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Ways of Relating to the Mathematics of the Past

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Ways of Relating to the Mathematics of the Past[1]

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Abstract

Historians of mathematics, by definition, look at mathematics of the past. But mathematicians, too, often look at mathematics of the past; mathematicians of the past themselves often looked very closely at mathematics of their own past. Is their relationship to the past the same as that of the historians? Is every view of the past an historical view? Indeed, is every historical view historical in the same way? Or is it possible that there are different kinds of relationships to the mathematics of the past? This paper will suggest that there are in fact a variety of such relationships. It will try to catalog some of these, without judgment as to whether they are necessarily correct or legitimate. It will also raise the question as to whether mathematics educators interested in the history mathematics have their own distinct relationship with the mathematics of the past or are aligned with one type or another.

Introduction

I would like to begin this piece autobiographically. When I started to work on the history of mathematics, I confess I did not give much thought as to the nature of the subject. I liked mathematics and I liked history. I liked biography and I also liked the Greeks. So I thought it might be fun to study a mathematician like Apollonius of Perga. I worked with the historian of mathematics Sabetai Unguru. Because of his own work and his own example,

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I very quickly began pondering what it really meant to do the history of mathematics and what it meant to be an historian of mathematics. By the time I finished my Ph.D., I could make some distinctions: I could divide historians of mathematics into a mathematician type, such as Zeuthen or van der Waerden, a historian type, like Sabetai Unguru, and, perhaps, a postmodern type (see [28]).

These distinctions are useful, but they are also quite crude, especially since the last, the postmodern type, is not in fact a serious option. Later, when I did some work on Edmond Halley's reconstruction of Apollonius' last book of the *Conics*, I realized that a distinction only between a "mathematician type" and "historian type" could not be the whole picture. In working on Halley's reconstruction, I had to ask what it meant for Halley in 1710, when powerful new mathematical tools were being developed, tools which Halley himself had a part in making and certainly mastered—what it meant *then* for Halley to turn his attention to a work of *ancient* mathematics. Although his text was from the past, it was not clear Halley's endeavor could be characterized as purely historical in the modern sense; nor could it be termed as completely unhistorical, like the much earlier attempt to reconstruct *Conics*, Book VIII by the great Ibn al-Haytham. The more I considered the question the more it became evident not so much that Halley's way of treating ancient Greek mathematical works was *sui generis*—for it was not—but simply that there are many different kinds of relationships to mathematics of the past.

The main goal of this article is to chart out these many relationships. Of course I cannot discuss in depth all the ways one can stand with respect to the mathematics of the past and still keep to a reasonable length. But what I would like to do is to list some of the more important of them, almost as an expanded table of contents. I hope that by doing so I will at least make the value of such a typology of relationships plausible. In particular, I would like to suggest how this kind of typology can form a natural bridge between historiography and history, clarifying on the one hand ways in which one can treat mathematics of the past, while, on the other, clarifying how mathematicians of the past viewed their own sources and their own position with respect to them. As for the latter, the historical side of the equation, I should emphasize I am not looking at the actual relationship between mathematicians and their sources—a standard task for historians of mathematics. What I am interested in is how those who have concerned themselves with mathematics one way or the other might have viewed the past: I grant the line between how texts of the past were used by such people and how they stood with respect to the past can be quite fine, but I believe there is indeed a line.

Beyond the immediate goal of setting out this catalog, this paper also has an educational undercurrent. As one interested in the role of history of mathematics in mathematics education, I have long been preoccupied with the questions: What does the mathematics of the past mean to students and what *can* it mean to them? What should our relationship to the past, as educators, be? Is there a distinct relationship to our mathematical past that might be termed an educational relationship? And, if so, does it sit beside the others as an equal, or perhaps, does it need to be represented on a different axis, a different dimension of the relationships, to use a somewhat hackneyed expression? These are fundamental questions having to do with history of mathematics in mathematics education. Such fundamental issues have been discussed in mathematics education, but hardly enough (see [8] for a review). They are among the issues in the background of the present piece, as I have said, but naturally it would take us too far afield to discuss them at any length or depth here; still, I will describe one possible educational relationship—not necessarily one that I advocate, but one that can focus questions about an educational type. It may be that our job as educators is not to settle upon a fixed educational type, but to assure that questions like those above remain open enough so that our students may honestly consider alternatives and, ultimately, develop their own relationship to the past.

Historical and non-historical postures towards mathematics of the past

Looking over the totality of relationships or postures towards mathematics of the past, the first and most basic division one should make is between those that are historical and those that are non-historical. It is, actually, not always an easy distinction to make, especially when it comes to mathematics. There is, first, the seductive tendency to think of mathematics, at bottom, as ahistorical, that its true core contains ideas unconditioned by time or place or language. Then there is the seemingly commonsense view that if one speaks about the past at all one is *ipso facto* engaging in history. The former concerns the nature of mathematics; the latter, the more difficult for us, concerns the nature of history.

In this connection, the philosopher and historian Michael Oakeshott's view of the historical mode of experience useful. Oakeshott makes the point that although "History is certainly a form of experience in which what is experienced is, in some sense, past ... history is not the only past, and a clear view of the character of the past in history involves the distinction of this past from that in other forms of experience" [21, page 102]. The past that, in Oakeshott's view, stands in clearest contrast to the historical past is what he calls the "practical past" (page 103). This past, like the historical past, is an experience in the present, unavoidably; however, it is a past that is *for the sake* of the present. One might say that the subject of the "practical past" is the present even while it refers to the past. The subject of the historical past, however, is the past as distinguished from the present, the past in its own particularity (page 106). The "practical past" sounds very much like what Herbert Butterfield called "Whiggism" [1], but, unlike Butterfield who rejects the Whig interpretation as illegitimate, *tout court*, Oakeshott is willing to see the "practical past" as simply different from the "historical past". It is this aspect of Oakeshott's view that makes it particularly appropriate for thinking about relationships towards mathematics of the past, for it allows us to see the historical and the non-historical as two reference points for locating these relationships without judging them as necessarily legitimate or illegitimate.

I ought to mention in passing another similar distinction proposed by Ivor Grattan-Guinness [11, 12] specifically regarding mathematics, namely, that between "history" and "heritage". These are distinguished by their guiding questions: for history it is, "What happened?" or "Why did N happen?"; for heritage the question is "How did we get here?", where the answer, Grattan-Guinness playfully points out, is more often than not via, "The royal road to me". Like Oakeshott, Grattan-Guinness wants to make the point that referring to heritage as if it were history or history as if it were heritage is to fall into the trap of *ignoratio elenchi*, as Oakeshott would have put it. There is no doubt Grattan-Guinness's history-heritage dichotomy can be a useful tool for analyzing how mathematics of the past is treated; however, for my own purposes, it is not broad enough and, more importantly, it does not bring out explicitly enough how these different approaches to the past are truly different views of the past itself and place one in different relations to the past. For this reason, in my book on Apollonius with Sabetai Unguru, we spoke about going through a historical door or a mathematical door [4, page 404]. Each door leads into a very different world: "The mathematical and the historical approaches are antagonistic. Whoever breaks and enters typically returns from his escapades with other spoils than the peaceful and courteous caller" (page 406).

Mathematicians, Mathematician-Historians, Historians of Mathematics

But in dividing the historical and non-historical relationship with the past, we ought to take care not to make the discussion only one of good and bad history. One can enter the world of past mathematics with no real intention to interpret the past, that is, with no real intention of doing something like history; rather, returning to Oakeshott's "practical-past", a mathematician can see mathematics of the past as a resource, referring to it as one might refer to a past issue of a contemporary journal. For such a mathematician, the past is past, but only in name, something incidental. On the other hand, those who come through the mathematics door, yet who see themselves investigating history, in some way may live in the practical past, but they have a different relation to the past than those who use the past directly for mathematical work.

With that, we can see three initial categories with respect to the basic division between non-historical and historical postures towards mathematics of the past. Towards the non-historical pole, we have what I shall call simply "mathematicians", for they are just that, people who see themselves doing mathematics, not history. Towards the historical pole, we have "historians of mathematics", for these see themselves doing history and their mode of experiencing the past is historical in the Oakeshottian sense; that is, they relate to the past as something utterly apart from the present, the past as a problem. Ranging the middle we have "mathematician historians", for these are generally mathematicians who see themselves engaged in an historical enterprise and yet to a greater or lesser degree (and there are many such degrees!) see a continuity between the mathematics of the past and their own mathematical work.

Before discussing these categories and their subtypes, I should set out some caveats. First, what I call "mathematician" refers only to a type of relationship to the past: to be included in this type does not require one to be a mathematician in the usual sense, nor does it mean that if one is a mathematician by training one is necessarily a "mathematician" by type. Second, these types, in general, should not be viewed in an absolute way. They are, rather, like Max Weber's "ideal types"—merely means of analyzing relationships: no mathematician, including those I have chosen as illustrations, is completely summed up one category or another. Mathematicians, like all human beings, are complicated creatures! Third, although I follow a more or less chronological pattern, "mathematicians", "mathematician-historians", and "historians of mathematics" are not necessarily historical categories: one can be a "mathematician historian" today as well as in the 18^{th} century. Fourth, in most of the examples the mathematical past being considered by one person or another is that of Greece. This is as natural here as it would be were we discussing the history of philosophy. Nevertheless, as I hope will be clear, the relationships evoked in the context of these examples have little to do with the particular character of Greek mathematics: what should stand out is more the qualities of the beholder than the beheld. Finally, this whole piece has been written in a light and playful spirit. It should be read that way, though one should also keep in mind Plato's dictum that "playfulness and seriousness are sisters".

Mathematicians

With these caveats out of the way, let me begin with "mathematicians", of which I want to distinguish three subtypes.

The first includes what I call "mathematical colleagues". In this case, figures in the past are viewed as if they were contemporaries working in the same field and working, fundamentally, in the same way. Thus one feels fully justified referring to mathematicians of the past as Littlewood famously said of the Greek mathematicians, namely, as "Fellows of another college" (quoted in [13, page 81]); mathematicians of the past, like one's colleagues, are useful for gaining insights into one's present mathematical research. This is the kind of relationship one sees in Apollonius' references to Euclid and Pappus' references to Apollonius. The example of Apollonius is less problematic than that of Pappus because of the relatively small amount of time separating Apollonius and Euclid, probably somewhat more than a half a century. However, whatever chronology one chooses to use, it is safe to assume that they were not contemporaries. Yet, when Apollonius refers to Euclid's attempts to solve the locus problem, Apollonius does not see himself having the benefit of modern methods but only of more powerful propositions, which he himself discovered. In other words, Apollonius views Euclid working on the same problems and within the same basic framework as he is: his criticism of Euclid is a little like the criticism of colleagues in the literature review of a research paper. Apollonius's "historical" comments then, are really just a way of presenting his own accomplishments and originality within the same context as Euclid. Indeed, Jaap Mansfeld [20], discussing the nature of Greek mathematical introductions, makes the point several times that "historical" remarks are a routine part of setting out the *skopos* or theme of the work. In this light too, one must view Pappus's comments at the start of Book VII of the *Collection* where he castigates Apollonius for not giving Euclid enough credit in connection to the locus problem as a way of clarifying the theme of Pappus's own present work, even though it appears to be a matter of historical judgment. After all, Pappus's purposes were not historical: he was expounding a body of knowledge for the benefit of "... those who want to acquire a power in geometry that is capable of solving problems set to them" (translation from [15, page 82]). Euclid and Apollonius are the central figures for Pappus's non-historical project.

The next type, "mathematical treasure hunters", is very similar to the first in that such see themselves working within the same framework and on the same type of problems as the mathematicians of the past to which they refer. The difference stems from the great span of time separating them from the latter. The effect of this span of time is that the mathematics of the past is for them to some extent lost and needs to be found or recovered. This type actually can be divided further, for there are those that see themselves as somehow inferior to the mathematicians of the past and those who see themselves as their equals. Both, however, seek to find lost treasures. But the word "treasure" needs to be qualified, for it must not be thought of as simply something one happens upon. A mathematical treasure is a thing to be understood; effort is required from the treasure hunter to piece together the lost mathematical text. Thus, mathematical treasure hunters see themselves continuing or completing the work of the ancients, whether or not this is truly the case, as Sabra discussed in his famous paper on Islamic science [22].

In fact, it is certain streams of Islamic mathematics I have particularly in mind here, and the image of "treasure hunters" actually comes from imagining those 9^{th} century brothers, the Banū Mūsā, searching the world for ancient mathematical manuscripts. But a more nuanced example is Ibn al-Haytham (965–1040), at least with respect to his reconstruction of Apollonius's *Conics*, Book VIII, mentioned earlier. Al-Haytham called his work a *Completion of the Conics* because, as he puts it in his introduction, "When we studied this work [the *Conics*], investigated the notions in it, and went through the seven books many times, we found that it lacked notions, which this work *should not leave untreated* [my emphasis]" (translation from [14, page 134]). The argument for the rationale of the reconstruction was, in effect, that these missing notions were worthy of Apollonius, so that not appearing in the extant books of the *Conics*, they had to have appeared in the lost book. What is important though is that Ibn al-Haytham could consider himself able to judge what was worthy of Apollonius because he saw himself as a fellow mathematician and one whose own thoughts about conics were consistent with any of those Apollonius might have thought. As Hogendijk says, "Ibn al-Haytham's supposition [regarding the "notions" he judged to have been necessarily included in Book VIII] is that Apollonius gave a complete treatment of certain classes of related problems. It seems to me that he based this supposition not on evidence in the Conics, but only on his implicit assumption that Apollonius's interests were identical to his own" [14, page 69]. Like the "mathematician" type, then, Ibn al-Haytham views Apollonius's and his own mathematical thought as coterminous intellectually. So, while "mathematical treasure hunters" may be looking back at a work of the past, they use it with an eye to producing a work on the frontier of new knowledge; the antiquity of original works is an almost incidental matter, except, perhaps, that one's present work *might* have been done previously before being lost; one might as well refer to the original works as the precious manuscripts of a brilliant colleague who has died and whose work was lost in a fire.

The next type, which I call "mathematical conquerors", is in the transitional area between "mathematicians" and "mathematician-historians". Unlike the "mathematical treasure hunters" who tend to feel either inferior or equal to the mathematicians of the past, "mathematical conquerors" see themselves as equal or superior to those ancient mathematicians. They see the mathematics of the past as an opportunity to highlight their own originality and power. More than an isolated theorem or set of theorems, they see themselves possessing general methods and approaches that allow them to open doors the ancients left shut: their power is conceived as power *over* the mathematicians of the past.

René Descartes (1596–1650), as one might guess, is a perfect example of this type. When he says there were "traces" (*vestigia*) of a general method in Pappus and Diophantos that they hid by "a certain pernicious cunning" (*perniciosa quadam astutitia*) (*Regulae ad Directionem Ingenii*, Reg. IV.5, see [3, page 376]), this is Descartes the "mathematical conqueror" speaking. The ancients may have possessed a general method, but it is Descartes himself who is cognizant of the true depth of the method, that *mathesis universalis*, which he identifies with algebra. He has no need to hide the method jealously, for the things revealed in his world are more potent than those of "that unsophisticated and innocent ancient time" (*rudi ista et pura antiquitate*). This sense of possessing a key which no locked door can resist is also behind the flurry of reconstructions at the beginning of the 17$^{\text{th}}$ century (I might add that one could explore the typology I have been discussing via the variety of reconstructive efforts in the history of mathematics and their motivations). Examples include François Viète's reconstruction of Apollonius's *On Tangencies*, his *Apollonius Gallus* of 1600, and Pierre de Fermat's reconstruction of Apollonius's *Plane Loci*, which he worked on from about 1628 to 1636. Regarding the latter, Michael Mahoney summed up the situation as follows:

> Fermat was no antiquarian interested in a faithful reproduction of Apollonius' original work; he was a working mathematician seeking to ferret out the analytic techniques he felt Apollonius had hidden. The *Plane Loci* was to serve as a means to an end rather than an end in itself [19, page 96].

But again, I want to emphasize that in "... ferret[ing] out the analytic techniques he felt Apollonius had hidden ... ", Fermat and others of his type were actually showing their own possession of powerful mathematical methods. In this sense, these reconstructions at the start of the 17$^{\text{th}}$ century were in fact pressing the development of mathematics forward.

The "mathematical conquerors" are, as I have said, a transitional category, not in time but in type. They are located in the general category of "mathematicians" because they see themselves engaged in an enterprise meant to further the development of the methods and ideas that they themselves are exploring in their own mathematical work. Undeniably, though, they also have a sense of the past and feel they are explaining the past. On the other hand, their sense of the past has the unambiguous character of a "practical past", again to use Oakeshott's term.

Mathematician-Historians

"Mathematician-historians", the next basic category, is perhaps the hardest to describe. These are generally well-trained mathematicians but who stand apart from "mathematicians" because of their fuller sense of the mathematics of the past as *past*. The mathematics of the past is still understood by them as continuous with present mathematics; however, it may be looked upon dispassionately because progress in contemporary mathematics does not *require* them to look back to the past. "Mathematician-historians", accordingly, do not see their mathematical work as utterly dependent on their understanding of past mathematical work; on the other hand, they see that mathematics as a discipline and, more pointedly, their own identities as mathematicians are elucidated by such understanding. For this reason, they do have a foot planted in the direction of history and deserve the word "historian" in their description—some to a greater degree than others. As with "mathematicians", there are three subtypes in this case.

The first subtype, furthest from the historical pole, I call, "moderators". Like all other types in this category, "moderators" are completely mindful of the advantages of the mathematics of their own day over older mathematics; they no longer need to prove the potency of algebraic methods, for example. However, they have studied older, usually classical works of mathematics, know them well, and respect their authors. They are not interested, accordingly, in conquering the mathematics of the past—for they no longer need to—but to moderate a conversation, as it were, between ancients and moderns.

It is in this group I place Edmond Halley (1656–1742), and I take him as my main example, though he is by no means the only figure who could serve the purpose: probably Barrow could also be considered "moderator", maybe even Newton. The time was right for such "moderators", for the battle between the "ancients and moderns", as immortalized in Swift's famous satire "The Battle of the Books", was raging, and where there are battles there will eventually be moderators. Halley's official appointment as Savilian Professor of Geometry at Oxford put him in the position of being a "moderator", since the professorship, besides the usual duties of a mathematics professor, required Halley to lecture on Euclid, Archimedes, and Apollonius. But Halley took up these responsibilities, it seems, willingly and not as a chore. Throughout his life he took great delight in classical works and in history, especially when he could relate his scientific knowledge to them (see [7, 2]). His identity as a moderator can be seen very clearly in the preface to his earlier translation of Apollonius's *Cutting-Off of a Ratio* and reconstruction of *Cutting-Off of an Area* (1706), which he opened by extolling the modern achievement of the "Algebra of Species", the "Arithmetic of Infinitesimals", and the "Fluxions", referring to the works of Viète, Wallis, and Newton, but then continued by urging that this should not in any way lessen the glory of the ancients who brought geometry to perfection (... *qui Geometriam ad eam provexere perfectionem*). This role as "moderator" can be seen also in the actual reconstructions themselves. In his reconstruction of *Conics* VIII, for example, he adopts Apollonius's voice as best as he can in the statements of the problems and their proofs; however, having completed the problem as he supposes Apollonius might have done, he adds his own solution. Thus Apollonius's problem 7 in Halley's reconstruction of *Conics*, Book VIII, reads:

> Given the axis and the latus rectum of the axis of a hyperbola, and given the ratio of conjugate diameters of the section, find the conjugate diameters *both in magnitude and in position* [my emphasis].

But then, having presented his proposal for Apollonius's solution, Halley adds:

> Since the difference between the squares on the conjugate diameters is always equal to the difference between the squares on the axes, though, we can give this solution to the problem in a fairly expedient way (*modo satis expedito*), *but without the position of the diameters* [my emphasis].

As in this case, these additions almost always emphasize the magnitudes of lines only and not their position, that is, they emphasize aspects of the problem concerning relationships of quantities alone and thus lend themselves to the kind of analytic tools of his own modern mathematical world. There is no hint that these alternatives are meant to show how a modern like himself can outdo Apollonius: it is a kind of dialogue.

The next two subtypes are closely related. They are the "privileged observers" and "mathematical critics". What distinguishes both of these subtypes is that they not only deem their modern mathematical knowledge to be superior to the mathematics of the past but also believe it provides them special power in interpreting the past. In this way, they have a certain kinship with "mathematical conquerors"; however, their end is not to *prove* the superiority of their mathematical ideas but to take advantage of them to piece together the past. For the "privileged observers" the latter is the principal goal. For the "mathematical critics", understanding is not enough: one must also show how mathematics of the past has made a positive contribution to present day mathematical truth and how it has not, where it was right and where it was wrong, by modern standards.

I take Hieronymus Georg Zeuthen (1830–1920) as a good representative of the subtype "privileged observers", although I could have as easily chosen André Weil or Bartel Leendert van der Waerden. About Zeuthen, Lützen and Purkert say that he "... always stressed that he made his contributions to this field [history of mathematics] not as a historian but as a professional mathematician" [18, page 14]. In his most well-known historical work, *Die Lehre von den Kegelschnitten im Altertum* (1886), accordingly Zeuthen makes it clear that it is indeed his modern methods, his being a modern mathematician, that provides him with a privileged standpoint for understanding Apollonius's *Conics*, the main subject of his 1886 work. Thus he writes that a proper view of the work can be achieved only "... by employing modern means of representation", which, unfortunately, "... the exclusive preoccupation of the ancients with logical completeness conceals" [29, page *xii*]. But Zeuthen, by saying this, was not trying to make a case for his modern means of representation, that is, he was not using history as a vehicle for advancing an area of mathematics as would a "mathematical conqueror"; on the contrary, there is no reason to believe that Zeuthen ever thought he was doing anything but history.

An example of a "mathematical critic" is Clifford Truesdell (1919–2000). As an historian, Truesdell brought to his work immense and exacting mathematical and scientific insight, but his approach was generally to show where Euler, Lagrange, the Bernoullis, and the others he studied got it right and where they got it wrong—and right and wrong, in his view, were to be taken as absolutes: the same today as yesterday. This sounds like the subtype I called a "mathematical colleague"; I might have categorized Truesdell this way had he not presented what he was doing as *history* and distinguishable from his purely scientific work. Yet here we see how unclear the picture can be. For example, about history Truesdell has written:

> One of the main functions [the history of mathematical science] should fulfill is to help scientists understand some aspects of specific areas of mathematics about which they still don't fully know. What's more important, it helps them too. By satisfying their natural curiosity, typically present in everybody towards his or her own forefathers, it helps them indeed to get acquainted with their ancestors in spirit. As a consequence, they become able to put their efforts into perspective and, in the end, also able to give those efforts a more complete meaning" (in [10, page 21]).

Nothing could be a clearer picture of Oakeshott's "practical past" (or better Grattan-Guinness's "heritage") than that, and it is a picture that comes very close to the "mathematician" category of relationships with mathematics of the past. The case of Truesdell, thus, is also a good opportunity to remind the reader that the categories and subtypes I have described must be taken only as "ideal types", and not as complete descriptions of particular figures.

Historians of Mathematics

This brings us to the last category along the scale from non-historical and historical postures towards the past. That "historians of mathematics" is a separate category and just not another subtype of the last is of course a matter of controversy. A critical turn in the controversy was Sabetai Unguru's 1975 paper "On the Need to Rewrite the History of Greek Mathematics" [26], where among other things Unguru challenged the interpretation of ancient mathematics by means of modern mathematical tools and the claim "mathematician-historians" could make, on that basis, to their pursuing true historical work. For their part, by assuming a complete continuity between the mathematics of the past and that of present, it would stand to reason that a mathematician is best placed to understand and interpret the mathematics of the past, so that "mathematician-historians" should be "historians of mathematics", *par excellence*. The problem goes right back to the initial division, what I referred to as the basic division: the non-historical versus the historical relationship to the past. Michael Oakeshott who helped define this distinction also put his finger on the difficulty involved: "If the historical past be knowable, it must belong to the present world of experience; if it be unknowable, history is worse than futile, it is impossible" [21, page 107]. In any relationship with the past—as Oakeshott would agree—we dance with the present, and all the more so when it comes to *mathematics* of the past.

What distinguishes the "mathematician historians" from the "historians of mathematics" is that this uneasy relation between present and past is understood as a difficulty: for them, the past is a problem. "Historians of mathematics" take as their working assumption, a kind of null-hypothesis, that there is a discontinuity between mathematical thought of the past and that of the present. Faced with a mathematical text, "historians of mathematics" do not try to coordinate the text with the mathematics of the present, but to set it off from the present; they try to make it not more familiar but rather more strange, more foreign. They cannot make the past into present experience (and in this regard Collingwood might have been too optimistic about the goal of history) but they can try to make the pastness of the text palpable and, accordingly, bring out its own identity.

Even here there are subtypes. One subtype is the "philosophical historian of mathematics". Those belonging to this type, like all "historians of mathematics", have a clear view of the problematic connection between past and present. But "philosophical historians" pursue their thorough and precise historical work against the background of a more general philosophical framework. The link with that framework can be stronger or weaker, but, in general, it can be said their work exemplifies and is to an extent driven by their philosophical outlook. The main example I have in mind is Jacob Klein (1899–1978). Klein's *Greek Mathematical Thought and the Origin of Algebra* [16] is at once one of the most probing books on Greek mathematics and its early modern transformation and, at the same time, an embodiment of ideas Klein learned from Husserl. In particular, it puts into action the idea that history and philosophy become united in the attempt to "reactivate sedimented meanings". This Klein describes elsewhere as follows:

> This interlacement of original production and "sedimentation" of significance constitutes the true character of history. From that point of view there is only *one* legitimate form of history: the history of human thought. And the main problem of any historical research is precisely the disentanglement of all these strata of "sedimentation", with the ultimate goal of reactivating the "original foundations", i.e. of descending to the true beginnings, to the "roots", of any science and, consequently, of all prescientific conceptions of mankind as well. Moreover, a history of this kind is the only legitimate form of epistemology [17, page 78].

And Klein completes this passage by saying, "History, in this understanding, cannot be separated from philosophy", which expresses precisely the differentia of this subtype.

Now "historical historians of mathematics"—and Sabetai Unguru (b.1931) should be taken as a prime example—may share the philosophical outlook of a "philosophical historian of mathematics" such as Jacob Klein, but they differ in not having philosophy as their goal. They bear some similarity to "treasure hunters" in that they look for traces of lost mathematics. But for them it is not the mathematical work alone that has been lost; rather it is the very mathematical thought behind it. They aim to struggle with mathematical thought of the past as a kind of human thought recognizable somehow as mathematical but different than modern mathematical thought. Their main end, as I said above, is to make the pastness of past mathematical thought stand out in clear relief. As Unguru himself has put it:

> The historian of ideas does not discharge his obligation by showing merely the extent to which past ideas are like modern ideas. His main effort should be in the direction of showing the extent to which past ideas were unlike modern ones, irrespective of the fact that they might (or might not) have led to the modern ideas. This is a wise methodological tack, since it enables the historian to avoid reductive anachronism while channeling his historical empathy toward and understanding of the past in its own right [27, page 562].

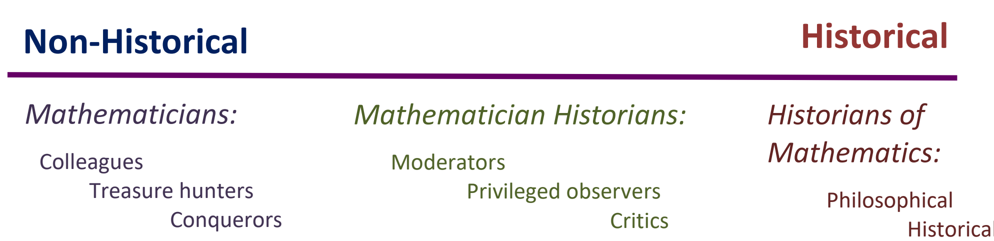

Figure 1: The scale from non-historical relationships to historical relationships.

Educationalist Historians of Mathematics

The types we have discussed to this point and their arrangement along the scale from the non-historical to historical extremes are represented in Figure 1. But we must now take up the question that is really of greatest importance to all of us here, namely: Is there a distinctively educational relationship to the past? Is there an "educational historian of mathematics"? If so, where should we place that type? Can it be placed among those in the last category, the "historians of mathematics"? This is possible: there are streams in mathematics education that look at cultural difference and are interested in showing different ways of thinking. This was the implicit position in [5, 6]. However, it is also possible and even likely that an "educational historian of mathematics" will be more at home among the "mathematician historians", or even the "mathematical conquerors" engaged in showing how the mathematics of the past proves the importance of the standard modern mathematics taught in the classroom or "mathematical treasure hunters" finding gems from history forgotten in the mathematics curriculum.

It is not by chance that the educational historian of mathematics can be placed at almost every position along the scale in Figure 1. Most of the historical figures we have discussed were active teachers in one way or another, and all were teachers in the sense that they wrote to communicate and enlighten. As already mentioned, Halley had to teach the classics of mathematics as part of his responsibilities as Savillian Professor of Geometry, and Jacob Klein shaped the program at St. John's College where students read Euclid, Apollonius, Newton, and Lobachevski. It is reasonable that that their particular relationship to history figured in their approach to teaching (and it certainly did in the case of Jacob Klein), and, conversely, how they thought historical texts could educate was directly related to their position in the non-historical-historical scale.

But the way teachers of mathematics teach mathematics is not only determined by their understanding of the nature of mathematics, historical or otherwise, but also by their own teaching goals and the kinds of problems they hope their teaching practices will solve. Relating to history in this context would again be an Oakeshottian "practical past", a past used a kind of tool only. In this sense, it seems to be located at the non-historical end of the scale I have discussed until now. Yet, the situation is not that simple, for one can treat history of mathematics as something to use but to use according to its specifically historical character. An educational relationship to the past might thus be located on a parallel axis to the one I have drawn; see Figure 2. With that possibility in mind, we can take Otto Toeplitz (1881–1940) as an example of an "educationalist historian of mathematics".

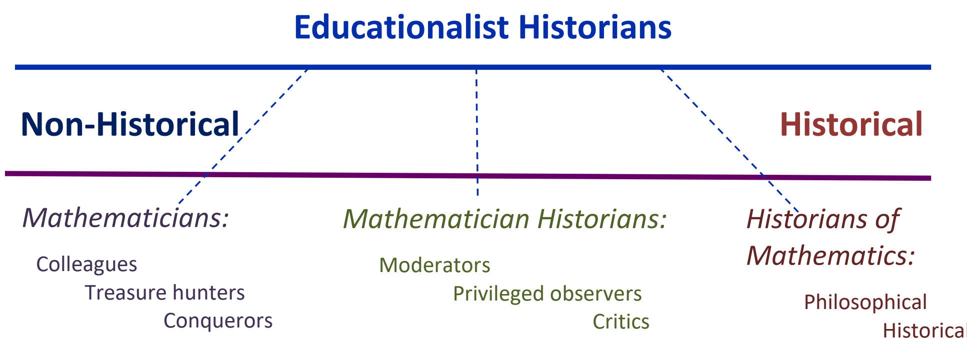

Figure 2: The parallel educationalist-historian axis.

Toeplitz's book *Calculus: The Genetic Approach* [24], originally published in German in 1949, teaches the calculus according to its historical development. It is probably the best-known example of the approach of mathematics teaching according to the educational "ontogeny recapitulates phylogeny" framework (see [23, 8]). Toeplitz set out his rationale for this book, however, not in the book itself but in an address in Düsseldorf in 1926 [25]. The primacy of his educational focus was already evident in his title, "The problem of university infinitesimal calculus courses and their demarcation from infinitesimal calculus in high schools"—no mention whatsoever of history, even though his genetic approach was the centerpiece of the address. Toeplitz was trying to solve specifically educational problems. This was most clear in what he called the "indirect genetic approach". This he described as "... the elucidation of didactic difficulties, I should say, didactical diagnosis and therapy (*didaktische Diagnose und Therapie*), on the basis of historical analyses, where these [historical analyses] serve only to turn [one's] attention in the right direction ([**?** , page 99], translation by Fried and Jahnke [9, page 308])". It was the indirect genetic approach which showed how history could serve as an educational *tool*. In fact, elsewhere in the address, Toeplitz wrote that teachers, to this end, do not have to refer to history explicitly. Toeplitz spoke too of a "direct genetic approach" that does bring history into the classroom directly and which is indeed what we see in Toeplitz's calculus text.

Conclusion

It is very easy to confound what I have tried to sketch here with a history of historiography, though, as I said at the outset, I certainly did want to connect historiography and the history of mathematics. This could be seen especially in light of the remarks about Zeuthen, Truesdell, Klein, and Unguru—since these figures all claim to be doing history *per se*. But part of what I wanted to impress upon the reader is that those who claim to do history per se are only one group of those who relate to mathematics of the past in some way. Mathematicians working on mathematics will also have a particular relation to their predecessors. Mathematics educators also have a relation to the past both implicitly and explicitly. They have an unavoidable implicit relation because they teach mathematics at a certain stage in the development of mathematics, the presuppositions about mathematics they bring to their teaching—what is interesting, what is useful, what is important—either continue or are set opposed to an older tradition. This kind of implicit relation is one shared by all those engaged in mathematics. But where mathematics educators bring history explicitly into their teaching, their relationship becomes a function of the kinds of didactic problems they want to solve. This is what we saw so clearly in Toeplitz. Those didactic considerations make this relationship different from the others. On the other hand, those didactic considerations include not only the very practical problems of the classroom, such as motivation, but also the question of what one wants ultimately students to know. If one does not wish to dictate the latter to students, then, it will be important for students to come to terms with their own way of relating to the mathematical tradition. In this way, the educational mathematical historian may actually be the one who uses history to teach the ways in which one can relate to history, that is, to encourage cognizance of all these types, to keep them all alive.

References

[1] Butterfield, H. (1931/1951). *The Whig Interpretation of History*. New York: Charles Scribner's Sons.

[2] Chapman, A. (1994). "Edmond Halley's Use of Historical Evidence in the Advancement of Science". *Notes and Records of the Royal Society of London*, **48**(2), pages 167–191.

[3] Descartes, R. (1908). *Oeurves de Descartes*. C. Adam and P. Tannery (eds). Paris: Leopold Cerf.

[4] Fried, M. N. and Unguru, S. (2001). *Apollonius of Perga's Conica: Text, Context, Subtext*. Leiden, The Netherlands: Brill Academic Publishers.

[5] Fried, M. N. (2001). "Can mathematics education and history of mathematics coexist?" *Science & Education*, **10**, pages 391–408.

[6] Fried, M. N. (2007). "Didactics and history of mathematics: Knowledge and self-knowledge". *Educational Studies in Mathematics*, **66**, pages 203–223.

[7] Fried, M. N. (2011). *Edmond Halley's Reconstruction of the Lost Book of Apollonius's Conics*. New York: Springer.

[8] Fried, M. N. (2014). "History of Mathematics and Mathematics Education". In Michael Matthews (ed.), *History, Philosophy and Science Teaching Handbook*, volume I, pages 669–705. New York: Springer.

[9] Fried, M. N. and Jahnke, H. N. (2015). "Otto Toeplitz's 1927 Paper on the Genetic Method in the Teaching of Mathematics". *Science in Context*, **28** (2), pages 285–295. (Included is an annotated translation by Fried and Jahnke of Toeplitz, "Das Problem der Universitätsvorlesungen über Infinitesimalrechnung und ihrer Abgrenzung gegenüber der Infinitesimalrechnung an den höheren Schulen", ("The Problem of University Courses on Infinitesimal Calculus and Their Demarcation from Infinitesimal Calculus in High Schools"), same issue, pages 297–310).

[10] Giusti, E. (2003). "Clifford Truesdell (1919-2000), Historian of Mathematics". *Journal of Elasticity*, **70**, pages 15–22.

[11] Grattan-Guinness, I. (2004a). "History or heritage? An Important Distinction in Mathematics and for Mathematics Education". *The American Mathematical Monthly*, **111**(1), pages 1–12.

[12] Grattan-Guinness, I. (2004b). "The Mathematics of the Past: Distinguishing its History from Our Heritage". *Historia Mathematica*, **31** (2004), pages 163–185.

[13] Hardy, G. H. (1992). *A Mathematician's Apology.* Cambridge: Cambridge University Press.

[14] Hogendijk, J. P. (1985). *Ibn al-Haytham's Completion of the Conics* (Sources in the History of Mathematics and Physical Sciences 7). New York: Springer-Verlag.

[15] Jones, A. (1986). *Pappus of Alexandria Book 7 of the Collection.* 2 parts. (Sources in the History of Mathematics and Physical Sciences, 8) New York: Springer-Verlag.

[16] Klein, J. (1968). *Greek Mathematical Thought and the Origin of Algebra.* Eva Brann (trans.) Originally published as "Die griechische Logistik und die Entstehung der Algebra" in *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik* (Abteilung B: Studien), 3 (fasc. 1, 1934), 18-105 and 3 (fasc. 2, 1936), pages 122–235), Cambridge, Mass.: The MIT Press.

[17] Klein, J. (1985). "Phenomenology and the History of Science". In R. B. Williamson and E. Zuckerman (eds.) *Jacob Klein: Lectures and Essays.* Annapolis, MD: St. John's College Press.

[18] Lützen, J. and Purkert, W. (1994). "Conflicting Tendencies in the Historiography of Mathematics: M. Cantor and H. G. Zeuthen". In Eberhard Knobloch and David E. Rowe (Eds.), *The History of Modern Mathematics. III: Images, Ideas, and Communities.* Boston: Academic Press.

[19] Mahoney, M. S. (1994). *The Mathematical Career of Pierre de Fermat 1601-1665*, 2nd edition. Princeton: Princeton University Press.

[20] Mansfeld, J., (1998). *Prolegomena Mathematica: From Apollonius of Perga to Late Neoplatonism: with an appendix on Pappus and the History of Platonism*, Leiden: Brill.

[21] Oakeshott, M. (1933). *Experience and Its Modes.* Cambridge: At the University Press.

[22] Sabra, A. I, (1987). "The Appropriation and Subsequent Naturalization of Greek Science in Medieval Islam". *History of Science*, **25**, pages 223–243.

[23] Schubring, G. (1978). *Das genetische Prinzip in der Mathematik-Didaktik* [*The genetic principle in mathematics education*]. Stuttgart: Klett.

[24] Toeplitz, O. (1963). *The Calculus: The Genetic Approach*. Chicago: University of Chicago Press.

[25] Toeplitz, O. (1927). "Das Problem der Universittsvorlesungen über Infinitesimalrechnung und ihrer Abgrenzung gegenüber der Infinitesimalrechnung an den hheren Schulen". *Jahresbericht der Deutschen Mathematiker-Vereinigung*, **36**, pages 88–100.

[26] Unguru, S. (1975). "On the Need to Rewrite the History of Greek Mathematics", *Archive for History of Exact Sciences*, **15**, pages 67–114.

[27] Unguru, S. (1979). "History of Ancient Mathematics: Some Reflections on the State of the Art". *Isis*, **70**(254), pages 555–565.

[28] Unguru, Sabetai and Fried, Michael N. (2007). "Apollonius, Davidoff, Rorty, and Zeuthen: From A to Z, or, What Else Is There?" *Sudhoffs Archiv*, **91**(1), pages 1–19.

[29] Zeuthen, H. G. (1886). *Die Lehre von den Kegelschnitten im Altertum* [*The Theory of Conic Sections in Antiquity*]. Andr. Fred. Host & Sohn (Kopenhagen). Facsimile Reprint Georg Olms 1996 (Hildesheim).