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THE NORM OF A TRUNCATED TOEPLITZ OPERATOR

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ABSTRACT. We prove several lower bounds for the norm of a truncated Toeplitz operator and obtain a curious relationship between the $H^2$ and $H^\infty$ norms of functions in model spaces.

1. INTRODUCTION

In this paper, we continue the discussion initiated in [6] concerning the norm of a truncated Toeplitz operator. In the following, let $H^2$ denote the classical Hardy space of the open unit disk $D$ and $K_\Theta := H^2 \cap (\Theta H^2)^\perp$, where $\Theta$ is an inner function, denote one of the so-called Jordan model spaces [2, 4, 7]. If $H^\infty$ is the set of all bounded analytic functions on $D$, the space $K^\infty_\Theta := H^\infty \cap K_\Theta$ is norm dense in $K_\Theta$ (see [2, p. 83] or [9, Lem. 2.3]). If $P_\Theta$ is the orthogonal projection from $L^2 := L^2(\partial D, |d\theta|^2)$ onto $K_\Theta$ and $\varphi \in L^2$, then the operator

$$A_\varphi f := P_\Theta(\varphi f), \quad f \in K^\infty_\Theta,$$

is densely defined on $K_\Theta$ and is called a truncated Toeplitz operator. Various aspects of these operators were studied in [3, 5, 6, 9, 10].

If $\|\cdot\|$ is the norm on $L^2$, we let

$$\|A_\varphi\| := \sup\{\|A_\varphi f\| : f \in K^\infty_\Theta, \|f\| = 1\}$$

and note that this quantity is finite if and only if $A_\varphi$ extends to a bounded operator on $K_\Theta$. When $\varphi \in L^\infty$, the set of bounded measurable functions on $\partial D$, we have the basic estimates

$$0 \leq \|A_\varphi\| \leq \|\varphi\|_\infty.$$

However, it is known that equality can hold, in nontrivial ways, in either of the inequalities above and hence finding the norm of a truncated Toeplitz operator can be difficult. Furthermore, it turns out that there are many unbounded symbols $\varphi \in L^2$ which yield bounded operators $A_\varphi$. Unlike the situation for classical Toeplitz operators on $H^2$, for a given $\varphi \in L^2$, there many $\psi \in L^2$ for which $A_\varphi = A_\psi$ [9, Thm. 3.1].

For a given symbol $\varphi \in L^2$ and inner function $\Theta$, lower bounds on the quantity (1) are useful in answering the following nontrivial questions:

(i) is $A_\varphi$ unbounded?

(ii) if $\varphi \in L^\infty$, is $A_\varphi$ norm-attaining (i.e., is $\|A_\varphi\| = \|\varphi\|_\infty$)?

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When $\Theta$ is a finite Blaschke product and $\varphi \in H^\infty$, the paper [6] computes $\|A_{\varphi}\|$ and gives necessary and sufficient conditions as to when $\|A_{\varphi}\| = \|\varphi\|_\infty$. The purpose of this short note is to give a few lower bounds on $\|A_{\varphi}\|$ for general inner functions $\Theta$ and general $\varphi \in L^2$. Along the way, we obtain a curious relationship (Corollary 5) between the $H^2$ and $H^\infty$ norms of functions in $K^\infty_{\Theta}$.

2. Lower bounds derived from Poisson’s formula

For $\varphi \in L^2$, let

$$\mathfrak{P}_\varphi(z) := \int_{\partial D} \frac{1 - |z|^2}{|\zeta - z|^2} \varphi(\zeta) \frac{|d\zeta|}{2\pi}, \quad z \in \mathbb{D},$$

be the standard Poisson extension of $\varphi$ to $\mathbb{D}$. For $\varphi \in C(\partial \mathbb{D})$, the continuous functions on $\partial \mathbb{D}$, recall that $\mathfrak{P}_\varphi$ solves the classical Dirichlet problem with boundary data $\varphi$. Also note that

$$k_\lambda(z) := \frac{1 - \Theta(\lambda)\Theta(z)}{1 - \lambda z}, \quad \lambda, z \in \mathbb{D},$$

is the reproducing kernel for $K^\infty_{\Theta}$ [9].

Our first result provides a general lower bound for $\|A_{\varphi}\|$ which yields a number of useful corollaries:

**Theorem 1.** If $\varphi \in L^2$, then

$$\sup_{\lambda \in \mathbb{D}} \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} \left| \int_{\partial \mathbb{D}} \varphi(z) \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda} \frac{|dz|}{2\pi} \right| \leq \|A_{\varphi}\|.$$  (3)

In other words,

$$\sup_{\lambda \in \mathbb{D}} \left| \int_{\partial \mathbb{D}} \varphi(z) d\nu_\lambda(z) \right| \leq \|A_{\varphi}\|$$

where

$$d\nu_\lambda(z) := \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} \left| \Theta(z) - \Theta(\lambda) \right|^2 \frac{|dz|}{2\pi}$$

is a family of probability measures on $\partial \mathbb{D}$ indexed by $\lambda \in \mathbb{D}$.

**Proof.** For $\lambda \in \mathbb{D}$ we have

$$\|k_\lambda\| = \sqrt{\frac{1 - |\Theta(\lambda)|^2}{1 - |\lambda|^2}},$$  (4)

from which it follows that

$$\|A_{\varphi}\| \geq \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} |\langle A_{\varphi}k_\lambda, k_\lambda \rangle|$$

$$= \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} |\langle \mathfrak{P}_\Theta \varphi k_\lambda, k_\lambda \rangle|$$

$$= \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} |\langle \varphi k_\lambda, k_\lambda \rangle|$$

$$= \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} \left| \int_{\partial \mathbb{D}} \varphi(z) \left| \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda} \right|^2 \frac{|dz|}{2\pi} \right|.$$

That the measures $d\nu_\lambda$ are indeed probability measures follows from (4).  \(\square\)
Now observe that if $\Theta(\lambda) = 0$, the argument in the supremum on the left hand side of (3) becomes the absolute value of the expression in (2). This immediately yields the following corollary:

**Corollary 1.** If $\varphi \in L^2$, then

$$
\sup_{\lambda \in \Theta^{-1}\{0\}} |(\Psi \varphi)(\lambda)| \leq \|A_\varphi\|,
$$

where the supremum is to be regarded as 0 if $\Theta^{-1}\{0\} = \emptyset$.

Under the right circumstances, the preceding corollary can be used to prove that certain truncated Toeplitz operators are norm-attaining:

**Corollary 2.** Let $\Theta$ be an inner function having zeros which accumulate at every point of $\partial \mathbb{D}$. If $\varphi \in C(\partial \mathbb{D})$ then $\|A_\varphi\| = \|\varphi\|_{\infty}$.

**Proof.** Let $\zeta \in \partial \mathbb{D}$ be such that $|\varphi(\zeta)| = \|\varphi\|_{\infty}$. By hypothesis, there exists a sequence $\lambda_n$ of zeros of $\Theta$ which converge to $\zeta$. By continuity, we conclude that

$$
\|\varphi\|_{\infty} = \lim_{n \to \infty} |(\Psi \varphi)(\lambda_n)| \leq \|A_\varphi\| \leq \|\varphi\|_{\infty}
$$

whence $\|A_\varphi\| = \|\varphi\|_{\infty}$. \qed

The preceding corollary stands in contrast to the finite Blaschke product setting. Indeed, if $\Theta$ is a finite Blaschke product and $\|\varphi\|_{\infty} \leq 1$, then it is known that $\|A_\varphi\| = \|\varphi\|_{\infty}$ if and only if $\varphi$ is the scalar multiple of the inner factor of some function from $K_\Theta$ [6, Thm. 2].

At the expense of wordiness, the hypothesis of Corollary 2 can be considerably weakened. A cursory examination of the proof indicates that we only need $\zeta$ to be a limit point of the zeros of $\Theta$, $\varphi \in L^\infty$ to be continuous on an open arc containing $\zeta$, and $|\varphi(\zeta)| = \|\varphi\|_{\infty}$.

Theorem 1 yields yet another lower bound for $\|A_\varphi\|$. Recall that an inner function $\Theta$ has a finite angular derivative at $\zeta \in \partial \mathbb{D}$ if $\Theta$ has a non-tangential limit $\Theta(\zeta)$ of modulus one at $\zeta$ and $\Theta' \Theta$ has a finite non-tangential limit $\Theta'(\zeta)$ at $\zeta$. This is equivalent to asserting that

$$
\frac{\Theta(z) - \Theta(\zeta)}{z - \zeta}
$$

has the non-tangential limit $\Theta'(\zeta)$ at $\zeta$. If $\Theta$ has a finite angular derivative at $\zeta$, then the function in (6) belongs to $H^2$ and

$$
\lim_{r \to 1^-} \int_{\partial \mathbb{D}} \left| \frac{\Theta(z) - \Theta(r \zeta)}{z - r \zeta} \right|^2 \frac{|dz|}{2\pi} = \int_{\partial \mathbb{D}} \left| \frac{\Theta(z) - \Theta(\zeta)}{z - \zeta} \right|^2 \frac{|dz|}{2\pi}.
$$

Furthermore, the above is equal to

$$
\lim_{r \to 1^-} \frac{1 - |\Theta(r \zeta)|^2}{1 - r^2} = |\Theta'(\zeta)| > 0.
$$

See [1, 8] for further details on angular derivatives. Theorem 1 along with the preceding discussion and Fatou's lemma yield the following lower estimate for $\|A_\varphi\|$.

**Corollary 3.** For an inner function $\Theta$, let $D_\Theta$ be the set of $\zeta \in \partial \mathbb{D}$ for which $\Theta$ has a finite angular derivative $\Theta'(\zeta)$ at $\zeta$. If $\varphi \in L^\infty$ or if $\varphi \in L^2$ with $\varphi \geq 0$, then

$$
\sup_{\zeta \in D_\Theta} \frac{1}{|\Theta'(\zeta)|} \int_{\partial \mathbb{D}} \varphi(z) \left| \frac{\Theta(z) - \Theta(\zeta)}{z - \zeta} \right|^2 \frac{|dz|}{2\pi} \leq \|A_\varphi\|.
$$
In other words,
\[ \sup_{\zeta \in D_\Theta} \left| \int_{\partial D} \varphi(z) d\nu_\lambda(z) \right| \leq \|A\varphi\|, \]
where
\[ d\nu_\lambda(z) := \frac{1}{|\Theta'(\zeta)|} \left| \frac{\Theta(z) - \Theta(\zeta)}{z - \zeta} \right| \frac{|dz|}{2\pi} \]
is a family of probability measures on \( \partial D \) indexed by \( \zeta \in D_\Theta \).

3. LOWER BOUNDS AND PROJECTIONS

Our next several results concern lower bounds on \( \|A\varphi\| \) involving the orthogonal projection \( P_\Theta : L^2 \to K_\Theta \).

**Theorem 2.** If \( \Theta \) is an inner function and \( \varphi \in L^2 \), then
\[ \frac{\|P_\Theta(\varphi) - \overline{\Theta(0)}P_\Theta(\Theta\varphi)\|}{(1 - |\Theta(0)|^2)^{1/2}} \leq \|A\varphi\|. \]

**Proof.** First observe that \( \|k_0\| = (1 - |\Theta(0)|^2)^{1/2} \). Next we see that if \( \varphi \in L^2 \) and \( g \in K_\Theta \) is any unit vector, then
\[ (1 - |\Theta(0)|^2)^{1/2} \|A\varphi\| \geq |\langle A\varphi k_0, g \rangle| \]
\[ = |\langle P_\Theta(\varphi k_0), g \rangle| \]
\[ = |\langle P_\Theta(\varphi) - \overline{\Theta(0)}P_\Theta(\Theta\varphi), g \rangle|. \]

Setting
\[ g = \frac{P_\Theta(\varphi) - \overline{\Theta(0)}P_\Theta(\Theta\varphi)}{\|P_\Theta(\varphi) - \overline{\Theta(0)}P_\Theta(\Theta\varphi)\|} \]
yields the desired inequality. \( \square \)

In light of the fact that \( P_\Theta(\Theta\varphi) = 0 \) whenever \( \varphi \in H^2 \), Theorem 2 leads us immediately to the following corollary:

**Corollary 4.** If \( \Theta \) is inner and \( \varphi \in H^2 \), then
\[ \frac{\|P_\Theta(\varphi)\|}{(1 - |\Theta(0)|^2)^{1/2}} \leq \|A\varphi\|. \] (7)

It turns out that (7) has a rather interesting function-theoretic implication. Let us first note that for \( \varphi \in H^\infty \), we can expect no better inequality than
\[ \|\varphi\| \leq \|\varphi\|_\infty \]
(with equality holding if and only if \( \varphi \) is a scalar multiple of an inner function). However, if \( \varphi \) belongs to \( K_\Theta^\infty \), then a stronger inequality holds.

**Corollary 5.** If \( \Theta \) is an inner function, then
\[ \|\varphi\| \leq (1 - |\Theta(0)|^2)^{1/2} \|\varphi\|_\infty \] (8)
holds for all \( \varphi \in K_\Theta^\infty \). If \( \Theta \) is a finite Blaschke product, then equality holds if and only if \( \varphi \) is a scalar multiple of an inner function from \( K_\Theta \).
Proof. First observe that the inequality
\[ \| \varphi \| \leq (1 - |\Theta(0)|^2)^{1/2} \| \varphi \|_{\infty} \]
follows from Corollary 4 and the fact that \( P_\Theta \varphi = \varphi \) whenever \( \varphi \in K_\Theta \). Now suppose that \( \Theta \) is a finite Blaschke product and assume that equality holds in the preceding for some \( \varphi \in K_\Theta^\infty \). In light of (7), it follows that \( \| A_\varphi \| = \| \varphi \|_{\infty} \).

From [6, Thm. 2] we see that \( \varphi \) must be a scalar multiple of the inner part of a function from \( K_\Theta \). But since \( \varphi \in K_\Theta^\infty \), then \( \varphi \) must be a scalar multiple of an inner function from \( K_\Theta \).

□

When \( \Theta \) is a finite Blaschke product, then \( K_\Theta \) is a finite dimensional subspace of \( H^2 \) consisting of bounded functions [3, 5, 9]. By elementary functional analysis, there are \( c_1, c_2 > 0 \) so that
\[ c_1 \| \varphi \| \leq \| \varphi \|_{\infty} \leq c_2 \| \varphi \| \]
for all \( \varphi \in K_\Theta \). This prompts the following question:

**Question.** What are the optimal constants \( c_1, c_2 \) in the above inequality?

4. Lower bounds from the decomposition of \( K_\Theta \)

A result of Sarason [9, Thm. 3.1] says, for \( \varphi \in L^2 \), that
\[ A_\varphi \equiv 0 \iff \varphi \in \Theta H^2 + \overline{\Theta H^2}. \]  
(9)
It follows that the most general truncated Toeplitz operator on \( K_\Theta \) is of the form \( A_{\psi, \chi} \) where \( \psi, \chi \in K_\Theta \). We can refine this observation a bit further and provide another canonical decomposition for the symbol of a truncated Toeplitz operator.

**Lemma 1.** Each bounded truncated Toeplitz operator on \( K_\Theta \) is generated by a symbol of the form
\[ \varphi = \underbrace{\psi \Theta}_{\in H^2} + \underbrace{\chi \overline{\Theta}}_{\in \overline{zH^2}} \]  
(10)
where \( \psi, \chi \in K_\Theta \).

Before getting to the proof, we should remind the reader of a technical detail. It follows from the identity \( K_\Theta = H^2 \cap \Theta \overline{zH^2} \) (see [2, p. 82]) that
\[ C : K_\Theta \to K_\Theta, \quad Cf := \overline{zf}_\Theta, \]
is an isometric, conjugate-linear, involution. In fact, when \( A_\varphi \) is a bounded operator we have the identity \( CA_\varphi C = A_\varphi^* \) [9, Lemma 2.1].

**Proof of Lemma 1.** If \( T \) is a bounded truncated Toeplitz operator on \( K_\Theta \), then there exists some \( \varphi \in L^2 \) such that \( T = A_\varphi \). We claim that this \( \varphi \) can be chosen to have the special form (10). First let us write \( \varphi = f + zg \) where \( f, g \in H^2 \). Using the orthogonal decomposition \( H^2 = K_\Theta \oplus \Theta H^2 \), it follows that \( \varphi \) may be further decomposed as
\[ \varphi = (f_1 + \Theta f_2) + z(g_1 + \Theta g_2) \]
where \( f_1, g_1 \in K_\Theta \) and \( f_2, g_2 \in H^2 \). By (9), the symbols \( \Theta f_2 \) and \( \overline{\Theta(zg_2)} \) yield the zero truncated Toeplitz operator on \( K_\Theta \). Therefore we may assume that
\[ \varphi = f + zg \]
for some \( f, g \in K_\Theta \). Since \( Cg = \overline{gz}_\Theta \), we have \( zg = (Cg)\overline{\Theta} \) and hence (10) holds with \( \psi = f \) and \( \chi = Cg \).

□
Corollary 6. Let $\Theta$ be an inner function. If $\psi_1, \psi_2 \in K_\Theta$ and $\varphi = \psi_1 + \psi_2\overline{\Theta}$, then
\[ \|\psi_1 - \frac{\Theta(0)}{1 - |\Theta(0)|^2} \psi_2\| \leq \|A_\varphi\| . \]

Proof. If $\varphi = \psi_1 + \psi_2\overline{\Theta}$, then, since $\psi_1, \psi_2 \in K_\Theta$ and $\psi_2\overline{\Theta} \in \mathbb{zH}^2$, we have
\[ P_\Theta(\varphi) - \frac{\Theta(0)}{1 - |\Theta(0)|^2} P_\Theta(\Theta \varphi) = \psi_1 - \frac{\Theta(0)}{1 - |\Theta(0)|^2} \psi_2. \]
The result now follows from Theorem 2. $\square$

References


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