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A LOCAL INVERSION PRINCIPLE OF THE NASH–MOSER TYPE

ALFONSO CASTRO† AND J. W. NEUBERGER‡

Abstract. We prove an inverse function theorem of the Nash–Moser type. The main difference
is that we use continuous steepest descent while Moser uses a combination of Newton-type iterations
and approximate inverses. We bypass the loss of derivatives problem by working on finite dimensional
subspaces of infinitely differentiable functions.

Key words. Nash–Moser methods, inverse function theorem, continuous steepest descent

AMS subject classifications. Primary, 34B15; Secondary, 35J65

1. Introduction. Inverse function theorems are fundamental tools for the study
of solutions to nonlinear equations. Proofs depend on iteration arguments. When a
nonlinear equation comes from a partial differential equation, it often happens that
the operators under consideration do not have enough regularity and the iteration
process is defined only for a few steps. This is known as loss of derivatives. In
order to overcome this difficulty, iteration techniques have been designed to allow the
iteration to lead to a limit. These are known as generalized inverse function theorems.
They generally depend on having a scale of intermediate spaces between the domain
and the range of the nonlinear operator under consideration. Most notable of such
results is the one proven by J. Moser in [5] and [6] (see also [2]). The reader is referred
to [3] for a sharper version of the results of [5].

Here we prove an inverse function theorem (Theorem 2.1 below) using finite
dimensional subspaces of infinitely differentiable functions, which makes the phe-
nomenon of loss of derivatives immaterial. In addition we only assume the operators
to have a first order derivative. This is in contrast with the results of [5] and [3] where
Newton-like iteration techniques require extensive use of the properties of the
quadratic remainder (see (10)). In [12] another inverse function theorem is proven
without assumptions on the quadratic remainder. Our use of continuous steepest de-
scent has roots in [1], [7], [8], and [9]. For applications of generalized inverse function
theorems to elliptic systems the reader is referred to [4] and [11].

2. Main result. For the sake of simplicity we present our main result in the
context of a particular class of Sobolev spaces. However, the general principle applies
to many other scales of spaces sharing the general properties of the Sobolev spaces
that are defined in the next paragraph. The reader is referred to [2] for various other
scales of spaces.

Let each of m and n denote a positive integer. For each nonnegative integer ρ let
$C^ρ$ denote the set of ρ-differentiable functions $u : R^n \to R^m$ that are 2π-periodic in
each of its n independent real variables. The norm in $C^ρ$ is given by

$$
|u|_ρ = \max\{|u(x)|; x \in R^n\} + \max\{|D^αu(x)|; x \in R^n, |α| = ρ\};
$$

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here, \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is an \( n \)-tuple of nonnegative integers and \(|\alpha| = \alpha_1 + \cdots + \alpha_n\). Similarly for \( r \geq 0 \) we define \( H^r \) as the Sobolev space of functions \( u : \mathbb{R}^n \to \mathbb{R}^m \) of the form

\[
(2) \quad u(x) = \sum_{j \in \mathbb{Z}^n} c_j e^{ij \cdot x}
\]

such that

\[
(3) \quad \|u\|^2 = \sum_{j \in \mathbb{Z}^n} |c_j|^2 + \sum_{j \in \mathbb{Z}^n} |j^{2r}|^2 |c_j|^2 < \infty,
\]

where \(|(j_1, \ldots, j_n)|^2 = j_1^2 + \cdots + j_n^2\). Actually \( H^r \) is also defined for \( r < 0 \) provided that the expression in (2) is understood in the sense of distributions and

\[
(4) \quad \|u\|^2 = |c_0|^2 + \sum_{j \in \mathbb{Z}^n \setminus \{0\}} |j^{2r}|^2 |c_j|^2 < \infty.
\]

For \( r \geq 0 \) the inner product in \( H^r \) is given by

\[
(5) \quad \left\langle \sum_{j \in \mathbb{Z}^n} c_j e^{ij \cdot x}, \sum_{j \in \mathbb{Z}^n} d_j e^{ij \cdot x} \right\rangle_r = \sum_{j \in \mathbb{Z}^n} (1 + |j|^{2r}) c_j \bar{d}_j.
\]

Let \( \epsilon > 0 \) and \( F : \{x \in C^1; |x|_1 < \epsilon\} \to C^0 \) be a continuous function such that \( F(0) = 0 \). Typically \( F \) is a first order differential operator. We assume that for \( u, v \in C^1 \), with \(|u|_1 < \epsilon\), the limit

\[
(6) \quad \lim_{t \to 0} \frac{F(u + tv) - F(u)}{t} = F'(u)v
\]

exists and is a continuous function of \( v \) and that \( F'(\cdot)v \) defines a continuous function from \( C^1 \) into \( C^0 \), for each \( v \in C^1 \). We also assume that there exist \( \lambda \in \mathbb{R} \), positive constants \( k_1, k_2, k_3 \), and \( l > (n/2) + 1 \) with

\[
(7) \quad \langle F'(u)v, v \rangle_0 \geq k_1 \|v\|_0^2
\]

for all \( u, v \in C^1 \);

\[
(8) \quad \langle F'(u)v, v \rangle_1 \geq k_2 \|v\|^2 - k_3 \|v\|_0^2
\]

for \( u \in C^1, |u|_1 \leq \epsilon, v \in H^l \); and for each positive integer \( \rho \) there exists \( M_\rho \) such that

\[
(9) \quad \|F(u)\|_\rho \leq M_\rho \|u\|_{\rho + 1} \quad \text{for} \quad |u|_1 < \epsilon.
\]

Without loss of generality we may assume that \( k_1 < k_2 \).

In [6], it is shown that the equation \( F(u) = g \) has a solution when the operators \( F'(u) \) have approximate inverses, and

\[
(10) \quad \|F(u + v) - F(u) - F'(u)v\|_0 \leq M \|v\|_0^{2-\beta} \|v\|_1^\beta \quad \text{for} \quad |u|_1 < \epsilon, \ |u+v|_1 < \epsilon,
\]

where \( M \) is a constant independent of \( u, v \) and \( 0 \leq \beta < 1 \). In addition, it is shown that (7) and (8) with \( l = \lambda \) are sufficient for \( F'(u) \) to have an approximate inverse.

Here we prove the following theorem.
Theorem 2.1. If (6)–(9) hold and $l > (n/2) + 1$, then there exist $\delta > 0$ such that, if $\|g\|_l < \delta$, then the equation

$$ F(y) = g $$

has a solution.

Proof. For each positive integer $k$, let $X_k$ denote the linear subspace of $H^l$ of functions of the form

$$ u(x) = \sum_{\|j\|^2 \leq k^2} c_j e^{ij \cdot x}. $$

Let $P_k \equiv P$ denote the orthogonal projection of $H^l$ onto $X_k$. An elementary Fourier series argument shows that $P$ is also an orthogonal projection of $H^0$ onto $X_k$ and

$$ \langle Pu, v \rangle_0 = \langle u, Pv \rangle_0 \quad \text{for all} \quad u, v \in H^l. $$

Now let $\Lambda \in (n/2 + 1, l)$. Since $X_k$ is finite dimensional and a subset of $C^1$, by (6) there exists a bounded differentiable function $\tilde{F} : X_k \rightarrow X_k$ such that $PF(u) = \tilde{F}(u)$ if $\|u\|_{X} < \epsilon/2$, $u \in X_k$. Hence the initial value problem

$$ z'(t) = -\tilde{F}(z(t)) + P(g), \quad t \geq 0, \quad z(0) = 0, $$

has a solution defined on $[0, \infty)$.

Let us see that, for $\|g\|$ small enough, $|z(t)|_1 < \epsilon/2$ for all $t \geq 0$. In fact, let $w(t) = P(\tilde{F}(z(t)) - g)$. Thus

$$ (\|w(t)\|_0^2)' = 2 \langle w(t), P\tilde{F}'(z(t))z'(t) \rangle_0 $$

$$ = -2 \langle Pw(t), \tilde{F}'(z(t))(P(w(t)))_0 $$

$$ \leq -2 k_1 \|w(t)\|_0^2. $$

In particular we see that the quantity $\|w(t)\|_0$ is a decreasing function of $t$. In addition, from (15) we have

$$ \|w(t)\|_0 \leq \|w(0)\|_0 e^{-k_1 t} = \|g\|_0 e^{-k_1 t}. $$

Now we estimate the $H^l$ norm of $w(t)$. In order to do so we observe that for each $(k, \lambda)$ there exists a positive constant $C(k, \lambda)$ such that

$$ \|x\|_X^l \geq C(k, \lambda) \|x\|^l_0 \quad \text{for all} \quad x \in X_k. $$

We note that $C(k, \lambda) \rightarrow 0$ as $k \rightarrow \infty$ when $\lambda < l$; otherwise $C(k, \lambda)$ can be taken to be equal to 1. From now on we restrict ourselves to the case $\lambda < l$; the case $\lambda \geq l$ is simpler. Thus we may assume that

$$ C(k, \lambda) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. $$

From (8), (13), (16), (17), and (18) we infer that for $k$ sufficiently large

$$ (\|w(t)\|_0^2)' = 2 \langle w(t), P\tilde{F}'(z(t))z'(t) \rangle_0 $$

$$ = -2 \langle Pw(t), \tilde{F}'(z(t))(P(w(t)))_0 $$

$$ \leq -2 k_2 \|w(t)\|_0^2 - k_3 \|w(t)\|_0^2 $$

$$ \leq -2 k_2 C(k, \lambda) \|w(t)\|_0^2 - k_3 \|g\|_0^2 e^{-2k_1 t}. $$
Thus for $k$ sufficiently large

$$\|w(t)\|^2 \leq (\|g\|^2 + k_3\|g\|^2_0/(k_1 - k_2 C(k, \lambda)))e^{-2k_2 C(k, \lambda)t}$$

(20)

$$\leq (\|g\|^2 + 2k_3\|g\|^2_0) e^{-2k_2 C(k, \lambda)t}$$

$$\equiv M(\|g\|_I) e^{-2k_2 C(k, \lambda)t},$$

where $M(\|g\|_I) \to 0$ as $\|g\|_I \to 0$. By interpolation properties of Sobolev space (see section I.2 in [6]) one has

$$\|w(t)\|_\Lambda \leq \|w(t)\|_0^{(1-\Lambda/L)} \|w(t)\|_L^{\Lambda/L} \leq M(\|g\|_I)^{\Lambda/(2l)} g\|_0^{(1-\Lambda/L)} e^{-k_1(1-\Lambda/L)t}.$$

Integrating (14) by (20) and (21), we see that there exist $\delta > 0$ such that if $\|g\|_I \leq \delta$, then

$$\|z(t)\|_\Lambda < \epsilon/3$$

for all $t \geq 0$.

Now letting $x_k \in X_k$ be an element in the $w$-limit set of the orbit defined by $z$, we see that $\|x_k\|_\Lambda \leq \epsilon/3$ and $\hat{F}(x_k) = PF(x_k) = P(g)$. Therefore by the Sobolev imbedding theorem we may assume that it converges to some element $x \in H^{\lambda_1}$, with $\lambda_1 \in ((N/2) + 1, l)$. By (9), $F(x_k)$ is bounded in $H^{\lambda_1-1}$. Using again that bounded sequences in $H^s$ have converging subsequences in $H^t$ if $s > t$, we may further assume that $\{F(x_k)\}$ converges in $H^{\lambda_2}$ with $\lambda_2 \in (n/2, \lambda_1 - 1)$. Recalling that, by Poincaré's inequality,

$$\|z\|_{\lambda_2-1} \leq C_k \|z\|_{\lambda_2}$$

for all $z \in X_k^\perp$,

with $C_k$ converging to zero as $k \to \infty$, we conclude that $\|(I - P)F(x_k)\|_{\lambda_2-1} \to 0$ as $k \to \infty$. This and the fact that $\{F(x_k)\}$ converges to $F(x)$ imply that $\{P(F(x_k))\}$ converges to $F(x)$ in $H^{\lambda_2}$. Hence in $H^{\lambda_2}$ we have

$$g = \lim P_k(g) = \lim P_k(F(x_k)) = F(x),$$

and this proves the theorem. □

REFERENCES


