1-1-1979

A Semilinear Dirichlet Problem

Alfonso Castro

Harvey Mudd College

Recommended Citation
A SEMILINEAR DIRICHLET PROBLEM

ALFONSO CASTRO

Introduction and notations. Let \( \Omega \) be a bounded region in \( \mathbb{R}^n \). In this note we discuss the existence of weak solutions (see [4, Section 2]) of the Dirichlet problem

\[
\begin{align*}
\Delta u(x) + g(x, u(x)) + f(x, u(x), \nabla u(x)) &= 0 & x \in \Omega \\
u(x) &= 0 & x \in \partial\Omega
\end{align*}
\]

where \( \Delta \) is the Laplacian operator, \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) and \( f : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R} \) are functions satisfying the Caratheodory condition (see [2, Section 3]), and \( \nabla \) is the gradient operator.

We let \( \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_m \leq \ldots \) denote the sequence of numbers for which the problem

\[
\begin{align*}
\Delta u(x) + \lambda u(x) &= 0 & x \in \Omega \\
u(x) &= 0 & x \in \partial\Omega
\end{align*}
\]

has nontrivial weak solutions.

The main result of this paper is:

Suppose the following two hypotheses hold.

1. The function \( g(x, u) \) admits a derivative with respect to \( u \), \( \partial g/\partial u : \Omega \times \mathbb{R} \to \mathbb{R} \), which satisfies the Caratheodory condition; furthermore, there exist \( \alpha, \alpha_1 \in \mathbb{R} \) and a positive integer \( N \) such that

\[
\lambda_N < \alpha \leq \partial g/\partial u(x, u) \leq \alpha_1 < \lambda_{N+1} \quad \text{for all } (x, u) \in \Omega \times \mathbb{R}
\]

2. There exist a constant \( \beta > 0 \) and a function \( c(x) \in L_2(\Omega) \) such that

\[
|f(x, u, y)|^2 \leq c(x) + \beta \|y\|^2
\]

for all \( (x, u, y) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \), where \( \| \| \) denotes the usual norm in \( \mathbb{R}^n \).

If

\[
\beta < (\min \{ 1 - \alpha_1/\lambda_{N+1}, \alpha/\lambda_N - 1 \})/\sqrt{\lambda_1}
\]

then (I) has a weak solution.

As a corollary of our main result we obtain bounds for the eigenvalues on \( (\lambda_N, \lambda_{N+1}) \) of a class of non-selfadjoint problems of the form:

\[
\begin{align*}
\Delta u(x) + \langle (a_1(x), \ldots, a_n(x)), \nabla u(x) \rangle + \lambda u(x) &= 0 & x \in \Omega \\
u(x) &= 0 & x \in \partial\Omega
\end{align*}
\]

where \( \langle , \rangle \) denotes the usual inner product in \( \mathbb{R}^n \) and \( a_1, \ldots, a_n \in L_\infty(\Omega) \).

Received November 3, 1977 and in revised form March 22, 1978. This research was done while the author was a Lefschetz researcher at Centro de Investigacion del IPN.
In [2, Theorem 1] and [4, Theorem 3.1] the problem (1) is considered and the existence of weak solutions is proved when \( f(x, u, y) = o(||y||) \) as \( ||y|| \to +\infty \). In [3, Theorem 3.4] the problem (1) is studied when \( \Omega \subset \mathbb{R} \) and \( f \) and \( g \) are permitted to depend on the second order derivatives. The results of [3] yield inequalities of the form (1.3) when \( \alpha_1 < \lambda_1 \). We denote in this paper by \( H^1 \) the Sobolev space \( H^{1,2}_0(\Omega) \) (see [1, p. 45]). We take as inner product in \( H^1 \) the bilinear form defined by

\[
\langle u, v \rangle_1 = \int_\Omega (\nabla u(\xi), \nabla v(\xi)) d\xi.
\]

We denote by \( || \cdot || \) the norm on \( H^1 \) and by \( || \cdot ||_0 \) the norm on \( L^2(\Omega) \). We let \( X \) denote the closed subspace of \( H^1 \) spanned by the eigenfunctions of (II) corresponding to eigenvalues \( \lambda_\varepsilon \) with \( \lambda_\varepsilon \leq \lambda_N \). We use the symbol \( \int \) to mean integral over \( \Omega \).

**Proofs.** From now on we assume that (1.1) and (1.2) hold. Let \( J : H^1 \times H^1 \to \mathbb{R} \) be defined by

\[
J(y, u) = \int_\Omega \left( ||\nabla u(\xi)||^2/2 - G(\xi, u(\xi)) - f(\xi, y(\xi), \nabla y(\xi))u(\xi) \right) d\xi,
\]

where \( G : \Omega \times \mathbb{R} \to \mathbb{R} \) is a continuous function such that \( \partial G/\partial u(x, u) = g(x, u) \) and \( G(x, 0) = 0 \). It is not difficult to see that for \( y, u, v \in H^1 \)

(2.1) \( \lim_{t \to 0} (J(y, u + tv) - J(y, u))/t = \int_\Omega \left( \langle \nabla u(\xi), \nabla v(\xi) \rangle - g(\xi, u(\xi))v(\xi) \right. \\
\left. - f(\xi, y(\xi), \nabla y(\xi))v(\xi) \right) d\xi. \)

Therefore, by Vainberg’s lemma (see [6, p. 63]), if (1.1) holds, the right hand side of (2.1) defines a continuous linear functional on \( v \in H^1 \). Hence, for each \( (y, u) \in H^1 \times H^1 \) there exist \( S(y, u) \in H^1 \) such that

(2.2) \( \lim_{t \to 0} J(y, u + tv) - J(y, u)/t = \langle u, v \rangle_1 + \langle S(y, u), v \rangle_1. \)

By (1.1), (1.2) and Vainberg’s Lemma (see [2, Proposition 4]) the functions \( u(\xi) \to g(\xi, u(\xi)) \) and \( y(\xi) \to f(\xi, y(\xi), \nabla y(\xi)) \) are continuous functions from \( H^1 \) into \( L_2(\Omega) \). Since, by Rellich’s principle, the inclusion of \( L_2(\Omega) \) into the dual space of \( H^1 \) is compact, \( S(y, u) \) is a compact function.

From (1.1) and the results of [5, Section 7] it follows that

\[
\Delta u(x) + g(x, u(x)) + f(x, y(x), \nabla y(x)) = 0 \quad x \in \Omega \\
u(x) = 0 \quad x \in \partial \Omega
\]

has a unique weak solution for each \( y \in H^1 \). Therefore, for each \( y \in H^1 \) there exists a unique \( \varphi(y) \in H^1 \) such that

(2.3) \( \langle \varphi(y) + S(y, \varphi(y)), v \rangle_1 = 0 \quad \text{for all } v \in H^1. \)
**Lemma 1.** The function \( \varphi : H^1 \to H^1 \) defined by (2.3) is compact.

*Proof.* First we show that \( \varphi \) is continuous. From the discussion in [5, Section 7] we see that if \( J_y : H^1 \to \mathbb{R} \) is defined by \( J_y(u) = J(y, u) \) then \( J_y \) is of class \( C^2 \). Let \( DJ_y(u) \) be the Hessian of \( J_y \) at \( u \). An elementary computation show that

\[
(DJ_y(u)v, v)_1 = \int \left( \| \nabla v \|^2 - \partial g / \partial u (\xi, u(\xi)) v^2(\xi) \right) d\xi.
\]

Following the arguments of [4, Section 7] we see that \( DJ_y(u) \) is a nonsingular Fredholm operator.

Let \( T : H^1 \times H^1 \to H^1 \) be defined by \( T(y, u) = u + S(y, u) \). Hence \( T \) is continuously differentiable with respect to \( u \) and \( \partial_u T(y, u) = DJ_y(u) \). Thus, by (2.3), for any \( y_0 \in H^1 T(y_0, \varphi(y_0)) = 0 \). By the foregoing argument \( \partial_u T(y_0, \varphi(y_0)) \) is nonsingular. Therefore, by the implicit function theorem there exist a neighborhood \( V \) of \( y_0 \) and a continuous function \( \psi : V \to H^1 \) such that \( T(y, \psi(y)) = 0 \) for all \( y \in V \). Consequently, by the uniqueness of \( \varphi(y) \), we have \( \varphi(y) = \psi(y) \) on \( V \), and this proves that \( \varphi \) is continuous.

Now we prove that \( \varphi \) is bounded on bounded sets. For \( y \in H^1 \), let \( \varphi_1(y) \) be the orthogonal projection of \( \varphi(y) \) on \( X \), and let \( \varphi_2(y) \) be \( \varphi(y) - \varphi_1(y) \). By (2.3) we have

\[
0 = \langle \varphi(y) + S(y, \varphi(y)), \varphi_2(y) - \varphi_1(y) \rangle_1.
\]

Hence,

(2.4) \[
0 = \| \varphi_2(y) \|^2_1 - \| \varphi_1(y) \|^2_1 - \int g(\xi, \varphi(y)(\xi))(\varphi_2(y)(\xi) - \varphi_1(y)(\xi))d\xi
\]

\[
- \int f(\xi, y(\xi), \nabla y(\xi))(\varphi_2(y)(\xi) - \varphi_1(y)(\xi))d\xi
\]

\[
\geq \| \varphi_2(y) \|^2_1 - \| \varphi_1(y) \|^2_1 - \sqrt{\lambda_1} \| g(\xi, 0) \|_0 \cdot \| \varphi(y) \|_1 - \lambda_1 \| \varphi_2(y) \|^2_0
\]

\[
+ \alpha \| \varphi_1(y) \|^2_0 - \left( \int f^2(\xi, y(\xi), \nabla y(\xi))d\xi \right)^{1/2} \cdot \sqrt{\lambda_1} \cdot \| \varphi(y) \|_1
\]

\[
\geq (1 - \alpha_1/\lambda_{N+1}) \| \varphi_2(y) \|^2_1 + (\alpha/\lambda_N - 1) \| \varphi_1(y) \|^2_1
\]

\[
- \sqrt{\lambda_1} \| g(\xi, 0) \|_0 \cdot \| \varphi(y) \|_1 - \left( \int f^2(\xi, y(\xi), \nabla y(\xi))d\xi \right)^{1/2} \cdot \sqrt{\lambda_1} \cdot \| \varphi(y) \|_1
\]

Thus, if \( m = \min \{1 - \alpha_1/\lambda_{N+1}, \alpha/\lambda_N - 1\} \) then we have

(2.5) \[
\sqrt{\lambda_1} \| g(\xi, 0) \|_0 + \sqrt{\lambda_1} \left( \int f^2(\xi, y(\xi), \nabla y(\xi))d\xi \right)^{1/2} \geq m \| \varphi(y) \|_1.
\]

Since, by (1.2), the Nemitski operator \( y(\xi) \to f(\xi, y(\xi), \nabla y(\xi)) \) maps bounded sets of \( H^1 \) into bounded sets of \( L_2(\Omega) \) we infer from (2.5) that \( \varphi \) is bounded on bounded sets.
Suppose \( \{y_n\} \) is a bounded sequence in \( H^1 \). Hence \( \{S(y_n, \varphi(y_n))\} \) contains a convergent subsequence \( \{S(y_{n_j}, \varphi(y_{n_j}))\} \). By (2.3), \( -\varphi(y_{n_j}) = S(y_{n_j}, \varphi(y_{n_j})) \). Therefore, \( \{\varphi(y_{n_j})\} \) is a convergent sequence. Consequently, \( \varphi \) is compact and the lemma is proved.

**Theorem 2.** If (1.1), (1.2) and (1.3) hold then the problem (1) has a weak solution.

**Proof.** By (1.2), there exists \( K \in \mathbb{R} \) such that

\[
(2.6) \quad \left( \int f^2(\xi, y(\xi), \nabla y(\xi))d\xi \right)^{1/2} \leq K^2 ||c(x)||_0 + \beta^2 ||y||^2
\]

for all \( y \in H^1 \). Combining (2.5) and (2.6) we have

\[
(2.7) \quad m||\varphi(x)||_1 \leq \sqrt{\lambda_1}||g(\xi, 0)||_0 + \sqrt{\lambda_1}K||c(x)||_0^{1/2} + \beta\sqrt{\lambda_1}||y||_1.
\]

Therefore, by (1.3), if \( R > 0 \) is big enough then the function \( \varphi \) maps the ball of center 0 and radius \( R \) into itself. Consequently, by Schauder’s fixed point theorem, \( \varphi \) must have a fixed point. Since any fixed point of \( \varphi \) is a weak solution of (1) the theorem is proved.

**Corollary 3.** If \( (\int (a_1^2(\xi) + \ldots + a_n^2(\xi))d\xi)^{1/2} \leq \beta \) then the problem (III) does not have eigenvalues in the open interval

\[
(\lambda_N(1 + \beta\sqrt{\lambda_1}), \lambda_{N+1}(1 - \beta\sqrt{\lambda_1})) = D.
\]

**Proof.** If \( \lambda \in D \), then following the proof of Theorem 2 we see that for any \( c(x) \in L_2(\Omega) \) the problem

\[
\Delta u(x) + \left( (a_1(x), \ldots, a_n(x)), \nabla u(x) \right) + \lambda u(x) = c(x) \quad x \in \Omega
\]

\[
u(x) = 0 \quad x \in \partial \Omega
\]

has a weak solution. Therefore by the Fredholm alternative (see [2, Proposition 1]) \( \lambda \) cannot be an eigenvalue of (III).

**References**


Centro de Investigacion del IPN, Mexico