Rolling Dice on a Date

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Rolling Dice on a Date

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Abstract

When a young mathematician faces the prospect of a date, all kinds of mathematics ensue. Here we explore her innovative way to keep the conversation going through rolling dice to decide which conversation starter to utilize. In the course of our exploration, we solve an interesting generating function problem.

1. Introduction

One co-author (who shall remain anonymous) recently faced the prospect of a date with a nice young man. Feeling a little bit anxious about her ability to keep the conversation going, she decided to use her mathematical skills in a proactive way. She developed a set of conversation starters, which she numbered and put on notecards. Her plan was to bring the notecards on the date, along with a set of dice. She would roll the dice and compute their sum to determine which conversation starter to use. This paper is the result of her plan.

Our young woman decided to have a set of $c = 365$ questions available, which she would number from 0 to $c - 1 = 364$. She decided to use $k$ dice, each having $n$ sides, with each die numbered from 0 to $n - 1$. Thus, upon rolling the $k$ dice each once, and adding the results, she could obtain sums between 0 and $k(n-1)$. Her first task therefore was to choose $(k, n)$ integer pairs such that $c - 1 = k(n - 1) = 364$. She found the possibilities given in Table 1.
She considered this plan, along with the pros and cons of carrying strange dice, carrying many dice, or rolling a die many times. She began to realize that, unless she brought a 365-sided die on her date night, some of her conversation starters would be more likely to be used than others.

2. Using generating functions

Suppose the young woman rolls $k$ dice, each having $n$ sides, with each die labeled from 0 to $n - 1$. The ordinary generating function governing the possible dice sums is given by

$$\left( \sum_{j=0}^{n-1} x^j \right)^k.$$

The resulting coefficient of $x^i$ in the polynomial expansion gives the number of ways that conversation starter $i$ can be chosen through this dice rolling procedure, for $i = 0, \ldots, c - 1$.

In rolling $k$ distinct dice, each having $n$ sides, the total number of possible outcomes (in terms of sequences of die results) is $n^k$. Thus, the probability

<table>
<thead>
<tr>
<th>$k$ dice</th>
<th>$n - 1$</th>
<th>$n$ sides</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>364</td>
<td>365</td>
</tr>
<tr>
<td>2</td>
<td>182</td>
<td>183</td>
</tr>
<tr>
<td>4</td>
<td>91</td>
<td>92</td>
</tr>
<tr>
<td>7</td>
<td>52</td>
<td>53</td>
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<tr>
<td>13</td>
<td>28</td>
<td>29</td>
</tr>
<tr>
<td>14</td>
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<td>26</td>
<td>14</td>
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<td>28</td>
<td>13</td>
<td>14</td>
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<tr>
<td>52</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>91</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>182</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>364</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 1: Dice rolling possibilities.
of conversation starter \( i \) being chosen is given by

\[
\frac{\text{coefficient of } x^i \text{ in } \left( \sum_{j=0}^{n-1} x^j \right)^k}{n^k}.
\] (1)

Sounds simple. But there are a lot of terms involved. If we were to naively multiply out the polynomial raised to the \( k \)th power, ignoring all the special structure, we would have \( n^k \) terms. Upon combining like terms, we would end up with \( k(n - 1) + 1 \) terms (one more than the degree of the resulting polynomial), and finding the coefficient of \( x^i \) as a function of \( i \) is not easy.

Math comes to the rescue. We now compute this coefficient of \( x^i \). Note that

\[
\left( \sum_{j=0}^{n-1} x^j \right)^k = \left( \frac{1 - x^n}{1 - x} \right)^k = (1 - x^n)^k (1 - x)^{-k}.
\]

In what follows we will make use of the binomial theorem: For any integer \( k \geq 0 \) and any real numbers \( a \) and \( b \),

\[
(a + b)^k = \sum_{h=0}^{k} \binom{k}{h} a^{k-h} b^h.
\]

Let \( a = 1 \) and \( b = -x^n \). We see that the polynomial expansion of \((1 - x^n)^k\) expands as a linear combination of terms \( 1, x^n, x^{2n}, \ldots, x^{kn} \), with the coefficient of \( x^{hn} \) being given by

\[
(-1)^h \binom{k}{h}.
\]

The polynomial expansion of \((1 - x)^{-k}\) contains terms of all non-negative integer degrees. Note by using a Taylor series about 0 that

\[
(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots
\]

and so

\[
(1 - x)^{-k} = (1 + x + x^2 + x^3 + \ldots)^k.
\]

The coefficient of \( x^g \) in the expansion of the expression on the right is known to be given by

\[
\left( \binom{k}{g} \right).
\]
where
\[
\binom{k}{g} = \binom{g + k - 1}{g}.
\]
See, for example, [1].

In what comes next, we will use the convolution formula: If \( f(x) = \sum_{e \geq 0} p_e x^e \) and \( g(x) = \sum_{e \geq 0} q_e x^e \), then for any \( i \geq 0 \), the coefficient of \( x^i \) in \( f(x)g(x) \) is given by
\[
\sum_{e=0}^{i} p_e q_{i-e}.
\]
Again see [1]. Let \( f(x) = (1 - x^n)^k \) and \( g(x) = (1 - x)^{-k} \). Note that the expansion of \( f(x) = (1 - x^n)^k \) only contains the powers of \( x \) that are non-negative multiples of \( n \), that is, \( 1, x^n, x^{2n}, \ldots \). Thus we define an index \( d \) related to \( e \) by \( e = nd \). When \( e = 0 \), we have that \( d = 0 \); when \( e = n \), we have that \( d = 1 \); when \( e = 2n \), we have that \( d = 2 \), and so on. When \( e = i \), the upper limit of the summation, we have that \( d = i/n \). Since we only want to sum over the relevant integer powers of \( x \) that are non-negative multiples of \( n \), the upper limit of our summation becomes the floor \( d = \lfloor i/n \rfloor \) so that \( e \) does not exceed \( i \). We have that the coefficient of \( x^i \) in \( \left( \sum_{j=0}^{n-1} x^j \right)^k \) is
\[
\sum_{d=0}^{\lfloor i/n \rfloor} (-1)^d \binom{k}{d} \binom{k}{i - nd}.
\]
Thus, the probability of conversation starter \( i \) being chosen is
\[
\frac{1}{n^k} \sum_{d=0}^{\lfloor i/n \rfloor} (-1)^d \binom{k}{d} \binom{k}{i - nd}.
\]
This formula is not simple. In fact, if we were to use the formula for every \( i \) value from 0 to \( k(n-1) \), in total we would end up doing \( O(k^2n) \) operations, assuming that we already have the binomial coefficients and the value of \( n^k \) stored.

In Figure 1, we show the probability mass functions for the probability of conversation starter \( i \) being chosen, for the \((k, n)\) pairs from Table 1. The curve with the highest peak corresponds to rolling 364 dice, each having
2 sides (labeled 0 and 1): \((k, n) = (364, 2)\). The curve with the second highest peak corresponds to \((k, n) = (182, 3)\), and so on. The horizontal line (uniformly distributed) corresponds to rolling one 365-sided die (labeled 0 through 364): \((k, n) = (1, 365)\). From the options in Table 1, the only way to obtain a uniform distribution for the usage of the 365 conversation starters is to carry a 365-sided die on the date. The more dice involved here (or equivalently the more die rolls), the less uniform the outcome (but can you find a way to prove this?).

3. Platonic solids and platonic dates

Considering the impracticality of having dice with such unusual numbers of sides, it might be more practical to consider some \(n\)-sided dice that can be commonly purchased, \(n = 4, 6, 8, 10, 12\), and 20. This list includes the platonic solids (tetrahedron, cube, octahedron, dodecahedron, and icosahedron).
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along with the commonly included 10-sided die. This results in potential $n - 1$ values of 3, 5, 7, 9, 11, and 19. The closest least common multiple to 365 for the most of these numbers is 315. Thus, we will use $c = 316$ as the new number of conversation-starters to be brought on the date. It becomes practical to consider the cases shown in Table 2, corresponding to use of the tetrahedron, the cube, the octahedron, and the 10-sided die, respectively.

<table>
<thead>
<tr>
<th>$k$ dice</th>
<th>$n - 1$</th>
<th>$n$ sides</th>
</tr>
</thead>
<tbody>
<tr>
<td>105</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>63</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>45</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>35</td>
<td>9</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 2: More dice rolling possibilities.

As we saw earlier, for a fixed $c$-value, the more dice involved, the less uniform the outcome. Among these options, the closest to uniform results will be obtained by rolling thirty-five 10-sided (“D10”) dice. Among the platonic options, forty-five octahedral dice may be used.

As for the young woman and her date? Let’s just say it went platonic too. By the end of this mathematics project she told her co-author that the project was more interesting than the date!

References