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A SuPERLINEAR DIRichLET PROBLEM IN A BALL
WITH JUMPING NONLINEARITIES

ALFONSO CASTRO AND ALEXANDRA KUREPA

ABSTRACT. Let $p, \varphi : [0, T] \to \mathbb{R}$ be bounded functions with $\varphi > 0$. Let $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitzian function satisfying the superlinear jumping condition: (i) $\lim_{u \to -\infty} (g(u)/u) \in \mathbb{R}$, (ii) $\lim_{u \to \infty} (g(u)/u^{1+\rho}) = \infty$ for some $\rho > 0$, and (iii) $\lim_{u \to -\infty} (u/g(u))^{N/2} (N G(ku) - ((N - 2)/2)u \cdot g(u)) = \infty$ for some $k \in (0, 1]$ where $G$ is the primitive of $g$. Here we prove that the number of solutions of the boundary value problem $\Delta u + g(u) = p(||x||) + c\varphi(||x||)$ for $x \in \mathbb{R}^N$ with $||x|| < T$, $u(x) = 0$ for $||x|| = T$, tends to $+\infty$ when $c$ tends to $+\infty$. The proofs are based on the "energy" and "phase plane" analysis.

1. Introduction

In this paper we consider the existence of solutions to the Dirichlet problem

$$\Delta u + g(u) = p(||x||) + c\varphi(||x||), \quad x \in \Omega,$$

$$u = 0, \quad x \in \partial \Omega,$$

where $\Omega$ is the ball of radius $T$ in $\mathbb{R}^N$ centered at the origin, $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitzian function, $p \in L^2(\Omega)$, $c \in \mathbb{R}$, and $\varphi : [0, T] \to \mathbb{R}$ is a differentiable function with

$$\varphi > 0 \quad \text{on} \quad [0, T].$$

In addition we assume that the problem is superlinear with jumping nonlinearities, i.e., that there exist real numbers $M$ and $\rho > 0$ such that

$$\lim_{u \to -\infty} \frac{g(u)}{u} = M,$$

$$\lim_{u \to \infty} \frac{g(u)}{u^{1+\rho}} = \infty.$$

For the sake of simplicity of the proofs we assume, without loss of generality, that

$$g(0) = 0 \quad \text{and} \quad g \quad \text{is strictly increasing,} \quad M > 0,$$

$$p \in L^\infty(\Omega), \quad \text{and} \quad 1 \leq \varphi(t) \leq 2 \quad \text{for all} \quad t \in [0, T].$$

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In order to state our main results we introduce the following notations:

\[ L(\kappa, u) = NG(\kappa u) - \left( (N - 2)/2 \right) u g(u), \]
\[ L(\kappa) = \lim_{u \to \infty} L(\kappa, u)(u/g(u))^{N/2}, \]

where \( G(u) = \int_{0}^{u} g(v) \, dv \) and \( \kappa \in (0, 1] \).

Our main result is

**Theorem A.** If (1.2)-(1.5) hold and \( L(\kappa) = \infty \) for some \( \kappa \in (0, 1] \), then there exists a positive integer \( J \) and an increasing sequence \( \{ c_j : j = 1, J + 1, \ldots \} \) tending to \( \infty \) such that for \( c > c_j \) the equation (1.1) has a radially symmetric solution \( u_j \) with \( u_j(0) < 0 \), and \( u_j \) has \( j \) interior nodal surfaces. In particular if \( c > c_j \) then the equation (1.1) has \( j - J \) radially symmetric solutions with \( u(0) < 0 \).

Theorem A is in the spirit of studying boundary value problems for which the interval \( (\lim_{u \to \infty} (g(u)/u), \lim_{u \to \infty} (g(u)/u)) \) contains at least one eigenvalue of \(-\Delta\) with Dirichlet boundary conditions. Such a problem is called superlinear (resp. sublinear) if \( \lim_{u \to \infty} (g(u)/u) = \infty \) (resp. \( \lim_{u \to \infty} (g(u)/u) < \infty \)). The one-dimensional superlinear version of Theorem A is given in [5], and it motivated our result (see also [4 and 13]). For studies on the sublinear case we refer the reader to [11] and references therein.

The proof of Theorem A is based on the shooting method that we have also used in [3]. We study the singular initial value problem

\[ u'' + \frac{N - 1}{t} u' + g(u) = p(t) + c\varphi(t), \quad t \in [0, T], \]
\[ u(0) = d, \quad u'(0) = 0, \]

where \( d \in \mathbb{R} \). A simple argument based on the contraction mapping principle shows that (1.8) has a unique solution \( u(t, d, c) \) on the interval \([0, T]\) depending continuously on \((d, c)\) (see [3, Lemma 2.1]). Radially symmetric solutions of (1.1) are the solutions of (1.8) satisfying

\[ u(T, d, c) = 0. \]

We analyze the energy of the corresponding solutions, i.e., we analyze the function

\[ E(t, d, c) = \frac{(u'(t, d, c))^2}{2} + G(u(t, d, c)). \]

In order to count the number of zeros of the solutions to (1.8) we show that

\[ E(t, -c^\zeta, c) > 0 \]

for \( \zeta > (N+1)/(N+4) \) and \( c \) sufficiently large. In turn (1.11) implies that in the \((u, u')\) plane, for \( c \) sufficiently large and for \( d \leq -c^{(N+1)/(N+4)} \), a continuous argument function \( \psi(t, d, c) \) can be defined. The function \( \psi(t, d, c) \) is such
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that \( \psi(t, d, c) = i \pi + \pi/2 \) (\( i \) an integer) if and only if \( u(t, d, c) = 0 \). We show that (see Lemma 4.1 and Lemma 2.7)

\[
\limsup_{-d \to \infty} \psi(T, d, c) = S,
\]

and that

\[
\liminf_{c \to \infty} \psi(T, -c^\xi, c) = \infty,
\]

where \( S \in \mathbb{R} \) and \( \xi \in [(N + 1)/(N + 4), 1) \). The intermediate value theorem and \( (1.13) \) imply that given any integer \( j \) there exists \( c_j \) such that if \( c > c_j \) then

\[
\psi(T, -c^\xi, c) \geq j \pi + \pi/2.
\]

By combining \( (1.12), (1.14) \) and the intermediate value theorem we see that if \( c \) is sufficiently large and \( j \geq J := [(S - \pi/2)/\pi] + 1 \), then there exist numbers \( d_j < d_{j+1} < \cdots < d_j \) with

\[
\psi(t, d_i, c) = i \pi + \pi/2, \quad i \in \{J, J + 1, \ldots, j\}.
\]

Hence

\[
u_i(T, d_i, c) = 0, \quad i \in \{J, J + 1, \ldots, j\}.
\]

Thus if \( c \) is sufficiently large, then \( (1.1) \) has \( j - J \) radially symmetric solutions.

Remarks. (i) If \( \Omega \) is a ring of the form \( \{x \in \mathbb{R}^N : a \leq ||x|| \leq b\} \), then the equation in \( (1.8) \) is no longer singular and thus the problem reduces to the one studied in [5].

(ii) Condition \( (1.2) \) can be considerably weakened. For example, it is easy to verify that if \( \phi > 0 \) on \([0, \varepsilon)\) and \( \phi = 0 \) on \([\varepsilon, T]\), then Theorem A holds.

2. PHASE-PLANE ANALYSIS

Let \( (u(t, d, c), u'(t, d, c)) \neq (0, 0) \) for all \( t \in [0, \xi] \). By defining \( r^2(t, d, c) = u^2(t, d, c) + (u'(t, d, c))^2 \) we see that for \( d < 0 \) there exists a unique continuous argument function \( \psi(t, d, c) \), \( t \in [0, \xi] \), such that

\[
u(t, d, c) = -r(t, d, c) \cos \psi(t, d, c),
\]

\[
u'(t, d, c) = r(t, d, c) \sin \psi(t, d, c),
\]

\[
\psi(0, d, c) = 0.
\]

An elementary calculation shows that

\[
\psi'(t, d, c) = \sin^2 \psi(t, d, c)
\]

\[
- \left( g(u(t, d, c)) + \frac{N-1}{t} u'(t, d, c) - c \phi(t) - p(t) \right) \cos \psi(t, d, c).
\]

\[
r(t, d, c)
\]

From this formula follows:
Remark 2.1. If \( r(t, d, c) > 0 \) for all \( t \in [0, T] \), and \( \psi(\hat{t}, d, c) = (2k + 1)\pi/2 \) for some \( \hat{t} \in [0, T] \) and some integer \( k \), then \( \psi(t, d, c) > (2k + 1)\pi/2 \) for all \( t > \hat{t} \). In particular, \( \psi(t, d, c) > -\pi/2 \) for all \( t \in [0, T] \).

Now let \( \gamma \) be such that
\[
\gamma_1 := 1 - 12\varepsilon \leq \gamma \leq 1 - 10\varepsilon := \gamma_2,
\]
where \( \varepsilon \) is defined by
\[
\varepsilon = \frac{\rho}{4(6N + 6N\rho + 8 + 8\rho)}.
\]
Elementary calculations show that
\[
\begin{align*}
(a) & \quad 2\gamma > 2\gamma + (\gamma - 1)N > 2\gamma + \frac{3}{2}(\gamma - 1)N > 1 + \frac{1}{1 + \rho}, \\
(b) & \quad \gamma + \varepsilon < 1, \\
(c) & \quad \gamma > \varepsilon + \frac{1}{1 + \rho}, \\
(d) & \quad 3\gamma - 2 - \frac{\varepsilon}{2} > \frac{2\gamma}{2 + \rho}, \\
(e) & \quad 2\gamma - 1 > \frac{N + 1}{N + 4}.
\end{align*}
\]

Let \( u(t, d, c) := u(t) \) be a solution to (1.8). Since the following constant appears often in this section we let
\[
Q := Q(N) := \left[\frac{\sqrt{2N} + 512N}{2N}\right]^2.
\]
Throughout the paper \( k \) will denote a nonnegative integer.

Lemma 2.2. There exists \( C_1 \) such that if \( c > C_1 \), \( E(t_1) = c^{2\gamma} \), \( u(t_1) > -\alpha c^{2\gamma-1} \) and \( \psi(t_1) \in [2k\pi, 2k\pi + \pi/2] \), for some \( t_1 \in (0, T] \) and \( \alpha \in [1, 64N] \), then there exists \( t_2 \in [t_1, t_1 + 2\alpha\sqrt{N}c^{\gamma-1}] \), such that
\[
\psi(t_2) = 2k\pi + \pi/2,
\]
and
\[
\frac{1}{128} c^{2\gamma} \leq E(t_2) \leq \frac{(\sqrt{2N} + 8\alpha)^2}{2N} c^{2\gamma} \leq Q c^{2\gamma}.
\]
In particular \( E(t_2) = c^{2\gamma} \) with \( \gamma' \in [\gamma - (\ln 128)/(2 \ln c), \gamma + (\ln Q)/(2 \ln c)] \).

Proof. Let \( t > t_1 \) be such that \( u(s) \leq 0 \) for all \( s \in (t_1, t] \). Therefore, we have
\[
u'(t) = t^{-N+1} \left[ t_1^{N-1} u'(t_1) + \int_{t_1}^{t} s^{N-1} \left( c\varphi(s) + p(s) - g(u(s)) \right) ds \right]
\geq t^{-N+1} \int_{t_1}^{t} s^{N-1} (c - \|p\|_\infty) ds
\geq \frac{c - \|p\|_\infty}{N} \left[ t - (t_1)^N \right] \geq \frac{c - \|p\|_\infty}{N} (t - t_1).
\]
Thus, for \( c \) sufficiently large we infer
\[
u(t) \geq -\alpha c^{2\gamma-1} + \frac{c - \|p\|_\infty}{2N} (t - t_1)^2
\geq -\alpha c^{2\gamma-1} + \frac{c}{4N}(t - t_1)^2.
\]
This shows that for some \( t_2 \in [t_1, t_1 + 2\alpha \sqrt{N}c^{\gamma-1}] \) we have
\[
(2.7) \quad u(t_2) = 0,
\]
and \( u'(s) \geq 0 \) for all \( s \in (t_1, t_2) \). In particular
\[
(2.8) \quad \psi(t_2) = 2k\pi + \pi/2.
\]
By the continuity of \( u' \) there exists \( \tau < t_1 \) such that \( u'(\tau) = 0 \). Since \( u(t_1) \geq -\alpha c^{2\gamma-1} \), then from (1.3) for \( u \) sufficiently large, we have that \( |g(u)| \leq (M + 1)|u| \). Therefore, for \( c \) sufficiently large
\[
(2.9) \quad G(u(t_1)) \leq [(M + 2)/2] \alpha^2 c^{4\gamma-2}.
\]
Thus for \( c \) sufficiently large we have
\[
(2.10) \quad u'(t_1) \geq c^\gamma.
\]
Hence,
\[
c^\gamma \leq u'(t_1) = t_1^{-N+1} \int_\tau^{t_1} s^{N-1} (c\varphi(s) + p(s) - g(u(s))) \, ds \leq \frac{4}{N} c(t_1 - \tau),
\]
which implies that
\[
(2.11) \quad t_1 - \tau \geq Nc^{\gamma-1}/4.
\]
Therefore,
\[
u'(t_2) = t_2^{-N+1} \int_{t_1}^{t_2} s^{N-1} (c\varphi(s) + p(s) - g(u(s))) \, ds
\geq \frac{c - \|p\|_\infty}{N} (t_2 - \tau) \geq \frac{c}{8} c^{\gamma-1}.
\]
Thus
\[
(2.12) \quad E(t_2) \geq \frac{1}{128} c^{2\gamma}.
\]
Now, for \( t \in (t_1, t_2) \) we have
\[
u'(t) = t^{-N+1} \left[ t_1^{N-1} u'(t_1) + \int_{t_1}^{t} s^{N-1} (c\varphi(s) + p(s) - g(u(s))) \, ds \right]
\leq \sqrt{2}c^\gamma + t^{-N+1} \int_{t_1}^{t} s^{N-1} (2c + \|p\|_\infty + (M + 1)|u(t_1)|) \, ds
\leq \sqrt{2}c^\gamma + t^{-N+1} \int_{t_1}^{t} s^{N-1} (2c + \|p\|_\infty + (M + 1)\alpha c^{2\gamma-1}) \, ds
\leq \sqrt{2}c^\gamma + 4c \frac{2\alpha}{\sqrt{N}} c^{\gamma-1} \leq \frac{\sqrt{2N} + 8\alpha}{\sqrt{N}} c^\gamma,
\]
where we have used (1.3), the hypothesis of the lemma and the fact that $t_2 \in [t_1, t_1 + 2\alpha\sqrt{N}c^{y-1}]$. Since $E(t_2) = (u'(t_2))^2/2$ we see that for $c > C_1$ we have

$$(0.14) \quad E(t_2) \leq \frac{(\sqrt{2N} + 8\alpha)^2}{2N} c^{2y},$$

where $C_1$ is such that (2.6), (2.9) and (2.10) hold. This together with (2.8) and (2.12) proves the lemma.

**Lemma 2.3.** If $c > C_2$, $\psi(t) = 2k\pi$, $u(t) = -c^{2y-1}$, $u'(t) = 0$, with $t \in [0, T]$, then there exists $t_2 \in (t, t + 2\sqrt{N}c^{y-1})$ such that

$$\psi(t_2) = 2k\pi + \pi/2,$$

and

$$\frac{1}{16N}c^{2y} \leq E(t_2) \leq \frac{(\sqrt{2N} + 8)^2}{2N} c^{2y} \leq Qc^{2y}.$$

In particular $E(t_2) = c^{2y'}$ with $y' \in [\gamma - (\ln16N)/(2\ln c), \gamma + (\ln Q)/(2\ln c)]$.

**Proof.** Using the same arguments leading to the proof of (2.7) in Lemma 2.2 we get the existence of $t_2 \in (t, t + 2\sqrt{N}c^{y-1})$ satisfying $u(t_2) = 0$. Since for $u < 0$ sufficiently large $|g(u)| \leq (M + 1)|u| \leq (M + 1)c^{2y-1}$, on $[t, t_2]$ we have

$$u'(t) \leq (3c + (M + 1)c^{2y-1}) \frac{1}{N} t \leq \frac{4}{N} c(t - \tau).$$

Thus,

$$u(t) \leq -c^{2y-1} + \frac{2}{N} c(t - \tau)^2.$$

Hence, $t_2 - \tau \geq \sqrt{N}c^{y-1}/\sqrt{2}$, which leads to $u'(t_2) \geq c''\sqrt{2N}$, and

$$(2.15) \quad E(t_2) \geq \frac{1}{16N}c^{2y}.$$ 

On the other hand by replacing $t_1$ with $t$ in (2.11) and using the fact that $u'(\tau) = 0$, we obtain

$$(2.16) \quad E(t_2) \leq \frac{(\sqrt{2N} + 8)^2}{2N} c^{2y}.$$ 

Hence, the lemma is proven.

**Lemma 2.4.** There exists $C_3$ such that if $c > C_3$, $E(t_2) = c^{2y}$, and $\psi(t_2) \in [2k\pi + \pi/2, 2k\pi + \pi)$ for some $t_2 \in [Nc^{y-1}/4\sqrt{Q}, T]$, then there exists $t_3 \in (t_2, t_2 + 2c^{y-1})$ such that $\psi(t_3) = 2k\pi + \pi$, and

$$\frac{3}{4}c^{2y} \leq E(t_3) \leq 2c^{2y}.$$

In particular $E(t_3) = c^{2y'}$ with $y' \in [\gamma + (\ln\frac{3}{4})/(2\ln c), \gamma + (\ln 2)/(2\ln c)]$.

**Proof.** Without loss of generality we can assume that $u(t_2) \leq g^{-1}(4c)$. For $t > t_2$ such that $u(t) \leq g^{-1}(4c)$, and $\psi(s) \in [2k\pi + \pi/2, 2k\pi + \pi]$, for all
s \in (t_2, t), we have

\[ E(t) = E(t_2) + \int_{t_2}^{t} \left( -\frac{N-1}{s} u'(s) + c\varphi(s) + p(s) \right) u'(s) \, ds \]

(2.17)

\[ \leq c^{2\gamma} + 3cu(t). \]

Since

(2.18)

\[ G(u(t)) \leq (4c)g^{-1}(4c) \leq 16c^{1+1/(1+\rho))} \]

for \( c \) sufficiently large, from (2.17) using (2.4) we see that there exists a positive constant \( \kappa \) independent of \( (c, \gamma) \) such that \( (u'(t))^2 \leq \kappa c^{2\gamma} \). On the other hand

\[ E(t) \geq E(t_2) + \int_{t_2}^{t} \left( -\frac{4(N-1)\sqrt{\Omega R}}{c^{\gamma-1}N} c^\gamma \right) u'(s) \, ds \]

(2.19)

\[ \geq c^{2\gamma} - \frac{4(N-1)\sqrt{\Omega R}}{N} \frac{c^{1+1/(1+\rho))}} \geq \frac{3}{4} c^{2\gamma}, \]

for \( c \) sufficiently large (see (2.4)). From (2.18) and (2.19) we have

\[ (u'(t))^2 \geq \frac{3}{4} c^{2\gamma}. \]

Thus

(2.20)

\[ (4c)^{1/(1+\rho)} \geq u(t) \geq \frac{\sqrt{3}}{2} c^\gamma (t - t_2). \]

Hence, there exists \( t^* \in (t_2, t_2 + 6c^{(1/(1+\rho)) - \gamma}) \) such that

(2.21)

\[ u(t^*) = g^{-1}(4c) \quad \text{and} \quad u'(t^*) \geq \frac{\sqrt{3}}{2} c^\gamma. \]

For \( t > t^* \) with \( u'(s) \geq 0 \) for all \( s \in [t^*, t] \) we have

\[ u''(t) = -\frac{N-1}{t} u'(t) + c\varphi(t) + p(t) - g(u(t)) \leq -c. \]

Hence,

(2.22)

\[ u'(t) \leq u'(t^*) - c(t - t^*) \leq \frac{\sqrt{3}}{2} c^\gamma - c(t - t^*). \]

Thus, there exists \( t_3 \in (t^*, t^* + \sqrt{3}c^{\gamma-1}/2) \subset (t_2, t_2 + 2c^{\gamma-1}) \), (see (2.4)), such that

(2.23)

\[ u'(t_3) = 0 \]

for \( c \) sufficiently large, and \( u'(t) > 0 \) for all \( t \in [t^*, t_3) \). In particular

(2.24)

\[ \psi(t_3) = 2k\pi + \pi. \]

Moreover, for \( t \in [t_2, t_3] \) from (2.17) we have

(2.25)

\[ G(u(t)) \leq E(t) \leq c^{2\gamma} + 3cu(t). \]
Thus from (1.4) we infer
\[(2.26)\quad (u(t))^{2+\rho} - 3cu(t) \leq c^{2y}.\]
Now, if \((u(t))^{2+\rho} < 6cu(t)\), then
\[(2.27)\quad u(t) < 6c^{1/(1+\rho)}.\]
For \((u(t))^{2+\rho} \geq 6cu(t)\) from (2.26) we see that
\[(2.28)\quad u(t) \leq \frac{1}{3}c^{2y-1}.\]
By using (2.4), (2.27), and (2.28) for \(c\) sufficiently large we infer
\[(2.29)\quad u(t) \leq \frac{1}{3}c^{2y-1}.\]
Thus, replacing (2.29) in (2.25), for \(t \in [t_2, t_3]\) we obtain
\[(2.30)\quad E(t) \leq 2c^{2y}.\]
Hence, from (2.30) we have
\[(2.31)\quad u'(t) \leq c^y.\]
Imitating the arguments leading to (2.19) it is easy to see that
\[(2.32)\quad E(t_3) \geq \frac{3}{4}c^{2y},\]
for \(c > C_3\), where \(C_4\) is assumed to be such that (2.19), (2.23), (2.29) and (2.32) hold. Thus, (2.24), (2.30) and (2.32) prove the lemma.

**Lemma 2.5.** There exists \(C_4\) such that if \(c > C_4\), \(E(t_3) = c^{2y}\), and \(\psi(t_3) \in [2k\pi + \pi, 2k\pi + \frac{3}{2}\pi]\) for some \(t_3 \in [Nc^y/8\sqrt{Q}, T]\) then there exists \(t_4 \in [t_3, t_3 + 2c^{-e/2}]\) such that
\[\psi(t_4) = 2k\pi + \frac{3}{2}\pi.\]
Moreover,
\[\frac{3}{4}c^{2y} \leq E(t_4) \leq c^{2y}.\]
In particular \(E(t_4) = c^{2y'}\) with \(y' \in [y + (\ln \frac{3}{4})/(2 \ln c), y]\).

**Proof.** Since \(\psi(t_3) \in [2k\pi + \pi, 2k\pi + \frac{3}{2}\pi]\), we have \(u(t_3) > 0\) and \(u'(t_3) \leq 0\). Hence either
\[(2.33)\quad u(t_3) > g^{-1}(2c + \|p\|_\infty + c^{y+\epsilon}),\]
or
\[(2.34)\quad u(t_3) \leq g^{-1}(2c + \|p\|_\infty + c^{y+\epsilon}).\]
If (2.33) holds then for \(t \geq t_3\) with \(u(s) \geq g^{-1}(2c + \|p\|_\infty + c^{y+\epsilon})\) for all \(s \in [t_3, t]\) we have
\[(2.35)\quad u'(t) \leq t^{-N+1} \left[ t_3^{N-1} u'(t_3) + \int_{t_3}^t s^{N-1}(c\varphi(s) + p(s) - g(u(s))) \, ds \right]\]
\[\leq t^{-N+1} \int_{t_3}^t s^{N-1}(-c^{y+\epsilon}) \, ds\]
\[
= \frac{-c^{y+\epsilon}}{N} - [t - (t_3^N/t^{N-1})] \leq 0.
\]
Suppose that for all \( t \in (t_3, t_3(1 + c^{-\varepsilon/2})) \) we have \( u(t) > g^{-1}(2c + \|p\|_\infty + c^{y+\varepsilon}) \).

Hence, now for all \( \lambda \in (-\varepsilon, -\varepsilon/2) \), by (2.35) we infer

\[
(2.36) \quad u'(t_3(1 + c^{-\varepsilon/2})) \leq \frac{-c^{y+\varepsilon}}{N} t_3[(1 + c^\lambda) - (1 + c^\lambda)^{-N+1}]
\]

Thus

\[
(2.37) \quad u(t_3(1 + c^{-\varepsilon/2})) \leq u(t_3(1 + c^{-\varepsilon})) + \int_{t_3(1 + c^{-\varepsilon})}^{t_3(1 + c^{-\varepsilon/2})} -c^{2\gamma-1} \frac{N}{(2N+2\sqrt{Q})} ds
\]

Since \( E(t_3) = c^{2\gamma} \), we have that \( u(t_3) \leq G^{-1}(c^{2\gamma}) \). By (1.4) we see that for \( c \) sufficiently large \( g^{-1}(c^{2\gamma}) < c^{2\gamma/(1+\rho)} \). Hence, there exists \( C_4 \geq \|p\|_\infty \) such that for \( c > C_4 \)

\[
(2.38) \quad u(t_3) \leq c^{2\gamma/(2+\rho)}.
\]

Also, since \( \varepsilon > 0 \) and (2.4) holds, we can assume \( C_4 \) to be such that \( c^{-\varepsilon/2} - c^{-\varepsilon} > c^{-\varepsilon/2}/2 \) for \( c > C_4 \), and that there exists a constant \( k_1 > 0 \) such that for \( c > C_4 \)

\[
(2.39) \quad u(t_3(1 + c^{-\varepsilon/2})) \leq c^{2\gamma/(2+\rho)} - k_1 c^{3\gamma-2-(\varepsilon/2)} \leq 0,
\]

which is a contradiction. Thus for \( c > C_4 \) there exists \( t' \in (t_3, t_3(1 + c^{-\varepsilon/2})) \) such that

\[
(2.40) \quad u(t') = g^{-1}(2c + \|p\|_\infty + c^{y+\varepsilon}).
\]

This and (2.34) show that there exists \( t'' \in [t_3, t_3(1 + c^{-\varepsilon/2})] \) with

\[
(2.41) \quad u(t'') \leq g^{-1}(2c + \|p\|_\infty + c^{y+\varepsilon}).
\]

Now we estimate \( E(t) \) for \( t > t_3 \) with \( u'(s) \leq 0 \) for all \( s \in [t_3, t] \). Since

\[
(2.42) \quad \frac{dE(t)}{dt} = -\frac{N-1}{t} (u'(t))^2 + (c\varphi(t) + p(t))u'(t)
\]

we have

\[
(2.43) \quad E(t) \leq c^{2\gamma}.
\]

Therefore,

\[
(2.44) \quad |u'(t)| \leq \sqrt{2c^\gamma}.
\]

Hence,

\[
(2.45) \quad -\frac{N-1}{t} u'(t) + c\varphi(t) + p(t) \leq 4c,
\]
for \( c \) sufficiently large. If in addition \( u(t) \geq 0 \) then

\[
E(t) = E(t_3) + \int_{t_3}^{t} \left( -\frac{N - 1}{t} u'(s) + c\varphi(s) + p(s) \right) u'(s) \, ds
\]

\[(2.46)\]

\[
\geq c^{2y} + 4c(u(t) - u(t_3)) \geq c^{2y} - 4cu(t_3)
\]

\[
\geq c^{2y} - 4c^{1+(2y/(2+p))} \geq \frac{3}{4} c^{2y},
\]

where we have used (2.4) and (2.39). Thus from (2.43) and (2.44) we obtain

\[
(2.47) \quad \frac{3}{4} c^{2y} \leq E(t_2) \leq c^{2y}.
\]

Now, for \( t > t'' \) we have

\[
G(u(t)) \leq u\varphi(u) \leq g^{-1}(4c)4c
\]

\[
\leq 16c^{1+(1/(1+p))} \leq \frac{1}{4} c^{2y},
\]

(see (2.4)). Combining (2.47) and (2.48) we get

\[
(2.49) \quad u'(t) < -c^y
\]

for all \( t \geq t'' \) such that \( u(s) \geq 0 \) and \( u'(s) \leq 0 \) with \( s \in [t'', t] \). Hence

\[
(2.50) \quad u(t) \leq g^{-1}(2c + \|p\|_\infty + c^{y+\epsilon}) - c^y(t - t'')
\]

\[
\leq 4c^{-1/(1+p)} - c^y(t - t''),
\]

(see (2.4)) for \( c \) sufficiently large. From (2.30) we see that for some \( t_4 \in [t'', t'' + c^{(1/(1+p)) - y}] \subset [t_3, t_3 + 2c^{-\epsilon/2}] \) (see (2.4)) we have

\[
(2.51) \quad u(t_4) = 0,
\]

in particular \( \psi(t_4) = 2k\pi + \frac{3}{2}\pi \) for \( c > C_4 \), where \( C_4 \) is in addition assumed to be such that (2.46)-(2.50) hold. This and (2.47) prove the lemma.

Lemma 2.6. There exists \( C_5 \) such that if \( c > C_5 \), \( \psi(t_4) \in [2k\pi + \frac{3}{2}\pi, 2(k+1)\pi] \), \( E(t_4) = c^{2y} \), with \( t_4 \in [Nc^{y-1}/8\sqrt{Q}, T] \), and \( u(t_4) > -\alpha c^{2y-1}, \alpha \in [1, 64N] \), then for some \( t_5 \in [t_4, t_4 + \sqrt{2}c^{y-1}] \) we have \( \psi(t_5) = 2(k+1)\pi \). Moreover,

\[
-(\alpha + 4)c^{2y-1} \leq u(t_5) \leq -\frac{1}{64\sqrt{Q} + 16}c^{2y-1}.
\]

Proof. Let now \( t > t_4 \) be such that \( u'(s) \leq 0 \) for all \( s \in (t_4, t) \). The existence of such \( t \)'s follows from the fact that \( \psi(t_4) \in [2k\pi + \frac{3}{2}\pi, 2(k+1)\pi] \). Since

\[
u''(s) = -\frac{N - 1}{s} u'(s) + c\varphi(s) + p(s) - g(u(s)) \geq \frac{\xi}{2},
\]

we have

\[
u'(t) \geq u'(t_4) + \frac{\xi}{2}(t - t_4) \geq -\sqrt{2E(t_4)} + \frac{\xi}{2}(t - t_4)
\]

\[
\geq -\sqrt{2c^y} + \frac{\xi}{2}(t - t_4).
\]
Therefore, there exists \( t_5 \in [t_4, t_4 + 2\sqrt{2}c^{y-1}] \) with \( u'(t_5) = 0 \) and \( u'(s) \leq 0 \) for all \( s \in [t_4, t_5] \). Thus

\[
\psi(t_5) = 2(k + 1)\pi,
\]

and \( \psi(s) \in [2k\pi + \frac{\pi}{2}, 2(k + 1)\pi] \), for all \( s \in [t_4, t_5] \). Furthermore,

\[
(2.53) \quad u'(s) = u(t_4) + \int_{t_4}^{t} u'(s) \, ds \geq \alpha c^{2y-1} - \sqrt{2}c^y 2\sqrt{2}c^{y-1} - (\alpha + 4)c^{2y-1}.
\]

By (1.3) there exist constants \( M_1 > 0 \), and \( M_2 \) such that \( G(u) \leq M_1 u^2 + M_2 \) for \( u \leq 0 \). Let \( C_5 \) be such that for \( c > C_5 \) we have

\[
(2.54) \quad G \left( -\alpha c^{2y-1} \right) \leq M_1 (\alpha + 4)c^{2y-2} + M_2 \leq \frac{c^{2y}}{2}.
\]

Thus

\[
(2.55) \quad G(u(t_4)) \leq G(u(t_5)) \leq M_1 (u(t_5))^2 + M_2 \leq \frac{c^{2y}}{2}.
\]

Since \( E(t_4) = c^{2y} \), from (2.56) we have

\[
(2.56) \quad G(u(t_4)) \leq G(u(t_5)) \leq M_1 (u(t_5))^2 + M_2 \leq \frac{c^{2y}}{2}.
\]

Since \( E(t_4) = c^{2y} \), from (2.56) we have

\[
(2.57) \quad u'(t_4) \leq -c^{y},
\]

for \( c \) sufficiently large. By the continuity of \( u' \) we see that there exists \( \bar{t} \in (t_4, t_5) \) such that

\[
(2.58) \quad u'\left(\bar{t}\right) = \frac{-c^{y}}{2}.
\]

Since \( |u'(t_4)| \leq \sqrt{2}c^{y} \), and \( |g(u)| \leq (M + 1)|u| \) for \( u \leq 0 \) sufficiently large, then for \( t \in [t_4, \bar{t}] \) we have

\[
(2.59) \quad u''(t) \leq \frac{8(N-1)\sqrt{O}}{c^{y-1}N} \sqrt{2}c^{y} + 3c + (M + 1)(\alpha + 4)c^{2y-1} \leq (16\sqrt{O} + 4)c,
\]

for \( c \) sufficiently large. Thus by integrating (2.59) on \( [t_4, \bar{t}] \), and using (2.57) and (2.58) we have that \( \bar{t} - t_4 \geq c^{y-1}/2(16\sqrt{O} + 4) \). Hence

\[
(2.60) \quad u(t_5) \leq u(\bar{t}) = u(t_4) + \int_{t_4}^{\bar{t}} u'(s) \, ds \leq 0 - \frac{c^{y}}{2}(\bar{t} - t_4) = -\frac{c^{2y-1}}{64\sqrt{O} + 16}.
\]

From (2.54) and (2.60) we see that if \( c > C_5 \), where \( C_5 \) is in addition assumed to be such that (2.59) holds, then the lemma is proven.

The next lemma summarizes the results of this section.
Lemma 2.7. Given any positive integer \( j \) there exists a real number \( m_j \) such that if \( c > m_j \), \( E(t, d, c) > 0 \) for all \( t \in [0, T] \), \( E(t_1, d, c) = c^{2y} \), for some \( t_1 \in [Nc^{y-1}/4, T] \), \( \gamma \in [(5\gamma_1 + 3\gamma_2)/8, (3\gamma_1 + 5\gamma_2)/8] \), and either

\begin{enumerate}[(A)]
    \item \( \psi(t_1, d, c) \in [2k\pi + \frac{3}{2}, 2k\pi + \frac{3}{2}] \), or
    \item \( \psi(t_1, d, c) \in [2k\pi + 2k\pi + \frac{3}{2}] \cup [2k\pi + \frac{3}{2}, 2(k+1)\pi] \), and \( u(t_1, d, c) \geq -c^{2y-1} \),
\end{enumerate}
then \( \psi(T, d, c) > j\pi + \pi/2 \).

Proof. We split the proof into four cases depending on the quadrant where \( \psi(t_1, d, c) \) lies.

Case I. Suppose that \( \psi(t_1, d, c) \in [2k\pi, 2k\pi + \pi/2] \). By (2.4) there exists \( C_5 \) such that for \( c > C_5 \) we have that \( (2\sqrt{2} + 2\alpha\sqrt{N} + 2)c^{2y-1} < c^{\epsilon/2} \). Thus by applying consecutively Lemmas 2.2 (or 2.3), 2.4, 2.5, 2.6, we see that if \( c > C := \max\{C_1, \ldots, C_5\} \) then there exist \( t_2 < t_3 < t_4 < t_5 \) such that

\begin{equation}
    t_5 \in [t_1, t_1 + k^*c^{-(\epsilon/2)}] .
\end{equation}

and

\begin{equation}
    \psi(t_5, d, c) = 2(k + 1)\pi .
\end{equation}

Moreover, from Lemma 2.2. we have

\begin{equation}
    E(t_2, d, c) = c^{2y'} ,
\end{equation}

with

\begin{equation}
    -\frac{\ln 128}{2\ln c} + \gamma' \leq \gamma' \leq \gamma + \frac{\ln Q}{2\ln c} .
\end{equation}

Also, by combining (2.64) with the results from Lemmas 2.4 and 2.5 we see that \( E(t_4, d, c) = c^{2y''} \), with

\begin{equation}
    \gamma - \frac{2\ln(3/4) + \ln 128}{2\ln c} \leq \gamma'' \leq \gamma + \frac{\ln 2 + \ln Q}{2\ln c} .
\end{equation}

Now by Lemma 2.6 (since in this case \( \alpha = 0 \)) we have that \( u(t_5) = -c^{2y-1} \) with

\begin{equation}
    \gamma - \frac{\ln(72(64\sqrt{Q} + 16))}{2\ln c} \leq \gamma \leq \gamma + \frac{\ln 8Q}{2\ln c} ,
\end{equation}

or

\begin{equation}
    -k'c^{2y-1} := -8Qc^{2y-1} \leq u(t_5) \leq -[72(64\sqrt{Q} + 16)]c^{2y-1} .
\end{equation}

Let now

\begin{equation}
    m_j := \max \left\{ \exp \left( \frac{4j\ln k'}{3(\gamma_2 - \gamma_1)} \right), \exp \left( \frac{4j\ln K'}{3(\gamma_2 - \gamma_1)} \right), (j/T)^{2/\epsilon}, C \right\} .
\end{equation}
From (2.67) and (2.68) it follows that
\[
\frac{5\gamma_1 + 3\gamma_2}{8} - \frac{3\gamma_2 - 3\gamma_1}{8} \leq \gamma \leq \frac{3\gamma_1 + 5\gamma_2}{8} + \frac{3\gamma_2 - 3\gamma_1}{8}.
\]
Also, from (2.68) we have
\[
t_5 - t_1 \leq T/j.
\]
Now, we observe that \( t_5 = \tau \), and \( \gamma = \gamma \) satisfy the hypothesis of Lemma 2.3. Thus, iterating the above argument \( j \) times we see that there exist \( t_5 < t_6 < \cdots < t_{j+3} < T \), with \( \psi(t_{j+3}, d, c) = 2(k+j)\pi \). Hence, by Remark 2.1. we have that
\[
\psi(T, d, c) > 2(k+j)\pi - \frac{\pi}{2} \geq 2j\pi - \frac{\pi}{2} \geq j\pi + \frac{\pi}{2},
\]
and that proves Case I.

**Case II.** If \( \psi(t_1, d, c) \in [2k\pi + \pi/2, (2k+1)\pi] \), then by applying Lemmas 2.4, 2.5, 2.6. we see that there exists \( t_5 \) that satisfies (2.61), (2.62), (2.67), (2.69), and thus, we are in Case I.

**Case III.** If \( \psi(t_1, d, c) \in [(2k + l)\pi, (2k + l)\pi + \pi/2] \), then we can apply Lemmas 2.5 and 2.6, and Case III reduces to Case I.

**Case IV.** If \( \psi(t_1, d, c) \in [(2k + l)\pi + \pi/2, 2(k+l)\pi] \), then by applying Lemma 2.6 we are in Case I, and that concludes the proof of Lemma 2.7.

### 3. Energy Analysis

Throughout this section we use \( K, k', K', k, k_1, k_2 \) to denote various constants independent of \((c, \gamma)\).

Let now \( u(t) := u(t, d, c) \) be a solution to (1.8). Suppose that there exists \( \hat{t} \in [Nc^{y-1}/4, T] \), with \( \gamma \in [\gamma_1, \gamma_2] \), such that
\[
E(\hat{t}, d, c) = c^2\gamma, \quad E(t) > 0 \text{ for all } t \in [0, \hat{t}] \text{ and } u(\hat{t}) \geq -c^{2y-1}.
\]
Arguing as in the proof of Lemma 2.7 (using Lemmas 2.2, 2.4 and 2.5 if \( \psi(\hat{t}, d, c) \in [2k\pi, 2k\pi + \pi/2] \), Lemmas 2.4 and 2.5 if \( \psi(\hat{t}, d, c) \in [2k\pi + \pi/2, (2k+1)\pi] \), Lemma 2.5 if \( \psi(\hat{t}, d, c) \in [(2k + l)\pi, (2k + l)\pi + \pi/2] \), and Lemmas 2.6, 2.3, 2.4 and 2.5 if \( \psi(\hat{t}, d, c) \in [(2k + l)\pi + \pi/2, 2(k+l)\pi] \)), we see that there exists \( \tau_1 \in [\hat{t}, \hat{t} + Kc^{-(e/2)}] \) such that \( u(\tau_1) = 0, u'(\tau_1) < 0 \), and
\[
k'c^{2y} \leq E(\tau_1, d, c) \leq K'c^{2y}.
\]
Now we prove by induction

**Lemma 3.1.** Let \( \hat{t} \) be as in (3.1) and let \( \hat{t} \leq \tau_1 < \hat{t}_1 < \tau_2 < \cdots < \tau_i < \hat{t}_i < \tau_{i+1} < \cdots < t \) denote the zeroes of \( u(t) \) for \( t > \tau_1 \) such that \( E(s, d, c) > 0 \) for \( s \in [\tau_1, t] \). If for some \( \kappa \in (0, 1) \) \( L(\kappa) = \infty \), then there exists \( c^* \) such that if \( c > c^* \) then
\[
k'c^{2y+(\gamma-1)N} \leq \tau_i^N E(\tau_i, d, c) \leq \tau_i^N [k''c^{2y} + (i-1)c^{3y-1}] \leq K'c^{2y}.
\]
In particular \( E(t, d, c) > 0 \) for all \( t \in [0, T] \).
Proof. Since \( \tau_1 \geq i \geq c^{y-1} \) from (3.2) we have (3.3) for \( i = 1 \). Also, from (3.2) we see that
\[
(3.4) \quad k' c^{2\gamma+\gamma(N-1)} \leq E(\tau_1, d, c) \leq K' c^{2\gamma}.
\]
Thus, as in (2.54) we obtain the existence of \( a \in [\tau_1, \tau_1 + Kc^{y-1}] \) such that \( u'(a) = 0 \) and \( u(a) > -((\alpha + 4)c^{y-1}) \). Furthermore, since \( u''(t) \leq Kc^{1-\gamma} c^\gamma + 3c + (M + 1)|u(a)| \leq Kc \) for \( t \in [\tau_1, a] \) we infer \( 0 = u'(a) \leq -c^{\gamma + (\gamma - 1)(N/2)} + Kc(a - \tau_1) \). Therefore \( a - \tau_1 \geq Kc^{\gamma - 1 + (\gamma - 1)(N/2)} \).

Also, by the continuity of \( u' \) we see that there exists \( \hat{a} \in [\tau_1, a] \) such that \( u'((\hat{a}) = 1/2 u'(\tau_1) \). Since \( u'(\hat{a}) = u'(\tau_1) + \int_{\tau_1}^\hat{a} u''(s) \, ds \leq u'(\tau_1) + Kc(\hat{a} - \tau_1) \) we have \( \hat{a} - \tau_1 \geq Kc^{\gamma - 1 + (\gamma - 1)(N/2)} \). Thus
\[
(3.5) \quad u(a) \leq u(\hat{a}) = u(\tau_1) + u'(\hat{a})(\hat{a} - \tau_1) \leq -Kc^{2\gamma - 1 + (\gamma - 1)N}.
\]
Arguing as in (2.6)-(2.7) we see that \( \hat{a} < a + Kc^{y-1} \). In particular there exists a unique \( b \) (\( u \) is convex on \( (\tau_1, \hat{a}) \)) such that \( u(b) = -c^{1/(1+p)} \). Since
\[
(3.6) \quad u(a) \leq u(\tau_1) + \int_{\tau_1}^b u'(s) \, ds \leq -Kc^{2\gamma - 1 + (\gamma - 1)N} + (b - a)Kc^\gamma,
\]
we infer
\[
(3.7) \quad b - a \geq Kc^{\gamma - 1 + (\gamma - 1)N},
\]
where we have also used the fact that \( 2\gamma + (\gamma - 1)N - 1 > 1/(1 + p) \). Hence, as in (2.12) we obtain that \( E(\hat{a}, d, c) \geq Kc^{2\gamma + (\gamma - 1)N} \). Imitating the arguments used in Lemma 2.4 and Lemma 2.5 we show the existence of \( \hat{\tau}_i < \tau_i' < \tau_i^* > \tau_i'' < \tau_i+1 \) such that \( u(\tau_i') = u(\tau_i'') = K \gamma^{-1}(4c) \), \( u'(\tau_i') > 0 \), \( u'(\tau_i'') < 0 \), and \( u'(\tau_i^*) = 0 \). In addition, we obtain that
\[
(3.8) \quad \tau_i' - \tau_i \leq Kc^{1/((1+p)) - \gamma - (\gamma - 1)(N/2)},
\]
and \( E(\tau_i^*, d, c) \geq Kc^{2\gamma + (\gamma - 1)N} \). Therefore, as in Lemma 2.4 we infer
\[
(3.9) \quad \tau_i+1 - \tau_i'' \leq Kc^{1/((1+p)) - \gamma - (\gamma - 1)(N/2)}.
\]
Since \( 1/(1 + p) \) \( - \gamma - (\gamma - 1)(N/2) < \gamma + (\gamma - 1)N - 1 \) (see (2.4a)) we have
\[
(3.10) \quad \left| \int_a^b Kcr^{N-1} u(r) \, dr \right| > \left[ \int_{[\tau_i, \tau_i'] \cup [\tau_i'', \tau_i+1]} Kcr^{N-1} u(r) \, dr \right].
\]
Multiplying (1.8) by \( r^N u' \) and integrating over \( [\tau_i, \tau_i+1] \) we obtain
\[
\tau_i+1 E(\tau_i+1, d, c) - \tau_i E(\tau_i, d, c)
+ \int_{\tau_i}^{\tau_i+1} \left[ \frac{N-2}{2} r^{N-1} (u'(r))^2 - Nr^{N-1} G(u(r)) \right] \, dr
= \int_{\tau_i}^{\tau_i+1} [c\varphi(r) + p(r)] r^N u'(r) \, dr,
\]
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where we have integrated by parts the term \( \int_{\tau_i}^{\tau_{i+1}} r^{N-1} u''(r) u'(r) \) dr. Similarly, multiplying (1.8) by \( r^{N-1} u \) and integrating over \([\tau_i, \tau_{i+1}]\) we infer

\[
\int_{\tau_i}^{\tau_{i+1}} r^{N-1} (u'(r))^2 \, dr = \int_{\tau_i}^{\tau_{i+1}} r^{N-1} \{ g(u(r)) u(r) - [c\varphi(r) + p(r)] u(r) \} \, dr.
\]

By replacing (3.10) in (3.9) we obtain

\[
\tau_{i+1}^N E(\tau_{i+1}, d, c) - \tau_i^N E(\tau_i, d, c)
\]

\[
= \int_{\tau_i}^{\tau_{i+1}} r^{N-1} \left[ NG(u(r)) - \frac{N-2}{2} g(u(r)) u(r) \right] \, dr 
+ \int_{\tau_i}^{\tau_{i+1}} r^{N-1} [c\varphi(r) + p(r)] \left[ ru'(r) + \frac{N-2}{2} u(r) \right] \, dr.
\]

Integrating by parts the last term we get

\[
\tau_{i+1}^N E(\tau_{i+1}, d, c) - \tau_i^N E(\tau_i, d, c) 
\]

\[
\geq \int_{\tau_i}^{\tau_{i+1}} r^{N-1} \left[ NG(u(r)) - \frac{N-2}{2} g(u(r)) u(r) \right] \, dr 
- \frac{c}{2} \int_{\tau_i}^{\tau_{i+1}} \left( \frac{N}{2} + 1 \right) r^{N-1} u(r) \, dr 
\geq \int_{[\tau_i, \tau_i'] \cup [\tau_i'', \tau_i]} r^{N-1} \left[ NG(u(r)) - \frac{N-2}{2} g(u(r)) u(r) - k c u(r) \right] \, dr 
+ \int_{\tau_i}^{\tau_{i+1}} r^{N-1} \left[ NG(u(r)) - \frac{N-2}{2} g(u(r)) u(r) - k c u(r) \right] \, dr 
- k c \int_a^b r^{N-1} u(r) \, dr.
\]

Since \( g \) is an increasing function for \( u > 0 \) we have

\[
G(u) = \int_0^{\kappa u} g(s) \, ds + \int_{\kappa u}^u g(s) \, ds \geq G(\kappa u) + (1 - \kappa) u g(\kappa u).
\]

Thus, from the assumption that \( L(\kappa) = \infty \), there exists \( k_1 \geq 0 \) such that for \( u \geq k_1 \) (\( k_1 \) is chosen so that \( NG(\kappa u) - ((N-2)/2) g(u) u \geq 0 \) for \( \kappa u \geq k_1 \))

\[
NG(u) - \frac{N-2}{2} g(u) u \geq NG(\kappa u) + N \left( \frac{1 - \kappa}{\kappa} \right) k u g(\kappa u) - \frac{N-2}{2} g(u) u
\]

\[
\geq N \left( \frac{1 - \kappa}{\kappa} \right) k u g(\kappa u) \geq N \left( \frac{1 - \kappa}{\kappa} \right) G(\kappa u)
\]

\[
\geq \left( \frac{1 - \kappa}{\kappa} \right) \frac{N-2}{2} g(u) u = k_2 g(u) u.
\]
Thus if \( c \) is large enough and \( \tau'_i, \tau''_i \) are chosen so that \( g(u(\tau'_i)) = g(u(\tau''_i)) = kc/k_2 \) then

\[
\int_{\tau'_i}^{\tau''_i} r^{N-1} \left[ NG(u(r)) - \frac{N-2}{2} g(u(r))u(r) - ck u(r) \right] dr \geq 0.
\]

From (1.3) it follows that \( L(1, u) \) is bounded below. Thus, (3.8), (3.14) and (3.12) yield

\[
\tau^N_{i+1} E(\tau^N_{i+1}, d, c) \geq \tau^N_i E(\tau_i, d, c) \geq Kc^{2\gamma+(\gamma-1)N}.
\]

On the other hand, since \( \phi \) is differentiable we have

\[
E(\tau_{i+1}) \leq E(\tau_i) + \int_{\tau_i}^{\tau_{i+1}} (c\phi(s) + p(s))u'(s) \, ds
\]

\[
\leq E(\tau_i) - c \int_{\tau_i}^{\tau_{i+1}} \phi'(s)u(s) \, ds \leq E(\tau_i) + Kc \int_{\tau_i}^{\tau_{i+1}} |u(s)| \, ds
\]

\[
\leq E(\tau_i) + Kcc^{2\gamma-1}c^{-\gamma}.
\]

Hence inductively we obtain

\[
E(\tau_{i+1}) \leq E(\tau_1) + (i-1)Kc^{3\gamma-1}.
\]

Since \( \tau_{i+1} - \tau_i \geq Kc^{\gamma-1} \), we see that \( i \leq (T/(Kc^{\gamma-1})) \). Therefore

\[
E(\tau_{i+1}) \leq E(\tau_1) + Kc^{2\gamma} \leq K'c^{2\gamma},
\]

which together with (3.15) proves (3.3).

Let \( \gamma_0 := (\gamma_1 + \gamma_2)/2 \), where \( \gamma_1 \) and \( \gamma_2 \) are as in (2.3).

**Lemma 3.2.** Let \( \tau \in [0, T] \) be such that \( u'(\tau) \geq 0 \) and \( u(\tau) \leq -c^{2\gamma_0-1} \). There exist \( C \) such that if \( c > C \) then for some

\[
\tau_1 \in \left( \sqrt{2N} \left( \frac{|u(\tau)|}{3c + (M+1)|u(\tau)|} \right)^{1/2}, 2\sqrt{N} \left( \frac{|u(\tau)|}{c} \right)^{1/2} \right)
\]

we have \( u(\tau_1 + \tau) = 0, \, u(t) < 0 \) for all \( t \in [\tau, \tau + \tau_1] \), and \( E(\tau_1, d, c) \geq c^{2\gamma_0}/64N \).

**Proof.** Since \( g \) is an increasing function, for \( t > \tau \) we have

\[
u'(t) \geq t^{-N+1}c/2 \int_{\tau}^{t} s^{N-1} ds \geq \frac{c}{2N}(t - \tau).
\]

Hence \( u(t) \geq u(\tau) + (c/4N)(t - \tau)^2 \). Therefore, \( \tau_1 \leq 2\sqrt{N}|u(\tau)|/c^{(1/2)} \) exists such that \( u(\tau_1 + \tau) = 0 \). Similarly using that for \( u < 0 \) sufficiently large \( |g(u)| \leq (M+1)|u| \) we have \( u'(t) < t(3c + (M+1)|u(\tau)|)/N \). Thus, \( u(t) \leq u(\tau) + t^2(3c + (M+1)|u(\tau)|)/(2N) \). This proves that

\[
\tau_1 \geq \sqrt{2N} \left( \frac{|u(\tau)|}{3c + (M+1)|u(\tau)|} \right)^{1/2} \geq \frac{\sqrt{2N}}{4} c^{\gamma_0-1}.
\]
Hence,
\[ u'(\tau_1) \geq \frac{c - \|p\|_\infty}{4N} \sqrt{2N} c^{y_0-1} \geq \frac{1}{4\sqrt{2N}} c^{y_0}. \]

Therefore, \( E(\tau_1, d, c) \geq c^{2y_0}/64N \), and that concludes the proof of the lemma.

**Lemma 3.3.** Let \( C \) be as in Lemma 3.2, and \( c > C \). If \( \bar{\tau} \in [0, T] \) is the largest number such that \( E(\bar{\tau}, d, c) = c^{2y_0}/64N \), and \( E(t, d, c) > 0 \) for all \( t \in [0, \bar{\tau}] \) then \( u(\bar{\tau}) \geq -c^{2y_0-1} \).

**Proof.** Suppose that \( u(\bar{\tau}) < -c^{2y_0-1} \). From Lemma 3.2 for all \( t > \bar{\tau} \) with \( u(t) < 0 \) we have
\[
\begin{align*}
\frac{\sqrt{2N}}{2} c^{1-y_0} & \frac{1}{4\sqrt{2N}} c^{y_0} + c\varphi(t) + p(t) - g(u(t)) \geq \frac{c}{2} \left( 1 - \frac{N-1}{2N} \right). \\

\end{align*}
\]
If \( u'(\bar{\tau}) < 0 \) we let \( \bar{t}_0 \) denote the smallest number greater than \( \bar{\tau} \) such that \( u'(\bar{t}_0) = 0 \), and if \( u'(\bar{\tau}) > 0 \) we let \( \bar{t}_0 = \bar{\tau} \). Of course, \( u(\bar{t}_0) \leq u(\bar{\tau}) < -c^{2y_0-1} \).

Since for all \( s > \bar{t}_0 \) with \( u < 0 \) on \((\bar{t}_0, s)\) we have that \(|u'(s)| \leq c^{y_0}/4\sqrt{2N} \), hence \( u(t) < 0 \) for \( t \in (\bar{t}_0, \bar{t}_0 + 4\sqrt{2N} c^{y_0-1}) \). Thus, for \( t = \bar{t}_0 + 4\sqrt{2N} c^{y_0-1} \) we have
\[
\begin{align*}
u'(t_0 + 4\sqrt{2N} c^{y_0-1}) = u'(t_0) + \int_{t_0}^{t_0 + 4\sqrt{2N} c^{y_0-1}} u''(s) \, ds \\
\geq \frac{c}{2} \left( 1 - \frac{N-1}{2N} \right) 4\sqrt{2N} c^{y_0-1} > \frac{1}{4\sqrt{2N}} c^{y_0},
\end{align*}
\]
which contradicts the assumption that \( \bar{\tau} \) is the largest number such that \( E(\bar{\tau}, d, c) = c^{2y_0}/64N \) and \( E(t, d, c) > 0 \) for all \( t \in [0, \bar{\tau}] \). Hence the lemma is proven.

Now we summarize the above results in the following lemma.

**Lemma 3.4.** If \( L(\kappa) = \infty \) for some \( \kappa \in (0, 1) \), then there exists \( C^* \) such that if \( c > C^* \) and \( d \leq -c^{2y_0-1} \) then \( E(t, d, c) > 0 \) for all \( t \in [0, T] \).

**Proof.** From Lemma 3.2 (taking \( \tau = 0 \)) we see that either \( E(t, d, c) \geq c^{2y_0}/64N \) for all \( t \in [0, T] \), or there exists \( \bar{\tau} \geq \tau_1 > 1/4 c^{y_0-1} \) such that \( E(\bar{\tau}, d, c) = c^{2y_0}/64N \) and \( E(t, d, c) > 0 \) for all \( t \in [0, \bar{\tau}] \). Also by Lemma 3.3 we have \( u(\bar{\tau}) \geq -c^{2y_0-1} \). Hence, taking \( y' = y_0 - (\ln 64N)/(2\ln c) \) we have \( E(\bar{\tau}, d, c) = c^{2y'} \), and
\[
u(\bar{\tau}) \geq -c^{2y_0-1} = -c^{2(y_0-y')} c^{2y'-1} = -64N c^{2y'-1} \equiv -\alpha c^{2y'-1}.
\]

Let now \( c^* \) be such that if \( c > c^* \) then \((\ln 64N)/(2\ln c) < 1/4(y_2 - y_1)\). By applying consecutively the lemmas from section 2, depending on the quadrant where \( u(\bar{\tau}) \) lies we obtain the existence of \( \tau_1 \in [\bar{\tau}, \bar{\tau} + Kc^{-(y_2/2)}] \) such that \( u(\tau_1) = 0, \ u'(\tau_1) < 0 \) and (3.2) holds. Now, directly from Lemma 3.1 by taking \( C^* = \max(c^*, C, c) \) we obtain the proof of the lemma.
4. Separation of zeroes

The following lemma is a version of the Sturm comparison theorem for singular linear differential equations, and it proves our main estimate on the separation of zeroes.

Lemma 4.1. There exists $H := H(c) > 0$ such that if $u$ satisfies (1.8), $u(t^0) < -H$ and $u'(t^0) = 0$, for some $t^0 \in [0, T]$, then $u(t) < 0$ for all $t \leq t^0 + \frac{1}{2}(M + 1)^{-1/2}$. Moreover, if for some $t^0 \in [0, T]$, $u(t^0, d, c) = 0$ and $u(t^0, d, c) < -H$, then

$$\psi(T, d, c) \leq \psi(t^0, d, c) + \pi/2 + 2J(t^0)\pi,$$

where $J(t^0)$ is the greatest integer less than $2(M + 1)^{1/2}(T - t^0)$. In particular taking $t^0 = 0$ we have

$$\limsup_{-d \to -\infty} \psi(T, d, c) \leq \frac{\pi}{2} + 2J(0) := S.$$

Proof. From (1.3)–(1.4) we see that there exists a real number $M^* < 0$ such that

$$g(s) - (M + 1)s \geq M^* \quad \text{for all } s \in \mathbb{R}.
$$

Thus, for $A(t) := p(t) + c\varphi(t) - g(u(t)) + (M + 1)u(t)$, we have

$$A(t) \leq \|p\|_\infty + c\|\varphi\|_\infty - M^* := H.
$$

By rewriting (1.8) we obtain

$$u''(t) + \frac{N - 1}{t}u'(t) + (M + 1)u(t) = A(t),
$$

$$u(t^0) = -h, \quad u'(t^0) = 0,
$$

where $h \in \mathbb{R}$ is such that $h > H$. Let now

$$F(t) = \frac{1}{2}[(u'(t))^2 + (M + 1)u^2(t)].
$$

From (4.3) and (4.4) we see that

$$dF/dt \leq A(t)u'(t) \leq H(2F(t))^{1/2}.
$$

Integrating (4.5) on $[t^0, t]$ we obtain

$$|u'(t)| \leq (2F(t))^{1/2} \leq (2F(t^0))^{1/2} + H(t - t^0)
$$

$$\leq (M + 1)^{1/2}h + H(t - t^0).
$$

Let $\tau > t^0$ be such that $u(\tau) = 0$, and $u(t) < 0$ for all $t \in [t^0, \tau]$. Hence, from (4.6) we have

$$0 = u(\tau) = u(t^0) + \int_{t^0}^{\tau} u'(s)\, ds
$$

$$\leq u(t^0) + (M + 1)^{1/2}h(t - t^0) + \frac{1}{2}H(t - t^0)^2
$$

$$\leq h[-1 + (M + 1)^{1/2}(t - t^0) + \frac{1}{2}(t - t^0)^2].$$
Therefore
\[ t - t^0 \leq \frac{1}{2}(M + 1)^{-1/2} := \Lambda. \]
Hence, \( u(t) < 0 \) for all \( t \in [t^0, t^0 + \Lambda] \), and thus in particular
\[ \psi(t^0 + \Lambda, d, c) \leq \psi(t^0, d, c) + \pi/2. \]
Iterating this argument we see that
\[ \psi(t^0 + 2\Lambda, d, c) \leq \psi(t^0, d, c) + \pi/2 + 2\pi, \]
and
\[ \psi(t^0 + i\Lambda, d, c) \leq \psi(t^0, d, c) + \pi/2 + (i - 1)2\pi, \]
with \( i \in \{0, 1, \ldots, J(t^0)\} \). In particular
\[ \psi(T, d, c) \leq \psi(t^0 + J(t^0)\Lambda, d, c) + 2\pi \]
\[ \leq \psi(t^0, d, c) + \pi/2 + (J(t^0) - 1)2\pi + 2\pi \]
\[ = \psi(t^0, d, c) + \pi/2 + J(t^0)2\pi, \]
and that proves the lemma.

5. Proof of Theorem A

By Lemma 3.4 we see that there exists \( C^* \) such that if \( c > C^* \) and \( d \leq -c^{2\gamma_0 - 1} \) then \( E(t, d, c) > 0 \) for all \( t \in [0, T] \). Therefore, if \( c > m_j \) then by Lemma 2.7 we have that
\[ \psi(T, -c^{2\gamma_0 - 1}, c) \geq j\pi + \pi/2. \]
Let \( J := [(S - \pi/2)/\pi] + 1 \), where \([x]\) denotes the largest integer less than or equal to \( x \). If \( j \geq J \) then \( j\pi + \pi/2 > S \). Hence by Lemma 4.1, (5.1), and the intermediate value theorem if \( c > C_j = \max\{m_j, C^*\} \) then there exist numbers \( d_j < d_{j+1} < \cdots < d_j \), such that
\[ \psi(T, d_i, c) = i\pi + \pi/2, \quad i \in \{J, J + 1, \ldots, j\}. \]
Hence, \( u_i(x) = u(\|x\|, d_i, c) \) is a solution to (1.1) with exactly \( i \) interior nodal surfaces. Thus, Theorem A is proven.

References


Department of Mathematics, University of North Texas, Denton, Texas 76203

Department of Mathematics, Texas Christian University, Fort Worth, Texas 76129