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Two Quick Combinatorial Proofs

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Theorem. *If $\sum a_n$ is a divergent series of positive reals, then there exists a sequence $\epsilon_1, \epsilon_2, \ldots$ of positive numbers that converges to zero, but for which $\sum \epsilon_n \cdot a_n$ still diverges.*

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$. We first show $\sum_{k=1}^{\infty}(s_{k+1} - s_k)/s_{k+1}$ diverges. For any $m \in \mathbb{N}$, choose $n \in \mathbb{N}$ such that $s_{n+1} > 2s_m$. Since $\{s_k\}_k^{\infty}$ is non-decreasing,

$$\begin{aligned}\sum_{k=m}^{n} \frac{s_{k+1} - s_k}{s_{k+1}} &\geq \sum_{k=m}^{n} \frac{s_{k+1} - s_k}{s_{n+1}} \\ &= \frac{1}{s_{n+1}}[(s_{m+1} - s_m) + (s_{m+2} - s_{m+1}) + \cdots + (s_{n+1} - s_n)] \\ &= \frac{s_{n+1} - s_m}{s_{n+1}} \\ &> \frac{s_{n+1} - \frac{1}{2}s_{n+1}}{s_{n+1}} = \frac{1}{2}.\end{aligned}$$

Thus, the partial sums of the series $\sum_{k=1}^{\infty} \frac{s_{k+1}-s_k}{s_{k+1}}$ do not form a Cauchy sequence, and so $\sum_{k=1}^{\infty} \frac{s_{k+1}-s_k}{s_{k+1}} = \infty$. Since $s_{k+1} - s_k = a_{k+1}$,

$$\sum_{k=1}^{\infty} \frac{s_{k+1} - s_k}{s_{k+1}} = \sum_{k=2}^{\infty} \frac{a_k}{s_k}.$$

Now let $\epsilon_k = 1/s_k$. Then $\epsilon_k \to 0$ and $\sum_{k=2}^{\infty} \epsilon_k a_k = \infty$. ∎

It is worthwhile to make students realize that there is no specific series that can be used to establish a boundary between the set of all divergent positive-termed series and the set of all convergent series. Readers interested in a more detailed historical development of this topic may wish to consult [**2**].

References

1. R. R. Goldberg, *Methods of Real Analysis*, 2nd. ed., John Wiley & Sons, 1976, 71–72.
2. K. Knopp, *Theory and Application of Infinite Series*, Hafner, 1947, 290–291.

Two Quick Combinatorial Proofs of $\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2$.

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A standard exercise in mathematical induction in many discrete mathematics classes is to prove the identity $\sum_{k=1}^{n} k^3 = n^2(n+1)^2/4$. Alternative proofs are possible that allow this identity to be appreciated from different perspectives. For instance, in [**2**] seven different geometric proofs are presented.

However, since $n^2(n+1)^2/4$ is equal to $\binom{n+1}{2}^2$, it seems only natural that a simple combinatorial proof should be possible. We present two such proofs. Specifically, we find sets S and T where $|S| = \sum_{k=1}^n k^3$ and $|T| = \binom{n+1}{2}^2$, then exhibit a bijection (i.e., a one-to-one, onto function) between them.

Let S denote the set of 4-tuples of integers from 0 to n whose last component is strictly bigger than the others; that is,

$$S = \{(h, i, j, k) \mid 0 \le h, i, j < k \le n\}.$$

For $1 \le k \le n$, there are k^3 ways to choose h, i, j given the last component k. Hence, $|S| = \sum_{k=1}^n k^3$.

Let T denote the set of ordered pairs of two element subsets of $\{0, \ldots n\}$, which may be expressed as

$$T = \{((x_1, x_2), (x_3, x_4)) \mid 0 \le x_1 < x_2 \le n, \quad 0 \le x_3 < x_4 \le n\}.$$

Clearly $|T| = \binom{n+1}{2}^2$.

To see that S and T have the same size, we find a bijection $f : S \to T$ between these sets. Specifically,

$$f((h, i, j, k)) = \begin{cases} ((h, i), (j, k)), & \text{if } h < i \\ ((j, k), (i, h)), & \text{if } h > i \\ ((i, k), (j, k)), & \text{if } h = i \end{cases}$$

is a bijection since the cases $h < i$, $h > i$, and $h = i$ are mapped onto ordered pairs $((x_1, x_2), (x_3, x_4))$ where $x_2 < x_4$, $x_2 > x_4$, and $x_2 = x_4$, respectively. Thus, $|S| = |T|$.

A simpler correspondence arises when we interpret $\binom{n+1}{2}$ as the number of ways to choose two elements from $\{1, \ldots, n\}$ with repetition allowed. This time we let

$$S = \{(h, i, j, k) \mid 1 \le h, i, j \le k \le n\},$$

which has size $|S| = \sum_{k=1}^n k^3$, and let

$$T = \{((x_1, x_2), (x_3, x_4)) \mid 1 \le x_1 \le x_2 \le n, \quad 1 \le x_3 \le x_4 \le n\},$$

which has size $\binom{n+1}{2}^2$. Here, our bijection $g : S \to T$ has just two cases:

$$g((h, i, j, k)) = \begin{cases} ((h, i), (j, k)) & \text{if } h \le i \\ ((j, k), (i, h-1)) & \text{if } h > i. \end{cases}$$

The first case maps onto those $((x_1, x_2), (x_3, x_4))$ where $x_2 \le x_4$, and the second case maps onto those where $x_2 > x_4$. Hence g is a bijection, and $|S| = |T|$.

Another combinatorial approach to this identity is utilized in [**1**] and [**3**] using the set S from our first proof. By conditioning on the number of 4-tuples in S with 2, 3 and 4 distinct elements, it follows that $\sum_{k=1}^n k^3 = \binom{n+1}{2} + \binom{n+1}{3}6 + \binom{n+1}{4}3!$, which algebraically simplifies to $n^2(n+1)^2/4$. Our motivation in this note was to avoid the use of algebra and arrive at $\binom{n+1}{2}^2$ in a purely combinatorial way.

We leave the reader with the challenge of finding a combinatorial proof of

$$\sum_{k=1}^n k^2 = \frac{1}{4}\binom{2n+2}{3}.$$

References

1. George Mackiw, A combinatorial approach to sums of integer powers, *Mathematics Magazine* **73** (2000) 44–46.
2. Roger B. Nelsen, *Proofs Without Words*, Mathematical Association of American, 1993.
3. Marta Sved, Counting and recounting, *The Mathematical Intelligencer* **5** (1983) 21–26.

A Generalization of the Mean Value Theorem for Integrals

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Let $f(x)$ be a continuous function on $[a, b]$. The Mean Value Theorem for Integrals asserts that there is a point c in (a, b) such that $\int_a^b f(x)\,dx = f(c)(b-a)$. Unlike the proof of the Mean Value Theorem for derivatives, the proof of the Mean Value Theorem for Integrals typically does not use Rolle's Theorem. In this note, we use Rolle's Theorem to introduce a generalization of the Mean Value Theorem for Integrals. Our generalization involves two functions instead of one, and has a very clear geometric explanation.

Theorem. If $f(x)$ and $g(x)$ are continuous functions on $[a, b]$, then there is a value c in (a, b) such that

$$\int_a^c f(t)\,dt + \int_c^b g(t)\,dt = f(c)(b-c) + g(c)(c-a).$$

Proof. Let $h(x)$ be the function defined on $[a, b]$ as

$$h(x) = (x-b)\int_a^x f(t)\,dt + (x-a)\int_x^b g(t)\,dt.$$

Since $f(x)$ and $g(x)$ are both continuous on $[a, b]$, the function $h(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Furthermore, $h(a) = 0$ and $h(b) = 0$. So, by Rolle's Theorem, there is a value c in (a, b) such that $h'(c) = 0$. By the Second Fundamental Theorem of Calculus,

$$h'(x) = (x-b)f(x) + \int_a^x f(t)\,dt - (x-a)g(x) + \int_x^b g(t)\,dt,$$

so from $h'(c) = 0$, we have

$$\int_a^c f(t)\,dt + \int_c^b g(t)\,dt = f(c)(b-c) + g(c)(c-a).$$

The geometric interpretation of this theorem (Figure 1) is that the sum of the area under f's graph on $[a, c]$ and the area under g's graph on $[c, b]$ equals the sum of the areas of two rectangles, one with base $[c, b]$ and height $f(c)$ and the other with base $[a, c]$ and height $g(c)$. ∎

Letting $g(x) = f(x)$ and $g(x) = 0$, we get the following respective corollaries.