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On Minimal Completely Regular Spaces Associated With a Given Ring of Continuous Functions

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ON MINIMAL COMPLETELY REGULAR SPACES ASSOCIATED
WITH A GIVEN RING OF CONTINUOUS FUNCTIONS

Melvin Henriksen

1. INTRODUCTION

Let $C(X)$ denote the ring of all continuous real-valued functions on a completely regular space $X$. If $X$ and $Y$ are completely regular spaces such that one is dense in the other, say $X$ is dense in $Y$, and every $f \in C(X)$ has a (unique) extension $\tilde{f} \in C(Y)$, then $C(X)$ and $C(Y)$ are said to be strictly isomorphic. In a recent paper [2], L. J. Heider asks if it is possible to associate with the completely regular space $X$ a dense subspace $\mu X$ minimal with respect to the property that $C(\mu X)$ and $C(X)$ are strictly isomorphic.\(^1\)

In this note, Heider's question is answered in the negative. It is shown, moreover, that if $\mu X$ exists, then it consists of all of the isolated points of $X$, together with those nonisolated points $p$ of $X$ such that $C(X - \{p\})$ and $C(X)$ fail to be strictly isomorphic. Thus, if $\mu X$ exists, it is unique.

2. PRELIMINARY REMARKS

Let $C(X)$ denote the ring of all continuous real-valued functions on a completely regular space $X$. Let $C^*(X)$ denote the subring of all bounded $f \in C(X)$. The following known facts are utilized below.

(2.1) Corresponding to each completely regular space $X$, there exists an essentially unique compact space $\beta X$, called the Stone-Čech compactification of $X$, such that (i) $X$ is dense in $\beta X$, and (ii) every $f \in C^*(X)$ has a (unique) extension $\tilde{f} \in C^*(\beta X) = C(\beta X)$. Thus $C^*(X)$ and $C(\beta X)$ are isomorphic. (See, for example, [3] or [4, Chapter 5].)

(2.2) There exists an essentially unique subspace $\nu X$ of $\beta X$ such that (i) $X$ is a $Q$-space, (ii) $X$ is dense in $\nu X$, and (iii) every $f \in C(X)$ has a (unique) extension $\tilde{f} \in C(\nu X)$. Thus $C(X)$ and $C(\nu X)$ are isomorphic. (For the definition of $Q$-space, and a proof of this theorem, see [1] or [3].)

(2.3) If $X$ and $Y$ are completely regular spaces such that $C(X)$ and $C(Y)$ are isomorphic, then $Y$ is homeomorphic to a dense subspace of $\nu X$ such that every real-valued function continuous on this subspace has a (unique) continuous extension over $\nu X$. [3, Theorem 65.]

(2.4) If $Z$ is any compact space, and $f$ is any continuous mapping of $X$ into $Z$, then there exists a (unique) continuous extension $\tilde{f}$ of $f$ over $\beta X$ into $Z$. (See [5, Theorem 88].)

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1 Since the writing of this paper, Heider's problem has been generalized and solved independently by J. Daly and L. J. Heider.
In the oral presentation of [2], Heider asked "... whether or not to each completely regular space \( X \), there is associated a completely regular space \( \mu X \) such that \( \mu X \) and \( v(\mu X) \) are homeomorphic, and \( \mu X \subset Y \subset vX \) for every completely regular space \( Y \) such that \( vY \) is homeomorphic to \( vX \)." By considering the special case \( Y = X \) in Heider's formulation, we see at once that \( \mu X \subset X \). Moreover, since \( v(\mu X) \) and \( vX \) are homeomorphic, it follows from (2.3) that \( \mu X \) is homeomorphic to a dense subspace of \( X \) all of whose continuous real-valued functions have continuous extensions over \( X \). Thus, it is natural to identify \( \mu X \) with its image in \( X \) under this homeomorphism; this identification leads to the formulation of Heider's problem given in the Introduction, namely: does there exist a dense subspace \( \mu X \) of \( X \) which is minimal with respect to the property that \( C(\mu X) \) and \( C(X) \) are strictly isomorphic? 

We conclude this section with a definition.

**Definition.** If \( X \) is a completely regular space, let \( \eta X \) denote the union of the set of isolated points of \( X \) and the set of nonisolated points \( p \) of \( X \) such that \( C(X \sim \{p\}) \) and \( C(X) \) fail to be strictly isomorphic.

Thus, by (2.3), a nonisolated point \( p \) of \( X \) fails to be in \( \eta X \) if and only if every \( f \in C(X \sim \{p\}) \) has a (unique) continuous extension over \( X \).

### 3. Uniqueness of \( \mu X \)

We begin this section with a theorem which will be used below, and which we believe to be of some independent interest.

**Theorem 3.1.** If \( Y \) is a dense subspace of a completely regular \( X \) such that the rings \( C(Y) \) and \( C(X) \) (respectively, \( C^*(Y) \) and \( C^*(X) \)) are strictly isomorphic, then, for any (nonisolated) point \( p \in Y \), the rings \( C(Y \sim \{p\}) \) (respectively, \( C^*(Y \sim \{p\}) \) and \( C^*(X \sim \{p\}) \)) are strictly isomorphic.

**Proof.** Except for the part of the theorem in parentheses, it is enough, by (2.3), to show that every \( f \in C(Y \sim \{p\}) \) has a (unique) extension \( F \in C(X \sim \{p\}) \). As for the part in parentheses, it will be evident from the construction that if \( f \in C^*(Y \sim \{p\}) \), then \( F \in C^*(X \sim \{p\}) \).

Let \( \{U_{\alpha}\}_{\alpha \in A} \) be a base of neighborhoods in the space \( X \) of \( p \). The index set \( A \) becomes a directed set if we let the statement \( \beta \geq \alpha \) mean that \( U_{\beta} \subset U_{\alpha} \). Since \( X \) is completely regular, for each \( \alpha \in A \), there exists an \( i_{\alpha} \in C(X) \) such that \( i_{\alpha}(x) = 1 \) for \( x \in X \sim U_{\alpha} \), and \( i_{\alpha} \) vanishes on a neighborhood of \( p \). (To see this, let \( h_{\alpha} \in C^*(X) \) be such that \( h_{\alpha}(X \sim U_{\alpha}) = 1 \), and \( h_{\alpha}(p) = -1 \). Then let \( i_{\alpha}(x) = \max(h_{\alpha}(x), 0) \) for every \( x \in X \).) Let \( f \) be the function defined on \( Y \) by letting \( f_{\alpha}(y) = i_{\alpha}(y)f(y) \) for every \( y \in Y \sim \{p\} \), and by letting \( f_{\alpha}(p) = 0 \). Clearly, \( f_{\alpha} \in C(Y) \), and \( f_{\alpha}(y) = f(y) \) for all \( y \) outside of \( U_{\alpha} \). Now, by hypothesis (and (2.3)), \( f_{\alpha} \) has a unique extension \( F_{\alpha} \in C(X) \).

For each \( x \in X \sim \{p\} \), the set \( \{F_{\alpha}(x)\}_{\alpha \in A} \) forms a real-valued net [4, Chapter 2]. For each \( x \in X \sim \{p\} \), let \( F(x) = \lim_{\alpha \in A} F_{\alpha}(x) \). This limit exists since, if \( U_{\alpha x} \) is a basic neighborhood of \( p \) disjoint from \( x \), it follows from \( \beta \geq \alpha_x \) that

\[
F_{\alpha x}(x) = F_{\beta}(x) = F(x).
\]

It is clear that \( F \) is an extension of \( f \). We will show next that \( F \in C(X \sim \{p\}) \), by verifying that \( F \) is continuous at each \( x_0 \in X \sim \{p\} \).

Let \( V_{x_0} \cap U_{\alpha_0} \) denote disjoint neighborhoods (in \( X \)) respectively of \( x_0 \) and \( p \). If \( x \in V_{x_0} \cap U_{\alpha_0} \), then for any \( \beta \geq \alpha_0 \), \( F(x) = F_{\beta}(x) \). Hence the continuity of \( F \) at \( x_0 \) follows from the continuity of \( F_{\beta} \) at \( x_0 \). This completes the proof of the theorem.
COROLLARY. If \( Y \) is a dense subspace of the completely regular space \( X \) then, for any (nonisolated) point \( p \in Y \), if \( \nu Y \) and \( \nu X \) (respectively, \( \beta Y \) and \( \beta X \)) are homeomorphic, then \( \nu(Y - \{p\}) \) and \( \nu(X - \{p\}) \) (respectively, \( \beta(Y - \{p\}) \) and \( \beta(X - \{p\}) \)) are homeomorphic.

It will be shown next that if \( \mu X \) exists, then it is unique.

**THEOREM 3.2.** If with the completely regular space \( X \) there is associated at least one dense subspace \( \mu X \) minimal with respect to the property that \( C(\mu X) \) and \( C(X) \) are strictly isomorphic, then \( \mu X \) is unique. In fact, \( \mu X = \eta X \).

**Proof.** It follows from the definition of \( \eta X \), and from the fact that \( \mu X \) is dense in \( X \), that each of these spaces contains all the isolated points of \( X \). Hence we need only consider the nonisolated points of \( X \). We will show first that \( \mu X \subset \eta X \).

Let \( p \) be a nonisolated point of \( X \) contained in \( \mu X \). By the minimality of \( \mu X \), there exists an \( f \in C(\mu X - \{p\}) \) with no continuous extension over \( \mu X \). But, by Theorem 3.1, \( f \) has an extension \( F \in C(X - \{p\}) \). If \( p \) were not in \( \eta X \), \( F \) would have a continuous extension over \( X \), whose restriction to \( \mu X \) would in turn be a continuous extension of \( f \) over \( \mu X \). Hence \( p \in \eta X \), whence \( \mu X \subset \eta X \).

Suppose there were a point \( p \in \eta X - \mu X \). If \( f \in C(X - \{p\}) \), then since \( C(\mu X) \) and \( C(X) \) are isomorphic, the restriction of \( f \) to \( \mu X \) has a continuous extension over \( X \). This latter would be a continuous extension of \( f \) over \( X \), contrary to the assumption that \( p \in \eta X \). Hence \( \mu X = \eta X \). This completes the proof of the theorem.

COROLLARY. A necessary and sufficient condition that \( \mu X \) exist (in which case it is equal to \( \eta X \)) is that \( \eta X \) be dense in \( X \) and that every \( f \in C(\eta X) \) have a (unique) extension \( f^* \in C(X) \).

As noted by Heider [2], \( \mu X = \eta X = X \), provided every point of \( X \) is a \( G_\delta \).

4. THE SUBSPACE \( \mu X \) NEED NOT EXIST

In this section we give an example of a completely regular space \( X \) such that \( \mu X \) does not exist. In fact, for this \( X \), \( \eta X \) is dense in \( X \), but \( C(\eta X) \) and \( C(X) \) are not isomorphic.

We begin by generalizing a result of Hewitt [3, p. 62].

**THEOREM 4.1.** Let \( Y \) be a noncompact completely regular space, and suppose that \( Y \subset X \subset \beta Y \) and that \( \beta Y - X \) has power less than \( \exp \exp N_0 \). Then \( \nu X = \beta X = \beta Y \).

**Proof.** We will show first that \( C(X) = C^*(X) \), thus verifying that \( \nu X = \beta X \). (See (2.1) and (2.2).) For any \( f \in C(X) \), let \( f^* \) denote its restriction to \( Y \). As noted in [1], \( f^* \) may be regarded as a continuous mapping of \( Y \) into the one-point compactification \( R \cup \{\infty\} \) of the real line \( R \). By (2.4), \( f^* \) has a (unique) continuous extension \( f^* \) over \( \beta Y \) into \( R \cup \{\infty\} \). Since \( Y \) is dense in \( X \), the function \( f^* \) is also an extension of \( f \).

Now the set \( G = \{ y \in Y : f^*(y) = \infty \} \) is a closed \( G_\delta \) of \( \beta Y \), and it is contained in \( \beta Y - X \subset \beta Y - Y \). Hewitt has shown [3, Theorem 49] that every nonempty closed \( G_\delta \) of \( \beta Y \) contained in \( \beta Y - Y \) has power at least \( \exp \exp N_\beta \). On the other hand it is evident, from the hypothesis, that \( G \) has power less than \( \exp \exp N_\beta \). Hence \( G \) is empty. So \( f^* \in C^*(Y) \), and it follows that \( f \in C^*(X) \). Thus \( \nu X = \beta X \).

Since \( X \) is dense in \( \beta Y \), and \( \beta Y \) is compact, in order to conclude that \( \beta X = \beta Y \) it suffices, by (2.1), to show that every \( f \in C^*(X) \) has a (unique) extension \( f^* \in C^*(\beta X) \). We may take \( f^* \) to be the (unique) extension over \( \beta Y \) of the restriction of \( f \) to \( Y \).

This completes the proof of the theorem.
Example. Let $Y$ be any completely regular space that admits unbounded continuous real-valued functions, and such that $\eta Y = Y$. (In particular, $Y$ could be any infinite discrete space.) Let $X = \beta Y$. For each $p \in \beta Y - Y$, it follows from Theorem 4.1 that $\nu(X - \{p\}) = X$. Hence, $\eta X \subset Y$, and since $\eta Y = Y$, it follows that $\eta X = Y$. But, although $\eta X$ is dense in $X$, no unbounded $f \in C(Y)$ has a continuous extension over the compact space $X$. Thus, by the corollary to Theorem 3.1, $\mu X$ does not exist.

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