follows that $J(R)$ contains a prime ideal?

In Section 2, I answer this question affirmatively. Indeed, if R is arithmetical and $J(R)$ is pseudoprime, then the set $N(R)$ of nilpotent elements of R is a prime ideal (Corollary 9). Along the way, necessary and sufficient conditions for $J(R)$ to contain a prime ideal are obtained.

In Section 3, I show that a class of rings introduced by A. Bouvier [1] are characterized by the property that every minimal prime ideal of R is contained in $J(R)$. The remainder of the section is devoted to rings with pseudoprime Jacobson radical that satisfy

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a variety of chain conditions. In particular, it is shown that if \( R \) is a Noetherian multiplication ring with pseudoprime Jacobson radical \( J(R) \), then \( J(R) \) contains a unique minimal prime ideal (Theorem 20), but there is a Noetherian semiprime ring \( R \) such that \( J(R) \) is pseudoprime and fails to contain a prime ideal (Example 21).

2. The ideal \( mI \) and pseudoprime ideals

As in [5], if \( I \) is an ideal of a ring \( R \), let

\[
mI = \cup \{ A(1 - i): i \in I \}
\]

where \( A(a) = \{ x \in R: ax = 0 \} \). In [5], the following assertions are proved.

1. Lemma (Jenkins-McKnight) If \( I \) and \( K \) are ideals of a ring \( R \) and \( I \subset K \), then

   (a) \( mI \) is an ideal of \( R \) contained in \( I \)
   
   (b) \( mI = \{ a \in R: I + A(a) = R \} \)
   
   (c) \( mI \subset mK \)
   
   (d) \( m(I + J(R)) = mI \).

Recall that the Jacobson radical \( J(R) \) of a commutative ring \( R \) with identity is the intersection of all the maximal ideals of \( R \), and that \( a \in J(R) \) if and only if \( (1 - ax) \) is a unit for every \( x \in R \) [11, Section 30].

Let \( U(R) \) denote the set of units of a ring \( R \), let \( M(R) \) denote the set of maximal ideals of \( R \), and let \( S(R) = \Sigma \{ mM : M \in M(R) \} \). By Lemma 1, \( S(R) = \Sigma \{ mI : I \) a proper ideal of \( R \} = \Sigma \{ A(1 - i) : i \in R \setminus U(R) \} \) is the smallest ideal containing \( A(1 - i) \) for every non unit \( i \in R \).

The next lemma indicates the importance of the ideals \( mI \) in the study of rings with pseudoprime Jacobson radical.

2. Lemma. The Jacobson radical \( J(R) \) of a ring \( R \) is pseudoprime if and only if \( S(R) \subset J(R) \).

Proof. To prove the lemma, it suffices to show that \( J(R) \) is pseudoprime if and only if \( mI \subset J(R) \) for every proper ideal \( I \) of \( R \).
If \( J(R) \) is pseudoprime, \( I \) is a proper ideal of \( J(R) \), and \( azmI \), there is an \( i\in I \) such that \( a(1 - i) = 0 \). But \( (1 - i)\notin J(R) \), so \( azJ(R) \).

Suppose, conversely, that \( mI \subset J(R) \) for every proper ideal \( I \) of \( R \), \( ab = 0 \), and \( b\notin J(R) \). Then there is a \( r\in R \) such that \( 1 - bx \) is not a unit. Thus \( a(1 - bx) = a \), so \( azm((1 - bx)R) \subset J(R) \).

Suppose \( I \) is a proper ideal of a ring \( R \) (which need not have an identity element). A proper prime ideal of \( R \) that fails to contain properly any other prime ideal of \( R \) is said to be a \textit{minimal prime ideal} of \( R \). Let \( \mathcal{P}(R) \) denote the set of minimal prime ideals of \( R \). It is well known that \( \bigcap \{P: P\notin \mathcal{P}(R)\} \) is the set \( N(R) \) of nilpotent elements of \( R \) \cite[p. 100]{11}, and that a prime ideal \( P \) is minimal if for every \( aeP \), there is a \( b\notin P \) such that \( absN(R) \) \cite[lemma 3.1]{5}.

If \( N(R) = \{0\} \), then \( R \) is called a \textit{semiprime} ring.

For any ideal \( I \) of \( R \), the \textit{radical} \( \sqrt{I} \) of \( I \) is the intersection of all the prime ideals of \( R \) containing \( I \). Equivalently, \( \sqrt{I} = \{a: a^n \in I\} \) for some positive integer \( n \). The next proposition describes \( \sqrt{mI} \) as an intersection of minimal prime ideals of \( R \).

3. \textbf{Proposition.} Suppose \( I \) is a proper ideal of \( R \) and \( P \) is a minimal prime ideal of \( R \)

\begin{itemize}
  \item[(a)] \( mI \subset P \) if and only if \( I + P \neq R \)
  \item[(b)] \( \sqrt{mI} = \bigcap \{P \in \mathcal{P}(R): I + P \neq R\} \)
  \item[(c)] If \( M \) is a maximal ideal of \( R \), then \( \sqrt{mM} = \bigcap \{P \in \mathcal{P}(R): P \subset M\} \)
  \item[(d)] If \( R \) is semiprime, then \( \sqrt{mI} = mI \).
\end{itemize}

\textbf{Proof of (a).} If \( I + P = R \), there is an \( i\in I \) and \( p\in P \) such that \( i + p = 1 \). Since \( P\notin \mathcal{P}(R) \), there is a \( q\notin P \) such that \( q(1 - i) = qpeN(R) \). Hence there is a positive integer \( n \) such that \( q^n(1 - i)^n = 0 \). By the binomial theorem \( (1 - i)^n = (1 - i') \) for some \( i'\in I \), so \( q^nemI\subset P \). We have shown that \( I + P = R \) implies \( mI \subset P \).

If, conversely, there is an \( a\in mI\subset P \), then \( a(1 - i) = 0 \) for some \( i\in I \). Since \( a\notin P \), \( 1 - i\notin P \) and \( I + P = R \). This completes the proof of (a).
To get (b) from (a), it suffices to show that $\sqrt{mI}$ is the intersection of all the minimal prime ideals containing it. It follows from [7, Theorem 10], that $mI$ is the intersection of all the prime ideals of $R$ such that $P/mI$ is a minimal prime ideal of $R/mI$. Suppose $a$ is an element of such a prime ideal $P$. Then there is a $b \notin P$ and a positive integer $n$ such that $(ab)^n \in mI$. Hence $a^n b^n (1 - i) = 0$ for some $i \in I$. Suppose $b^n (1 - i) \in P$. Now $b \notin P$, so $(1 - i) \in P$ since $P$ is a prime ideal. Thus $I + P = R$, and by (a), $mI \not\subset P$. This contradiction shows that $a^n [b^n (1 - i)] = 0$ and $b^n (1 - i) \notin P$. Hence $P \in \mathcal{D}(R)$ and (b) holds.

Clearly (c) follows from (b).

If $ae \in mI$, then $a^n e \in mI$ for some positive integer $n$. So there is an $i \in I$ such that $a^n (1 - i) = 0 = [a(1 - i)]^n$. Since $R$ is semiprime, $a(1 - i) = 0$ and $ae \in mI$. Thus (d) holds.

For any ring $R$, let $G(R)$ denote the multiplicative semigroup generated by $\{(1 - i) : i \in R \setminus \mathcal{U}(R)\}$ and let $T(R) = \{aeR : ax = 0$ for some $x \in G(R)\}$. Note that $T(R)$ is an ideal of $R$ which is proper if and only if $0 \notin G(R)$. Also, $S(R) \subset T(R)$. For, if $aeS(R)$, then there is a finite set $\{M_1, \ldots, M_n\}$ of maximal ideals, and elements $m_i \in M_i$ for $i = 1, \ldots, n$ such that $a \Sigma_{i=1}^n A(1 - m_i)$. Then $a \Pi_{i=1}^n (1 - m_i) = 0$, so $ae \in T(R)$.

4. Proposition. The following properties of a minimal prime $P$ of a ring $R$ are equivalent

(a) $P \subset J(R)$.
(b) $P \supset S(R)$.
(c) $P \not\subset T(R)$.

Proof. If $P \subset J(R)$ and $M$ is a maximal ideal of $R$, then $P \subset M$. Hence by Proposition 3, $mM \subset P$, so $S(R) = \Sigma \{mM : M \in \mathcal{M}(R)\} \subset P$. Thus (a) implies (b).

Suppose next that there is an $ae \in T(R) \setminus P$. Then there is an $x \in G(R)$ such that $ax = 0 \in P$. Since $a \notin P$, we have $x \notin P$. Since $x \in G(R)$, there is finite set $\{M_1, \ldots, M_n\}$ of maximal ideals of $R$ and elements $m_i \in M_i$ for $i = 1, \ldots, n$ such that $x = (1 - m_1) \cdots (1 - m_n) \in P$. Hence $(1 - m_i) \in P$ for some $i$, so $P + M_i = R$. By Proposition 3, $mM \not\subset P$ and therefore $P \not\subset S(R)$. Thus we have shown that (b) implies (c).
If \( P \supset T(R) \), then \( P \supset S(R) = mM \) for every maximal ideal \( M \) of \( R \). So, by Proposition 3, \( P \subset J(R) \). Thus (c) implies (a) and the proof of Proposition 4 is complete.

Since every proper ideal of \( R \) is contained in a prime ideal, the following corollary follows immediately from Proposition 4 and the remarks preceding it. It may also be derived easily from [2, Proposition 3.3].

5. COROLLARY. The Jacobson radical of a ring \( R \) contains a prime ideal if for every positive integer \( n \), whenever \( m_1, \ldots, m_n \) is a finite set of non units of \( R \), it follows that \( \Pi_{i=1}^n (1 - m_i) \neq 0 \).

Another easy consequence of Proposition 4 follows.

6. COROLLARY. If \( R \) is a ring with pseudoprime Jacobson radical \( J(R) \), and \( P \) is a minimal prime ideal of \( R \) such that \( P \supset J(R) \), then \( P = J(R) \).

PROOF. Since \( J(R) \) is pseudoprime and \( P \supset J(R) \), then \( P \supset J(R) \supset S(R) \). Hence by Proposition 4, \( P \subset J(R) \), so \( P = J(R) \).

The next theorem and its corollaries solves the problem posed by M. Larsen, W. Lewis, and R. Shores in [9, p. 233]. Recall that if \( I_1 \) and \( I_2 \) are proper ideals of a ring \( R \) and \( I_1 + I_2 = R \), then \( I_1 \) and \( I_2 \) are said to be co-maximal.

7. THEOREM. Suppose \( R \) is a ring with pseudoprime Jacobson radical.

(a) If \( S(R) \) contains a prime ideal \( P \), then \( P = S(R) \) is the unique minimal prime ideal of \( R \) contained in \( J(R) \).

(b) If \( \sqrt{mP} \) is a prime ideal, then \( P = N(R) \).

PROOF. The prime ideal \( P \) contains a minimal prime ideal \( P_0 \), and by Lemma 2, \( P_0 \subset P \subset S(R) \subset J(R) \). By Proposition 4, \( S(R) \subset P \), so \( P_0 = P = S(R) \). Using Proposition 4 again yields that \( S(R) \) is the unique minimal prime ideal contained in \( J(R) \), and (a) holds.

If \( \sqrt{mP} = Q \) is a prime ideal, then by Proposition 4, \( Q \in \mathcal{P}(R) \). But \( \sqrt{mP} \subset P \), so \( \sqrt{mP} = P \subset J(R) \). By Lemma 1, \( mP = m\sqrt{mP} = \{0\} \). Hence \( P = \sqrt{\{0\}} = N(R) \), and (b) holds.
8. Corollary. The following properties of a ring $R$ are equivalent.

(a) $J(R)$ is pseudoprime and there is a minimal prime ideal $P$ of $R$ co-maximal with every other minimal prime ideal of $R$

(b) $N(R)$ is a prime ideal.

Proof. If (a) holds, then $\sqrt{mP} = P \subset J(R)$ by Proposition 3 and Lemma 2. Hence $S(R)$ contains a prime ideal, so (b) holds by Theorem 7.

If (a) holds, then $N(R) \subset J(R)$ and $N(R)$ is the unique element of $\overline{P}(R)$. So (a) holds and Corollary 8 follows.

A ring is called arithmetical if its lattice of ideals is distributive. In [6, Corollary 2] C. Jensen has shown that incomparable prime ideals of an arithmetical ring are co-maximal. Hence we have:

9. Corollary. If the Jacobson radical $J(R)$ of an arithmetical ring $R$ is pseudoprime, then $N(R)$ is a prime ideal contained in $J(R)$.

3. Other classes of rings whose Jacobson radicals are pseudoprime

In [1], A. Bouvier calls a ring $R$ presimplifiable if whenever $x, y \in R$ and $xy = x$, then $x = 0$ or $y$ is a unit, and the studies factorization properties of such rings. By Lemma 1, $R$ is presimplifiable if and only if $S(R) = \{0\}$. These rings are characterized in the next theorem.

10. Theorem. The following properties of a ring $R$ are equivalent.

(a) $R$ is presimplifiable

(b) $mM \subset N(R)$ for every maximal ideal $M$ of $R$.

(c) Every minimal prime ideal of $R$ is contained in $J(R)$.

(d) Every proper divisor of $0$ in $R$ is contained in $J(R)$.

Proof. If $R$ is presimplifiable and $M$ is a maximal ideal of $R$, then $mM = \{0\} \subset N(R)$, so (a) implies (b).

If (b) holds, then $mM \subset N(R) \subset P$ for every $P \in \mathcal{P}(R)$ and maximal ideal $M$ of $R$. So, by Proposition 3, if $P \in \mathcal{P}(R)$, then $P$
is contained in every maximal ideal of $R$. That is, $P \subset J(R)$, so (c) holds.

Every proper divisor of 0 is contained in some minimal prime ideal of $R$ [4, Section 2], so (c) implies (d).

If (d) holds, then $A(a) \subset M \subset J(R)$ for every maximal ideal $M$ of $R$. Hence by Lemma 1, $mM = \{aeR: M + A(a) = R\} = \{0\}$, so $R$ is presimplifiable. This completes the proof of Theorem 10.

In the remainder of the paper, rings satisfying various chain conditions, and which have a pseudoprime Jacobson radical, are studied.

Suppose $R$ is a ring (which does not necessarily have an identity element). If $A \subset R$, let $h(A) = \{P \in P(R): A \subset P\}$, and if $S \subset P(R)$, let $k(S) = \bigcap\{P \in P(R): P \subset S\}$. If we call a subset $S$ of $P(R)$ closed if $S = h(k(S))$, then it is known that $P(R)$ becomes a Hausdorff topological space with $B = \{h(a): a \in R\}$ as a base for its open sets. Moreover, for any $aeR$, $h(A(a))\cap h(a) = \emptyset$ and $h(A(a))\cup h(a) = P(R)$, so the hull of each element of $R$ is both closed and open. Moreover, $P(R)$ and $P(R/N(R))$ are homeomorphic. [4, Section 2].

If $R$ is semiprime and for every $x,y \in R$, there is a $z \in R$ such that $A(x) \cap A(y) = A(z)$, $R$ is said to satisfy the annihilator condition, or to be an a.c.-ring. The following assertions are proved in [4, Theorem 3.4]. Recall that if $aeR$ is not a proper divisor of 0, then $a$ is called a regular element of $R$, and an ideal containing a regular element is called a regular ideal of $R$.

11. Lemma. (Henriksen and Jerison). The following properties of a semiprime ring (not necessarily with an identity element) are equivalent.

(a) $P(R)$ is compact and satisfies the annihilator condition.
(b) $\{h(a): a \in R\}$ is a base for the open subsets of $P(R)$.
If (a) holds, then
(c) $R$ has a regular element, and
(d) a proper ideal of $R$ is contained in a minimal prime ideal of $R$ if (and only if) it is not regular.

The next lemma is probably known, but does not seem to appear in the literature.
12. **Lemma.** If $R$ is a semiprime ring (not necessarily with an identity element) and $\mathcal{P}(R)$ is finite, then $R$ satisfies the annihilator condition.

**Proof.** By Lemma 11 (a,b,c), if $S \subseteq \mathcal{P}(R)$, there is an $a \in R$ such that $h(a) = S$. Hence if $x,y \in R$, there is a $z \in R$ such that $h(z) = h(x) \cap h(y)$. By [4, Lemma 3.1], since $R$ is semiprime, $A(z) = A(x) \cap A(y)$ and $R$ an a.c.-ring.

13. **Proposition.** The following properties of an a. c.-ring $R$ such that $\mathcal{P}(R)$ is compact are equivalent

(a) $J(R)$ contains a prime ideal of $R$.

(b) $S(R)$ is not a regular ideal.

**Proof.** If (a) holds, then $J(R)$ contains a $P \in \mathcal{P}(R)$, by Proposition 4, $S(R) \subseteq P$. But no element of a minimal prime ideal is regular, so (b) holds.

If (b) holds, then by Lemma 11 (c), $S(R)$ is contained in some $P \in \mathcal{P}(R)$. So by Proposition 4, (b) holds.

The following corollary is an immediate consequence of Lemma 12 and Proposition 13.

14. **Corollary.** If $R$ is a semiprime ring such that $\mathcal{P}(R)$ is finite, then $J(R)$ contains a prime ideal if and only if $S(R)$ is not a regular ideal.

15. **Remarks.** (a) The hypothesis of Corollary 14 is satisfied if $R$ is a semiprime ring that satisfies the ascending chain condition on annihilator ideals [7, Theorem 88], or if $R$ has few zero divisors in the sense of [10, p. 152].

(b) Since $N(R) \subseteq J(R)$ and $N(R) \subseteq P$ for every $P \in \mathcal{P}(R)$, it follows easily that $J(R)$ is pseudoprime (resp. $J(R)$ contains a prime ideal of $R$) if and only if $J(R/N(R))$ is pseudoprime (resp. $J(R/N(R))$ contains a prime ideal of $N(R)$).

Next, I examine consequences of the assumption that $mI$ is finitely generated. For any ideal $I$ of $R$ let $\mathcal{F}(I)$ denote the set of
finitely generated ideals \( F \) of \( I \) such that \( FI = F \). It is shown in [7, Theorem 76] that:

1. If \( F \in \mathcal{I}(1) \), there is an \( i \in I \) such that \( a(1 - i) = 0 \) for all \( a \in F \). That is, \( F \subseteq ml \).

Suppose \( I \) is an ideal of a ring \( R \). If \( ab \in I \) and \( a \notin I \) imply \( b \notin \sqrt{I} \), the \( I \) is called a primary ideal. The radical of a primary ideal is a prime ideal [13, p. 152]. If whenever \( A \) and \( B \) are ideals of \( R \), \( AB \subseteq I \), and \( A \neq I \) imply \( B^n \subseteq I \) for some positive integer \( n \), then \( I \) is called a strongly primary ideal. It is known that a primary ideal with finitely generated radical is strongly primary [13, p. 200, proof of 2)].

Let \( I^\omega = \bigcap_{in} I^n \), and note that if \( a \in ml \), there is an \( i \in I \) such that \( a = ai = ai^2 = \ldots = ai^n \) for every positive integer \( n \). Thus \( ml \subseteq I^\omega \).

16. Proposition. Suppose \( I \) is an ideal of a ring \( R \).

(a) If \( ml \) is finitely generated, then \( ml \) is the largest element of \( \mathcal{I}(1) \) and \( ml = A(1 - i) \) for some \( i \in I \).

(b) If \( I^\omega \) is finitely generated, then \( ml = I^\omega \) if and only if \( I^\omega ml = I^\omega \).

(c) If \( I^\omega \) is finitely generated and \( I^\omega ml \) is an intersection of strongly primary ideals, then \( ml = I^\omega \).

(d) If \( R \) is Noetherian, then \( ml = I^\omega \).

Proof. Since \( (ml)I = ml \), (a) follows from (1), and (b) follows from (a) and the fact that \( ml \subseteq I^\omega \).

Suppose \( I^\omega ml \) is contained in a strongly primary ideal \( Q \). If \( I \not\subseteq \sqrt{Q} \), then \( I^\omega \subseteq Q \) since \( Q \) is primary. If \( I \subseteq \sqrt{Q} \), then there is a positive integer \( n \) such that \( I^\omega \subseteq I^n \subseteq Q \) since \( Q \) is strongly primary. Hence \( I^\omega ml = I^\omega \) and (c) follows from (b).

Finally (d) follows from (c) since every ideal of a Noetherian ring is an intersection of (strongly) primary ideals [11, p. 199].

Proposition 16 (d) is also proved in [12, p. 49].

The next two examples show that some of the assumptions made in Proposition 16 (c) are necessary.
17. Example. An integral domain $D_1$ such that if $M$ is a maximal ideal of $D_1$ then $M^\circ M = J(D_1)$ is a prime ideal, but $mM \neq M^\circ$.

Let $D_1$ denote the ring of formal power series $a(x) = \sum_{n=0}^{\infty} a_n x^n$ with rational coefficients such that $a(0) = a_0$ is an integer. As is noted in [3, p. 162], $M$ is a maximal ideal of $D_1$ if and only if there is a prime integer $p$ such that $M_1 = pD_1$. Moreover $(pD_1)^\circ = \{a(x) \in D_1 : a(0) = 0\} = J(D_1)$, and, clearly $(pD_1)^\circ (pD_1) = (pP_1)^\circ$.

Since $D_1$ is an integral domain, $(pD_1)^\circ = \{0\} \neq (pD_1)^\circ$. Note that $(pD_1)^\circ$ is not finitely generated since for $n = 0, 1, 2, \ldots$, $\left(\frac{4}{2^n} x\right) D_1$ is a strictly ascending chain of ideals contained in $(pD_1)^\circ$.

18. Example. An integral domain with a prime ideal $P$ such that $P^\circ$ is both prime and principal, but $mP \neq P^\circ$.

If $D_1$ is the ring of Example 17, let $D_2 = D_1 [[y]]$ denote the ring of formal power series with coefficients in $D_1$. Let $P = \{a(y) = \sum_{n=0}^{\infty} a_n(x)y^n : a_n(x) \in D_1 \text{ for } n \geq 0 \text{ and } a_0(x) \in J(D_1)\}$. Thus $a(y) \in P$ if and only if when we write $a_n(x) = \sum_{n=0}^{\infty} a_{0n} x^n$, we have $a_{00} = 0$. It is easily verified that $P$ is a prime ideal, and $P^\circ = \{a(y) \in D_2 : a(0) = 0\} = yD_2$ is also a prime ideal. Since $D_2$ is an integral domain, $mP = \{0\} \neq yD_2 = P^\circ$. Note finally that $\sqrt{PP^\circ} = \sqrt{P} \cap P^\circ = \sqrt{P^\circ} = P^\circ$ is a prime ideal, but, by Proposition 16, $PP^\circ$ is not an intersection of strongly primary ideals.

The next proposition provides another sufficient condition for $J(R)$ to contain a prime ideal.

19. Proposition. Suppose $P$ is a minimal prime ideal of a ring $R$ such that

(i) $P$ is finitely generated, and

(ii) there is a maximal ideal $M \supseteq P$ and an ideal $B$ of $R$ for which $P = MB$.

Then:

(a) $\sqrt{mP} = P$ if $P = M$ and $mM = P$ if $P \neq M$.

(b) If $J(R)$ is pseudoprime, then it contains a unique minimal prime ideal of $R$.

Proof. If $P = M = MR$, then (a) holds by Proposition 3. If $P \neq M$, then $B \subseteq P$ since $P$ is prime, and $P = MB \subseteq MP \subseteq P$. Thus
P = MP, so P ⊆ mM by (1) and mM ⊆ P by Proposition 3. Hence P = mP and (a) holds in this case as well.

Part (b) follows from (a) and Theorem 7.

An ideal B of a ring R is called a multiplication ideal if whenever A is an ideal of R such that A ⊆ B, there is an ideal C of R such that A = BC. If every ideal of R is a multiplication ideal, then R is called a multiplication ring. The ring R is called an almost multiplication ring if every ideal with a prime radical is a power of its radical. The following facts are known.

(2) Every multiplication ring is an almost multiplication ring and every Noetherian almost multiplication ring is a multiplication ring [10, p. 216 and p. 213, Theorem 9.21].

(3) If P is a prime ideal and M is a maximal ideal of an almost multiplication ring such that P ⊆ M and P ≠ M, then P = mP. [13, p. 224, Ex. 9]

With the aid of (2) and (3) the following consequences of Proposition 19 follow.

20. Theorem. If the Jacobson radical J(R) of a ring R is a pseudoprime multiplication ideal and if every radical ideal of R contained in J(R) is finitely generated, then R is an integral domain or J(R) is a minimal prime ideal of R. In particular, the Jacobson radical of a Noetherian (almost) multiplication ring contains a unique minimal prime.

Proof. By Proposition 19 and (3), J(R) contains a unique minimal prime ideal P. Since J(R) is a multiplication ideal, if P ≠ J(R) there is an ideal B of R such that P = J(R)B. Since P is prime, B ⊆ P, so P = J(R)B ⊆ J(R)P ⊆ P, and hence P = J(R)P. Hence by (1) and Lemma 1, P ⊆ mJ(R) = {0}. Thus R is an integral domain. This completes the proof of the theorem.

The next example shows that a Noetherian ring may have a pseudoprime Jacobson radical which contains no prime ideal.

21. Example. A semiprime Noetherian ring R with pseudoprime Jacobson radical J(R) which has exactly three minimal prime ideals, none of which are in J(R). If F is any field, let T = F[x₁, x₂, x₃] denote
the ring of polynomials in three indeterminates \( x_1, x_2, x_3 \). Let

\[ I = x_1 x_2 T + x_1 x_3 T + x_2 x_3 T, \]

and let \( T^* = \left\{ \frac{a}{1 - i : a \in T, i \in \mathbb{I}} \right\} \) denote the quotient ring of \( R \) with respect to the multiplicative system \( \{ 1 - i : i \in \mathbb{I} \} \). Finally, let \( R = T^*/(x_1 x_2 x_3) T^* \), and let \( b = b + x_1 x_2 x_3 T^* \) for any \( b \in T^* \).

Since \( T \) is a Noetherian unique factorization domain, \( R \) is Noetherian, and each of its proper divisors of 0 is a multiple of \( x_1, x_2, \) or \( x_3 \). Clearly, also, \( I = x_1 x_2 R + x_1 x_3 R + x_2 x_3 R \subset J(R) \), and it follows that \( J(R) \) is pseudoprime. Since every element of a minimal prime ideal is a proper divisor of 0, the minimal prime ideals of \( R \) are \( P_i = \overline{x_i R} \) for \( i = 1, 2, 3 \), none of which are contained in \( J(R) \) since \( 1 - x_i \) is not a unit of \( R \). Finally, \( R \) is semiprime because \( P_1 \cap P_2 \cap P_3 = \{ 0 \} \).

In view of Example 22, the following proposition may not seem so special.

22. Proposition. If \( R \) is a ring with no more than two minimal prime ideals and \( J(R) \) is pseudoprime, then \( J(R) \) contains a prime ideal.

Proof. If \( R \) has exactly one minimal prime ideal, it must be \( N(R) \subset J(R) \). Suppose \( R \) two minimal prime ideals \( P_1, P_2 \). By Remark 15(b), we may assume that \( R \) is semiprime. By Proposition 3, if \( \text{McM}(R) \), then \( m \mathbb{M} \) is \( P_1, P_2 \) or \( P_1 \cap P_2 = \{ 0 \} \). Hence \( S(R) = \{ 0 \} \), or \( S(R) \) contains a prime ideal. In the first case, the conclusion follows from Theorem 10, and in the second case it follows from Theorem 7.

I conclude with an example that shows that the hypothesis of Proposition 22 can be satisfied for a ring \( R \) without \( R \) being presimplifiable.

23. Example. A semiprime Noetherian ring \( R \) with two minimal prime ideals such that \( J(R) \subset \mathbb{P}(R) \) and \( R \) is not presimplifiable. Let \( S \) denote the ring of formal power series with 0 constant term with coefficients from the ring of integers mod 2. Clearly \( S \) is Noetherian and \( J(S) = S \). If \( Z \) denotes the ring of integers, let \( R = S^* Z = \{ (a, n) : a \in S, n \in Z \} \) where for \( a_1, a_2 \in R, n_1, n_2 \in Z \), \( (a_1, n_1) + (a_2, n_2) = (a_1 + a_2, n_1 + n_2) \), and \( (a_1, n_1)(a_2, n_2) = (a_1 a_2 + n_2 a_1 + n_1 a_2, n_1 n_2) \). It is well known that \( R \) is a Noetherian ring with identity and the mapping
a → (a,0) is an injection of S onto a prime ideal $\hat{S}$ of R. It is easily verified that $S = J(R)$. Also since $(a,0)(0,2) = (0,0)$ for every $a \in S$, $J(R) = \hat{S}$ is a minimal prime ideal of R. By the same reasoning $P = \{(0,2n): n \in \mathbb{Z}\} \in P(R)$, and any other prime ideal of R contains a regular element. So $\mathcal{P}(R) = \{J(R), P\}$, and R is not presimplifiable since $P \notin J(R)$. Finally, R is semiprime since $P \cap J(R) = \{0\}$.

REFERENCES