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POSITIVE SOLUTIONS FOR A SEMILINEAR ELLIPTIC PROBLEM WITH CRITICAL EXPONENT

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1. Introduction

The solvability of boundary value problems of the form

$$\Delta u + f(u) = g(x) \text{ in } \Omega, \quad u(x) = 0 \text{ on } \partial \Omega, \quad (1.1)$$

where $\Omega$ is a smooth bounded region in $\mathbb{R}^N$, $N \geq 3$, and $\Delta$ is the Laplacian operator, depends on the growth of the nonlinearity $f$. We say that $f$ grows subcritically if there exists

$$q \in (1, (N + 2)/(N - 2))$$

such that $\limsup_{|u| \to \infty} |f(u)|/|u|^q < \infty$. If $\lim_{|u| \to \infty} (|f(u)|/|u|^{(N+2)/(N-2)}) \in \mathbb{R}$ then we say that $f$ grows critically. In order to apply to this problem compactness techniques such as those derived from the imbedding properties of the Sobolev spaces (see [1]) one realizes that $f$ must grow subcritically. Moreover, in [2], Pohozaev showed results for the subcritical case that do not extend to the critical case. Here we show, in particular, the existence of large positive solutions for small values of $g$ when $f$ grows critically, which is not the case when $f$ grows subcritically. For related problems with critical exponents the reader is referred to [3–8].

From now on we consider the boundary value problem (1.1) when $\Omega$ is the unit ball in $\mathbb{R}^N$, $f(u) = |u|^p$ with $p = 4/(N - 2)$, and $g(x) = -\lambda \in \mathbb{R}$. Our main result is the following theorem.

**Theorem 1.** There exists a continuous function $F: (0, \infty) \to (0, \infty)$ such that $u$ is a positive solution to (1.1) if and only if $\lambda = F(u(0))$. If $u_1$ and $u_2$ are positive solutions to (1.1) with $u_1(0) = u_2(0)$ then $u_1 \equiv u_2$. Moreover, $\lim_{d \to 0} F(d) = 0$, $\lim_{d \to \infty} F(d) = 0$, and there exists $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$ then (1.1) has exactly two solutions. In particular, $\{(\lambda, u); u > 0, u \text{ satisfies (1.1)}\}$ is connected.

The classical work of Gidas et al. [7] tells us that positive solutions in $\Omega$ are radially symmetric. This allows us to shift our study to the ordinary differential equation

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\[ u'' + \frac{N-1}{r} u' + |u|^pu + \lambda = 0, \quad r \in (0, 1), \]  
\[ u'(0) = 0, \]  
\[ u(1) = 0. \]  
Instead of considering directly this boundary value problem we study \( u(r, \lambda, d) \), the solution to the initial value problem (1.2), (1.3), and
\[ u(0) = d. \]  
In order to prove theorem 1 we establish that \( u_d(1, \lambda, d) < 0 \) and that for \( d > 0 \) large, \( u_d(1, \lambda, d) < 0 \) when \( u(\cdot, \lambda, d) \) satisfies (1.2), (1.3), (1.4).

2. PRELIMINARIES

First we establish "Pohazaev identity" (see [2, 9]) for the initial value problem (1.2), (1.3), (1.5). Given \( d \in \mathbb{R} \), and \( \lambda \in \mathbb{R} \), define
\[ E(r, \lambda, d) = \frac{(u'(r, \lambda, d))^2}{2} + \frac{(u(r, \lambda, d))^{p+2}}{p+2} + \lambda u(r, \lambda, d). \]  

**Lemma 2.** Let \( u(r, \lambda, d) \) be a solution to (1.2), (1.3) and (1.5). If \( 0 \leq \tilde{r} \leq r \), then
\[ r^{N-1}H(r) - (\tilde{r})^{N-1}H(\tilde{r}) = \frac{N+2}{2} \int_{\tilde{r}}^{r} \lambda s^{N-1}u(s) \, ds, \]  
where
\[ H(r) = rE(r) + \frac{N-2}{2} u(r, \lambda, d) u'(r, \lambda, d). \]

**Proof.** Multiplying (1.2) by \( r^N u'(r) \) and integrating over \([\tilde{r}, r] \), we obtain,
\[ r^N E(r) = \int_{\tilde{r}}^{r} s^{N-1} \left\{ N \left( \frac{u^{p+2}(s)}{p+2} + \lambda u \right) - \left( \frac{N-2}{2} \right) (u'(s))^2 \right\} \, ds + (\tilde{r})^N E(\tilde{r}). \]  
Similarly, multiplying (1.2) by \( r^{N-1} u(r) \) and integrating over \([\tilde{r}, r] \) we infer,
\[ \int_{\tilde{r}}^{r} s^{N-1}(u'(s))^2 \, ds = u'(r)u(r)r^{N-1} - u'(\tilde{r})u(\tilde{r})(\tilde{r})^{N-1} + \int_{\tilde{r}}^{r} s^{N-1}(u^{p+2}(s) + \lambda u) \, ds. \]
Replacing (2.4) in (2.3), we obtain (2.2). This completes the proof.

Taking \( \tilde{r} = 0 \) in (2.2) we get
\[ \frac{N+2}{2} \int_{0}^{\tilde{r}} \lambda s^{N-1}u(s) \, ds = \frac{r^N(u'(r))^2}{2} + \frac{r^N(u^{p+2}(r))}{p+2} + r^N \lambda u(r) + \frac{N-2}{2} r^{N-1} u'(r)u(r). \]  

**Corollary 3.** The problem (1.1) has no nonnegative solutions for \( \lambda \leq 0 \).
Proof. Taking \( r = 0 \), \( r = 1 \) in (2.2) we obtain
\[
\frac{(u'(1))^2}{2} = \frac{N}{2} \int_0^1 \lambda s^{N-1} u(s) \, ds.
\] (2.6)

Since \( u \) is positive, (2.6) yields \((u'(1))^2 \leq 0\) for \( \lambda \leq 0 \). Hence, \( u = 0 \). This completes the proof.

Now, for a positive solution \( u \) of (1.1), we define the function
\[
h(r) = \frac{-ru'(r)}{u(r)}, \quad r \in [0, 1). \quad (2.7)
\]

Clearly, \( h \) is continuous, and \( h(0) = 0 \). Since \( u(1) = 0 \), we see that \( \lim_{r \to 1^-} h(r) = \infty \). Furthermore, \( h \) is an increasing function. Indeed,
\[
h'(r) = \frac{-u'(r)u(r) - ru(r)u''(r) + r(u'(r))^2}{(u(r))^2}. \quad (2.8)
\]

Substituting (1.2) in (2.8) we have,
\[
h'(r) = \frac{(N-2)u'(r)u(r) - ru(r)u''(r) + r(u'(r))^2}{(u(r))^2}. \quad (2.9)
\]

Combining (2.5) and (2.9) we have,
\[
(1 - 2/(p + 2))ru^p(r) + (2/N)ru^p(r) \geq ru^p(r) > 0. \quad (2.11)
\]

Lemma 4. If \( u \) is a positive solution to (1.1), then there exists \( M_0 > 0 \) and a unique \( r \in (0, 1) \) such that \( u(r) = M_0 r^{-2/p} \). Moreover, if \( 0 < M < M_0 \) then there exists exactly two numbers \( r_1, r_2 \in (0, 1) \) such that \( u(r_i) = M(r_i)^{-2/p}, i = 1, 2 \).

Proof. Let \( r \in [0, 1] \) be such that \( M_0 = \max\{u(r)r^{2/p} : r \in [0, 1]\} = u(r)\tilde{r}^{2/p} \). Thus, the graph of \( u \) is tangent to the graph of \( M_0 r^{-2/p} \), at \( \tilde{r} \), and \( u(r) \leq M_0 r^{-2/p} \) for all \( r \in [0, 1] \).

Now for \( M < M_0 \) we show that the graph of \( u \) intersects the graph of \( M\tilde{r}^{-2/p} \) at exactly two points. Suppose \( 0 < r_1 < r_2 < r_3 < 1 \) are the first three numbers such that \( u(r_i) = M(r_i)^{-2/p}, i = 1, 2, 3 \). Since \( u \) is a decreasing function, \( u(r_1) < u(r_2) < u(r_3) \). Let \( Z = M\tilde{r}^{-2/p} \), then, we have \( Z(r_2) = u(r_2), Z'(r_2) > u'(r_2), Z(r_3) = u(r_3), \) and \( Z'(r_3) < u'(r_3) \). Hence,
\[
h(r_3) = \frac{-r_3 u'(r_3)}{u(r_3)} < \frac{-r_3 Z'(r_3)}{Z(r_3)} = \frac{2}{p},
\]
and,

$$h(r_2) = \frac{-r_2 u'(r_2)}{u(r_2)} > \frac{-r_2 Z'(r_2)}{Z(r_2)} = \frac{2}{p}.$$ 

However, then $h(r_3) < h(r_2)$ with $r_2 < r_3$, which contradicts that $h$ is an increasing function (see (2.11)). Assuming that $u(\bar{r})^{2/p} = u(\bar{r})^{2/p} = M_0$ we see that $h(\bar{r}) = h(\bar{r}) = 2/p$ which contradicts that $h$ is an increasing function. Hence, $\bar{r}$ is unique.

From (2.5) and the quadratic formula we obtain,

$$ru'(r) = -\frac{N-2}{2} u(r) \pm \frac{1}{2} A(r),$$

where

$$A(r) = \left\{ \left( N - 2 \right)^2 u^2(r) - \frac{8}{p + 2} r^2 u^{p+2}(r) - 8r^2 \lambda u(r) + 4(N + 2) \lambda \frac{1}{r^{N-3}} \int_0^r s^{N-1} u(s) \, ds \right\}^{1/2} \tag{2.13}$$

From (2.12) we have

$$\frac{2}{N-2} h(r) = 1 \pm \frac{1}{N-2} A(r). \tag{2.14}$$

Since $h(0) = 0$ and $\lim_{r \to 1^-} h(r) = \infty$ we see from (2.14) that for $r$ near zero

$$\frac{2}{N-2} h(r) = 1 - \frac{1}{N-2} A(r), \tag{2.15}$$

and for $r$ near 1,

$$\frac{2}{N-2} h(r) = 1 + \frac{1}{N-2} A(r). \tag{2.16}$$

The fact that $h$ is an increasing function together with (2.15), and (2.16) imply the existence of a unique $\hat{r}$ such that $(2/(N-2))h(\hat{r}) = 1$, that is, $A(\hat{r}) = 0$. Since $A(\hat{r}) = 0$ implies $h(\hat{r}) = 2/p$ and $\hat{r}$ is the only element in $[0, 1]$ for which $h(r) = 2/p$ we see that $\hat{r} = \bar{r}$. Using that $u(\bar{r}) = M_0 \bar{r}^{-2/p}$ and integrating (2.11) on $[0, \bar{r}]$ we obtain

$$M_0 \leq \left( \frac{N(N-2)}{2} \right)^{1/p}. \tag{2.17}$$

Lemma 5. If $\hat{r}$ is as above, then $\hat{r} \leq O(d^{-p/2}).$

Proof. Let $r_0 = d^{-p/2}$, and put $K_0 = r_0^2 u^p(r_0)$. We claim that $K_0 \geq (1 - 1/4N)^p$. Indeed, since

$$r^{N-1} u'(r) = -\int_0^r s^{N-1}(\lambda + u^{p+1}(s)) \, ds \geq -\frac{d^{p+1} + \lambda}{N} r^N,$$

it follows that $u'(r) \geq -(d^{p+1} + \lambda)/N r$. Integration over $[0, r_0]$ yields

$$u(r_0) \geq u(0) - \frac{d^{p+1} + \lambda}{2N} r_0^2 = d - \frac{d^{p+1} + \lambda}{2N} d^{-p} = \frac{2N - 1}{N} d - \frac{\lambda}{2N} d^{-p}.$$
Thus, for \( \lambda \in [0, 1] \) and \( d \) large, \( K_0^{1/p} r_0^{-2/p} = u(r_0) \geq ((2N - 1)/3N)d \). Hence, \( K_0^{1/p} \geq (2N - 1)/3N \) and the claim is established.

If \( \hat{r} \geq d^{-p/2} \), then integrating \( 2/(N - 2) h'(r) \) over \([r_0, \hat{r}]\) and using (2.11) we obtain

\[
1 = \frac{2}{N - 2} h(\hat{r}) \geq \frac{2}{N - 2} h(r_0) + \frac{4}{N(N - 2)} \int_{r_0}^{\hat{r}} K_0 r^{-2} \, dr.
\]

Hence

\[
\ln\left( \frac{\hat{r}}{r_0} \right) \leq \frac{N(N - 2)}{4K_0}, \quad \text{or equivalently, } \hat{r} \leq d^{-p/2} \exp\left( \frac{N(N - 2)}{4K_0} \right),
\]

which proves lemma 5.

**Lemma 6.** If \( u \) is a solution to (1.2), (1.3) and (1.5), then

\[
\frac{2}{p} du_d(r) = -\frac{N + 2}{2} \lambda u_\lambda(r) + \frac{N - 2}{2} u(r) + ru'(r). \tag{2.18}
\]

**Proof.** Let \( v(r) = \beta^{-2/p} u(r/\beta, \lambda, d) \). Thus,

\[
v'' + \frac{N - 1}{r} v' + v^{p+1} = \frac{1}{\beta^{2+2/p}} u'' \left( \frac{r}{\beta} \right) + \frac{1}{\beta^{1+2/p}} \cdot \frac{N - 1}{r} u' \left( \frac{r}{\beta} \right) + \frac{1}{\beta^{2+2/p}} u^{p+1} \left( \frac{r}{\beta} \right)
= \frac{1}{\beta^{2+2/p}} u'' \left( \frac{r}{\beta} \right) + \frac{1}{\beta^{1+2/p}} \cdot \frac{N - 1}{r} u' \left( \frac{r}{\beta} \right) + \frac{1}{\beta^{2+2/p}} u^{p+1} \left( \frac{r}{\beta} \right)
= \frac{1}{\beta^{2+2/p}} (-\lambda),
\]

which implies that \( v(r) = u(r, \lambda/\beta^{2+2/p}, d/\beta^{2/p}) \). Thus, for all \( \beta > 0 \),

\[
u(r/\beta, \lambda, d) = \beta^{2/p} u(r, \lambda/\beta^{2+2/p}, d/\beta).
\tag{2.19}
\]

Differentiating (2.19) with respect to \( \beta \) we obtain, for all \( r > 0, \lambda > 0, d > 0, \beta > 0 \),

\[
-\frac{r}{\beta^2} u' \left( \frac{r}{\beta}, \lambda, d \right) = \frac{2}{p} \beta^{2/p-1} u \left( r, \frac{\lambda}{\beta^{2+2/p}}, \frac{d}{\beta^{2/p}} \right) - \left( 2 + \frac{2}{p} \right) \beta^{-3} u_\lambda \left( r, \frac{\lambda}{\beta^{2+2/p}}, \frac{d}{\beta^{2/p}} \right)
- \left( 2 \frac{d}{p} \right) \beta^{-1} u_d \left( r, \frac{\lambda}{\beta^{2+2/p}}, \frac{d}{\beta^{2/p}} \right).
\]

In particular, taking \( \beta = 1 \), we obtain (2.18). This completes the proof.

Substituting (2.12) in (2.18) we have

\[
\frac{N - 2}{2} du_d(r) = -\frac{N - 2}{2} \lambda u_\lambda(r) \pm \frac{1}{2} A(r). \tag{2.20}
\]

In (2.20) the "+" sign is to be used on the interval \([0, \hat{r}]\), and the "-" sign on the interval \((\hat{r}, 1]\) (see (2.15) and (2.16)). The existence and uniqueness of the number \( \hat{r} \) are due to the fact that \( h(r) = -ru'/u \) is an increasing function.
LEMA 7. If $u$ is a positive solution to (1.2), (1.3) and (1.5) then
\[-\frac{1}{2N} \leq -\frac{r^2}{2N} \leq u_\lambda(r, \lambda, d) \leq 0 \quad \text{for } r \in [0, 1].\]

Proof. Let $z(r) = u_\lambda(r, \lambda, d)$. Thus,
\[z'' + \frac{N-1}{r} z' + (p+1) |u|^p z + 1 = 0, \quad z(0) = 0, \quad \text{and} \quad z'(0) = 0. \quad (2.21)\]
Let $G$ be defined by
\[G(r) = \frac{1}{2}(z')^2 + \frac{1}{2}(p+1)|u|^p z^2 + z. \quad (2.22)\]
Differentiating (2.22) we obtain
\[G'(r) = z'z'' + (p+1)|u|^p z \left[ z' + \frac{p}{2} u' z \right] + z' \]
\[= -\frac{(N-1)}{r} (z')^2 + \frac{1}{2}(p+1)|u|^p u' z^2 \leq 0, \quad (2.23)\]
where we have used that $u' \leq 0$.

Suppose, there is an $\tilde{r} > 0$ such that $z(\tilde{r}) = 0$. Then substituting in (2.22) yields $G(\tilde{r}) = \frac{1}{2}(z'(\tilde{r}))^2 \geq 0$, contradicting the fact that $G$ is a decreasing function. Therefore, $z \leq 0$ on $[0, 1]$. However, then
\[r^{N-1}z'(r) = - \int_0^r s^{N-1} (1 + (p+1)|u|^p u z) \, ds \geq \int_0^r s^{N-1} \, ds \]
and, hence, $z'(r) \geq -r/N$. Integrating over $[0, r]$ we obtain
\[z(r) \geq z(0) - \frac{r^2}{2N} = -\frac{r^2}{2N} \geq -\frac{1}{2N}. \]
This completes the proof.

From (2.20), lemma 7, and the definition of $\tilde{r}$ we obtain that
\[w(r) = u_d(r, \lambda, d) > 0, \quad \text{for } r \in [0, \tilde{r}]. \quad (2.24)\]

LEMA 8. If $u(r_2) = ((N - 2)/4(N + 2))^{1/p} r_2^{-2/p}$, with $0 < \hat{r} \leq r_2 < 1$ then $r_2 \leq O(d^{-p/2})$.

Proof. Since
\[ \frac{2}{N-2} (h(r_2) - h(\hat{r})) = \sqrt{\frac{6N-4}{N(N+2)}} + O(r^{2+2/p}), \]
and $(u(r))^p = \frac{1}{4}((N - 2)^3/(N + 2))(1/r^3)$, for $r \in (\hat{r}, r_2)$; from (2.11) we have
\[\sqrt{\frac{6N-4}{N(N+2)}} + O(r^{2+2/p}) \geq \frac{(N - 2)^2}{N(N+2)} \int_{\hat{r}}^{r_2} r^{-1} \, dr.\]
Therefore,
\[
\ln \left( \frac{r_2}{\hat{r}} \right) \leq \frac{\sqrt{N(N + 2)(6N - 4)}}{(N - 2)^2} = \delta(N),
\]
that is, \( r_2 \leq \hat{r} \exp(\delta(N)) \). Since from lemma 5, \( \hat{r} = O(d^{-p/2}) \), we conclude that \( r_2 = O(d^{-p/2}) \). This completes the proof.

Now we show that for \( d \) sufficiently large \( u_d(r_2) < 0 \). Indeed,
\[
\left( \frac{2}{p} \right) du_d(r_2) = -\lambda \frac{N + 2}{2} u_\lambda(r_2) - \frac{1}{2} A(r_2),
\]
and since \( u_\lambda(r) \geq -r^2/2N \) (see lemma 7) we have
\[
\left( \frac{2}{p} \right) du_d(r_2) \leq \lambda \frac{N + 2}{2N} r_2^2
\]
\[
- \sqrt{r_2^{-4/p} \left( \frac{(N - 2)^3}{4(N + 2)} \right)^{2/p} \frac{6N - 4}{N(N + 2)} - \lambda \left( 8 - \frac{4(N + 2)}{N} \right) \left( \frac{N - 2}{4(N + 2)} \right)^{1/p} r_2^{2 - 2/p}.}
\]
Simplifying the expression on the right we get
\[
\left( \frac{2}{p} \right) du_d(r_2) \leq \lambda \frac{N + 2}{4N} r_2^2 - \frac{N - 2}{2} r_2^{-2/p} \left( \frac{(N - 2)^3}{4(N + 2)} \right)^{1/p} \sqrt{\frac{6N - 4}{N(N + 2)}} + O(r_2^{2 - 2/p}). \tag{2.25}
\]
From lemma 8 we obtain
\[
\left( \frac{2}{p} \right) du_d(r_2) \leq \lambda \frac{N + 2}{2N} r_2^2 - O(d) \tag{2.26}
\]
and, hence,
\[
u_d(r_2) \leq -O(1) + O(d^{-1}) < 0, \quad \text{for } d \text{ sufficiently large.} \tag{2.27}
\]
From (2.24) and (2.27) we see that there exists an \( \bar{r} \in (\hat{r}, r_2) \) such that
\[
u_d(\bar{r}) = 0. \tag{2.28}
\]
Let \( \gamma(r) = r^{-(N - 2)/2} \). A straightforward calculation shows that \( \gamma'' + ((N - 1)/r) \gamma' + ((N - 2)/2r) \gamma = 0 \). Since \( u_d \) satisfies a linear differential equation, its zeros are nondegenerate. Since \( (p + 1) u''(r) \leq ((N - 2)/4r^3) \) for \( r \in [r_2, 1] \), and \( \gamma \) is positive on \((0, \infty)\), by the Sturm comparison theorem (see [10]) we see that \( u_d(\cdot, \lambda, d) \) cannot have two zeros in \([r_2, 1]\). The next three lemmas are devoted to proving that \( u_d(\cdot, \lambda, d) < 0 \) in \([r_2, 1]\).

**Lemma 9.** Let \( u \) be a positive solution to (1.2), (1.3) and (1.5). Then
\[
\lim_{d \to +\infty} \int_0^1 r^{N-1} u(r, \lambda, d) \, dr = 0.
\]
Proof. Let \( \varepsilon > 0 \) be such that for \( r > \hat{r} + \delta \), \( (2/(N - 2))h(r) \geq 1 + \varepsilon \) (see (2.16)). Thus,

\[
-\frac{u'}{u} \geq \frac{1}{2} (1 + \varepsilon)(N - 2) \frac{1}{r}.
\]

Integrating this over \([r_2, r]\) we get

\[
u(r) \leq U(r_2)r^{(1/2)(1+\varepsilon)(N-2)}r^{-(1/2)(1+\varepsilon)(N-2)},
\]

which in turn gives that

\[
u(r) \leq \left( \frac{(N - 2)^3}{4(N + 2)} \right)^{1/p} r^{(N-2)/2}r^{-(N-2)/2(1+\varepsilon)}.
\]

Therefore,

\[
\int_{r_2}^{r_1} r^{N-1} u(r) \, dr \leq \left( \frac{(N - 2)^3}{4(N + 2)} \right)^{1/p} r^{(N-2)/2}(r^{(N-2)/2(1+\varepsilon)}) \left( \int_{r_2}^{r_1} r^{(N-2)/2(1+\varepsilon)} \, dr \right)
\]

and, hence,

\[
\int_{r_2}^{r_1} r^{N-1} u(r) \, dr \leq \left( \frac{(N - 2)^3}{4(N + 2)} \right)^{1/p} \frac{2r_2^{(N-2)/2(1+\varepsilon)}}{2N - (1 + \varepsilon)(N - 2)}, \tag{2.29}
\]

which tends to zero as \( d \) tends to \(+\infty\), since \( r_2 \to 0 \) as \( d \to \infty \). Also, since

\[
\int_{r_2}^{r_1} r^{N-1} u(r) \, dr \leq \frac{d}{N} r_2^{N} = \frac{d}{N} (d^{-p/2})^{N} = \frac{1}{N} d^{-(2+N)/(N-2)},
\]

which tends to zero as \( d \to +\infty \). We conclude that

\[
\int_{0}^{1} r^{N-1} u(r) \, dr \to 0 \quad \text{as} \quad d \to +\infty.
\]

This completes the proof.

Lemma 10. If \( u \) is a positive solution to (1.2), (1.3) and (1.5) then

\[
\int_{0}^{1} r^{N-1} u^{p+1} \, dr \leq o(\sqrt{\lambda}).
\]

Proof. Taking \( r = 1 \) in Pohozaev's identity (2.5) we obtain

\[
(u'(1))^2 = \lambda(N + 2) \int_{0}^{1} r^{N-1} u(r) \, dr.
\]

Hence,

\[
\left( \int_{0}^{1} r^{N-1} u^{p+1} \, dr \right)^2 \leq \left( \int_{0}^{1} r^{N-1} (\lambda + u^{p+1}) \, dr \right)^2 = (u'(1))^2.
\]

This and lemma 9 yield

\[
\left( \int_{0}^{1} r^{N-1} u^{p+1} \, dr \right)^2 \leq \lambda(N + 2) \int_{0}^{1} r^{N-1} u(r) \, dr = o(\lambda).
\]
Thus we obtain
\[ \int_0^1 r^{N-1} u^{p+1} \, dr \leq o(\sqrt{\lambda}), \]
which proves the lemma.

**Lemma 11.** If \( u(r, \lambda, d) \) is a positive solution to (1.2), (1.3) and (1.5) then for \( \lambda \in [0, 1] \) we have
\[ \left\| u_\lambda(r, \lambda, d) + \frac{r^2}{2N} \right\|_\infty \leq o(\lambda^{p/(2(p+1)))} \]
and, hence,
\[ \lim_{d \to +\infty} \left\| u_\lambda(r, \lambda, d) + \frac{r^2}{2N} \right\|_\infty = 0. \]

**Proof.** From the definition of \( u_\lambda \) we have
\[ u_\lambda(r, \lambda, d) = -\int_0^r s^{-N+1} \int_0^s r^{N-1}(1 + (p + 1)|u|^p u_\lambda) \, dr \, ds \]
\[ = -\frac{r^2}{2N} - \int_0^r s^{-N+1} \int_0^s r^{N-1}(p + 1)|u|^p u_\lambda \, dr \, ds. \tag{2.30} \]
Since \(-u_\lambda \leq 1/2N\) (see lemma 7), from (2.30) we obtain that
\[ u_\lambda(t, \lambda, d) \leq -\frac{r^2}{2N} + \frac{p + 1}{2N} \int_0^1 s^{-N+1} \int_0^s r^{N-1} u^p \, dr \, ds. \]
Hence, it suffices to show that \( \int_0^1 r^{N-1} u^p \, dr \to 0 \) as \( d \to \infty \). Now by Hölders inequality we have
\[ \int_0^s r^{N-1} u^p \, dr \leq \int_0^1 r^{N-1} u^p \, dr \leq M \left( \int_0^1 r^{N-1} u^{p+1} \right)^{p/(p+1)}.
\]
Now using lemma 10 we obtain that
\[ \int_0^1 r^{N-1} u^p \, dr \leq o(\lambda^{p/(2(p+1)))}. \]
Therefore,
\[ \int_0^r s^{-N+1} \int_0^s r^{N-1}(p + 1)u^p(-u_\lambda) \leq \frac{1 + p}{2N} \int_0^r s^{-N+1} \int_0^s r^{N-1} u^p \, dr = o(\lambda^{p/(2(p+1)))} \]
and, hence, \( u_\lambda(r) \leq -r^2/2N + o(\lambda^{p/(2(p+1)))} \) which, in turn, implies
\[ \left\| u_\lambda(r, \lambda, d) + \frac{r^2}{2N} \right\|_\infty \leq o(\lambda^{p/(2(p+1)))}, \]
thus, completing the proof.
Now we show that $u_d(1, \lambda, d) \neq 0$. Consider the initial value problem (1.2), (1.3) and (1.5) for $\lambda = 0$. That is

$$v'' + \frac{N-1}{r} v' + |v|^p v = 0,$$

$$v(0) = d \quad \text{and} \quad v'(0) = 0.$$

That is we denote $u(r, d, 0)$ by $v(r, d)$. From Pohozaev's identity (2.5) and the quadratic formula we obtain

$$v' = -\frac{N-2}{2r} v - \frac{N-2}{2r} \sqrt{1 - \frac{4r^2}{N(N-2)}} v^p$$

for all $r > \hat{r} = K_1 d^{-p/2}$, where $K_1$ is a constant independent of $d$ (see lemma 5). Also, given an $\varepsilon > 0$, there exists a constant $K_2$ independent of $d$ such that for $r \geq K_2 d^{-p/2}$ we have

$$\frac{2}{N(N-2)} r^2 v^p \leq \varepsilon.$$  

Substituting (2.34) in (2.33) we get

$$v'(r) \leq -\sigma \frac{v}{r},$$

where $\sigma = \frac{1}{2} (N-2)(1 + \sqrt{1 - \varepsilon})$. Integrating this over $[r, 1]$ we obtain

$$v(r) \geq v(1) r^{-\sigma}.$$  

From this we infer

$$\int_{kd^{-p/2}}^1 r^{N-1} v(r) \, dr \geq (N - \sigma)^{-1} v(1) [1 - (kd^{-p/2})^{N-\sigma}].$$

Thus, by choosing $\varepsilon$ small enough and $d$ large enough we get

$$\int_0^1 r^{N-1} v(r) \, dr \geq v(1) \tau,$$

with $\tau < \frac{1}{2}$ but arbitrarily close to $\frac{1}{2}$. Suppose now that $u_d(1, \lambda, d) = 0$. Then using the mean value theorem we obtain

$$v(1) = u(1, 0, d) = u(1, \lambda, d) - \lambda u_\lambda(1, \hat{\lambda}, d)$$

for some $\hat{\lambda} \in [0, \lambda]$. From lemma 11 and the fact that $u(1, \lambda, d) = 0$, we obtain

$$v(1) = \lambda \left( \frac{1}{2N} - o(\lambda^{p/(2(1+p))}) \right).$$

From lemma 11 and the rescaling equation in (2.18), we obtain

$$u'(1, d, \lambda) = \lambda \left( \frac{N+2}{2} \right) \left( - \frac{1}{2N} + o(\lambda^{p/(2(1+p))}) \right).$$
From Pohozaev's identity and (2.5) we have
\[ u'(1, d, \lambda) = -\sqrt{\lambda(N + 2)} \int_0^1 r^{N-1} u(r) \, dr, \] (2.41)
and by the mean value theorem and lemma 3 we get
\[ u(r, \lambda, d) \geq v(r, d) - \frac{\lambda}{2N} r^2. \] (2.42)
Combining (2.40) and (2.41) we obtain
\[ \lambda(N + 2) \int_0^1 r^{N-1} u(r) \, dr = \left( \frac{N + 2}{2} - \lambda \right)^2 \left( \frac{1}{4N^2} + o(\lambda^{p/(2(1+p))}) \right). \]
That is,
\[ \int_0^1 r^{N-1} u(r) \, dr = \left( \frac{N + 2}{4} - \frac{\lambda}{2N} \right)^2 \left( \frac{1}{4N^2} + o(\lambda^{p/(2(1+p))}) \right). \] (2.43)
On the other hand, from (2.38), (2.39) and (2.42) we obtain
\[ \int_0^1 r^{N-1} u(r) \, dr \geq \int_0^1 r^{N-1} \left( v(r) - \frac{\lambda}{2N} r^2 \right) \, dr > v(1) - \frac{\lambda}{2N(N + 2)} \]
\[ = \lambda \left[ \frac{1}{2N} - o(\lambda^{p/(2(1+p))}) \right] \frac{\lambda}{2N(N + 2)}. \] (2.44)
Combining (2.43) and (2.44) we obtain
\[ o(\lambda^{p/(2(1+p))}) \geq \frac{(8\tau - 1)N^2 + (16\tau - 12)N - 4}{16N^2(N + 2)}. \] (2.45)
Since \( \tau \) can be chosen arbitrarily close to \( \frac{1}{2} \), we see that the numerator of the right-hand side can be made arbitrarily close to \( 3N^2 - 4N - 4 \), which is positive for \( N \geq 3 \). Hence, (2.45) cannot hold for small values of \( \lambda \) which is a contradiction. Thus, there exist \( D > 0 \) and \( \Lambda > 0 \) such that if \( u(\cdot, \lambda, d) \) is a positive solution to (1.2), (1.3), (1.5), \( \lambda \in (0, \Lambda) \) and \( d > D \) then
\[ u_d(1, \lambda, d) < 0. \] (2.46)

3. PROOF OF THEOREM 1

Since \( u_d(1, \lambda, d) < 0 \) (see lemma 7), the implicit function theorem implies that if \( S \) is a connected component of \( \{ (\lambda, d) ; u(1, \lambda, d) = 0, u(r, \lambda, d) > 0 \text{ for all } r \in [0, 1] \} \) then there exists a differentiable function \( F : (0, \infty) \to (0, \infty) \) such that \( S = \{ (F(d), d) ; d \in (0, \infty) \} \). Integrating (1.2) on \( [0, 1] \) we see that \( -d \leq -F(d)/2N \). Hence,
\[ \lim_{d \to 0} F(d) = 0. \] (3.1)
Let us see now that
\[ \lim_{d \to \infty} F(d) = 0. \] (3.2)
By lemma 9 we have $\lim_{d \to \infty} u(1/4, F(d), d) = 0$. Thus, if $\limsup_{d \to \infty} F(d) > 0$ then for some sequence $\{d_n\} \to \infty$ we have $\{F(d_n)/u(1/4, F(d), d)\} \to \infty$. Hence, because $u(\cdot, F(d_n), d_n)$ satisfies

$$u'' + \frac{N - 1}{r} u' + \left(\frac{|u|^p + F(d_n)}{u}\right) u = 0,$$

by the Sturm comparison theorem we see that $u(\cdot, F(d_n), d_n)$ must have a zero in $[1/4, 1)$. This contradicts that $u(\cdot, F(d_n), d_n)$ is positive in $(0, 1)$. Thus, (3.2) is proven.

Since $u(r, F(d), d) \leq d$ for all $r \in [0, 1]$, by the Sturm comparison theorem we see that for $d > 0$ small $u_d(r, F(d), d) > 0$ for all $r \in [0, 1]$. Hence, by the implicit function theorem there exists $\delta > 0$ and an increasing differentiable function $\phi: (0, \delta) \to (0, \infty)$ such that $u(\cdot, \phi(\lambda), d)$ is a solution to (1.2), (1.3), (1.5) if and only if $d = \phi(\lambda)$. Thus, if $S_1 = \{(F(d), d); d \in (0, \infty)\}$ and $S_2 = \{(F_1(d), d); d \in (0, \infty)\}$ are connected components of positive solutions to (1.2), (1.3), (1.5), by (3.1) we see that $d = \phi(F(d)) = \phi(F_1(d))$. Hence, $F(d) = F_1(d)$ for $d$ close to 0. Therefore, $S_1 = S_2$, which proves that the set of positive solutions to (1.1) is connected.

Differentiating $u(1, \lambda(d), d)) = 0$ with respect to $d$ we obtain

$$u_d(1, \lambda(d), d)) + u(1, \lambda(d), d)) \cdot \lambda'(d) = 0.$$

This, (2.46) and (3.2) imply that $F$ is a decreasing function in $(d, \infty)$, which proves that for $\lambda < F(D)$ the problem (1.1) has exactly one solution with $u(0) > D$. Since $\phi$ is an increasing function, so is $F$. Thus, if $\lambda \in (0, \phi(\delta))$ then (1.1) has exactly one small solution. Thus, if $0 < \lambda < \min[\delta/2, F(D), \min[F(d); d \in [\delta/2, F(D)]]]$ then the problem (1.1) has exactly two positive solutions.

REFERENCES

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