Combinatorial Interpretations of Spanning Tree Identities

Arthur T. Benjamin
Harvey Mudd College

Carl R. Yerger
Davidson College

Recommended Citation
Combinatorial Interpretations of Spanning Tree Identities

Arthur T. Benjamin and Carl R. Yerger

November 14, 2004

Abstract

We present a combinatorial proof that the wheel graph $W_n$ has $L_{2n} - 2$ spanning trees, where $L_n$ is the $n$th Lucas number, and that the number of spanning trees of a related graph is a Fibonacci number. Our proofs avoid the use of induction, determinants, or the matrix tree theorem.

1 Introduction

Let $G$ be a graph and let $\tau(G)$ be the number of spanning trees of $G$. In this paper we will present combinatorial proofs that determine $\tau(G)$ for the wheel graph and a related auxiliary graph. Two simple bijections will provide a direct explanation as to why the number of spanning trees for these graphs are Fibonacci and Lucas numbers.

Definition 1.1. For $n \geq 1$, the wheel graph $W_n$ has $n + 1$ vertices, consisting of a cycle of $n$ outer vertices, labeled $w_1, \ldots, w_n$, and a "hub" center vertex, labeled $w_0$, that is adjacent to all the $n$ outer vertices.

For example, $W_8$ is presented in Figure 1. The Lucas numbers are recursively defined by $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$.

Theorem 1.2. For $n \geq 1$, $\tau(W_n) = L_{2n} - 2$.

This result was first proved by Sedlacek in [5] and later by Myers in [3]. As part of Myers’ proof, he employs an auxiliary graph, denoted by $A_n$, that is similar to the wheel graph and presented in Figure 2. For $n \geq 2$, $A_n$ has $n + 1$ vertices and $2n + 1$ edges, consisting of a path of $n$ outer vertices, labeled $a_1, \ldots, a_n$, and a hub vertex $a_0$ that is adjacent to all $n$ outer vertices. In addition, $a_0$ has an extra edge connecting to $a_1$ and an extra edge connecting to $a_n$.

We label the two edges from $a_0$ to $a_1$ as red and blue, and do the same for the edges from $a_0$ to $a_n$. Let $f_n$ denote the $n$th Fibonacci number with initial conditions $f_1 = 1$, and $f_2 = 2$. 

1
Theorem 1.3. For $n \geq 2$, $\tau(A_n) = f_{2n+1}$.

One way to determine $\tau(A_n)$, as shown by Koshy [2], is to apply the matrix tree theorem [6], first proved by Kirchhoff, by computing the determinant of the $n$-by-$n$ tridiagonal matrix

$$A_n = \begin{bmatrix} 3 & -1 & 0 & \ldots & 0 \\ -1 & 3 & -1 & \ldots & 0 \\ 0 & -1 & 3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -1 \\ \end{bmatrix}.$$}

Expanding along the first row, and proceeding inductively, it follows that $\tau(A_n) = |A_n| = 3|A_{n-1}| - |A_{n-2}| = 3f_{2n-1} - f_{2n-3} = f_{2n+1}$.

The matrix tree theorem also indicates that $\tau(W_n)$ equals the determinant of the following matrix $n$-by-$n$ circulant matrix.
\[ B_n = \begin{bmatrix} 3 & -1 & 0 & \ldots & -1 \\ -1 & 3 & -1 & \ldots & 0 \\ 0 & -1 & 3 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & 0 & 0 & \ldots & -1 & 3 \end{bmatrix}. \]

Expanding \( |B_n| \) along its first row, we obtain \( |A_n| \) as one of its subdeterminants. Proceeding by induction and with a bit more computation (see [2]), \( \tau(W_n) = L_{2n} - 2 \) can then be obtained. In the next two sections, we give combinatorial proofs of Theorems 1.2 and 1.3 that are much more direct.

2 Combinatorial Proof of \( \tau(W_n) = L_{2n} - 2 \)

The Lucas number \( L_n \) counts the ways to tile a bracelet of length \( n \) and width 1 using \( 1 \times 1 \) squares and \( 1 \times 2 \) dominoes [1]. Equivalently, \( L_n \) is the number of matchings in the cycle graph \( C_n \). Observe that even cycle graphs \( C_{2n} \) have exactly two perfect matchings and thus \( L_{2n} - 2 \) imperfect matchings, such as the one in Figure 3.

![Figure 3: An imperfect matching of \( C_{8} \).](image)

Given an imperfect matching \( M \) (a subgraph of \( C_{2n} \) where every vertex \( c_i \) has degree 0 or 1), we construct a subgraph \( T_M \) of \( W_n \) as follows:

1. For \( 1 \leq i \leq n \), an edge exists from \( w_0 \) to \( w_i \) if and only if \( c_{2i-1} \) has degree 0 in \( M \).
2. For \( 1 \leq i \leq n \), an edge exists from \( w_i \) to \( w_{i+1} \) (where \( w_{n+1} \) is identified with \( w_1 \)) if and only if \( c_{2i} \) has degree 1 in \( M \).

The bijection is illustrated in Figure 4.

To see that \( T_M \) is a spanning tree of \( W_n \), suppose that \( M \) has \( x \) vertices of degree 1 and \( y \) vertices of degree 0; thus \( x + y = 2n \). Observe that vertices of degree 1 come in adjacent pairs and that if
Figure 4: An example of the bijection for $n = 4$.

$v_j$ has degree 0, then the next vertex of degree 0, clockwise from $v_j$, must be $v_k$, where $k$ and $j$ have opposite parity. Thus, $T_M$ will use exactly $x/2 + y/2 = n$ edges of $W_n$. Since $W_n$ has $n + 1$ vertices, we need only show that $T_M$ has no cycles. Suppose, to the contrary, that $T_M$ has a cycle $C$. Then, denoted by $w_0 w_i w_{i+1} \cdots w_k w_0$, must use two edges adjacent to $w_0$ (otherwise $M$ would be a perfect matching). Thus, $c_{2i-1}$ and $c_{2k-1}$ have degree 0 in $M$ and hence some vertex $c_{2j}$ must also have degree 0 where $c_{2j}$ is strictly between $c_{2i-1}$ and $c_{2k-1}$ on $C$. But when $c_{2j}$ has degree 0, there is no edge in $T_M$ from $w_j$ to $w_{j+1}$, yielding a contradiction. Hence no cycle $C$ exists on $T_M$ and so $T_M$ is a tree.

The process is reversible since a spanning tree $T$ of $W_n$ completely determines the degree sequence $d_1, d_2, d_3, \ldots, d_{2k}$ where $d_i \in \{0, 1\}$ is the degree of the vertex $c_i$ in a subgraph of $C_{2n}$. Since $w_0$ is not an isolated vertex of $T$, not all $d_k$ are equal to 1. We show that $C_{2n}$ has a unique matching that satisfies this degree sequence by showing that every string of 1s has even length: i.e., if $d_k = 0$, $d_{k+1} = d_{k+2} = \cdots = d_{k+j} = 1$, and $d_{k+j+1} = 0$, then $j$ must be even. For if $k = 2i - 1$ is odd and $j$ is odd then the tree $T$ would contain a cycle $w_0 w_i w_{i+1} \cdots w_{i+(j+1)/2} w_0$. If $k = 2i$ is even and $j$ is odd, then $T$ is not connected since the path $w_{i+1} w_{i+2} \cdots w_{i+(j+1)/2}$ is disconnected from the rest of $T$.

3 Combinatorial Proof of $\tau(A_n) = f_{2n+1}$

The Fibonacci number $f_n$ counts the ways to tile a $1 \times n$ rectangle using $1 \times 1$ squares and $1 \times 2$ dominoes [1]. Alternatively, $f_n$ counts the matchings of $P_n$, the path graph on $n$ vertices, whose vertices are consecutively denoted $p_1, \ldots, p_n$. Let $M$ be an arbitrary matching of $P_{2n+1}$. We construct a subgraph $T_M$ of $A_n$ as follows:

1. For $1 \leq i \leq n$, $T_M$ has an edge from from $a_0$ to $a_i$ if and only if
vertex \( p_{2i} \) has degree 0 in \( M \). (For \( i = 1 \) or \( n \), then this refers to the red edge.)

2. For \( 0 \leq i \leq n - 1 \), \( T_m \) has an edge from \( a_i \) to \( a_{i+1} \) if and only if \( p_{2i+1} \) has degree 1 in \( M \). (For \( i = 0 \), this refers to the blue edge.)

3. \( T_m \) has a blue edge from \( a_0 \) to \( a_n \) if and only if \( p_{2n+1} \) has degree 1 in \( M \).

Notice that these rules make it impossible for \( T_m \) to contain two edges from \( a_0 \) to \( a_1 \) or two edges from \( a_0 \) to \( a_n \). The bijection is illustrated in Figure 5.

![Figure 5: An example of the bijection for \( n = 4 \).](image)

Like before, we prove that \( T_M \) is a spanning tree of \( A_n \). Suppose that \( M \) has \( a \) and \( b \) vertices of degree 0 and 1 respectively; thus \( a + b = 2n + 1 \). Reasoning as before, \( M \) has \( b/2 \) odd vertices of degree 1 and \( (a - 1)/2 \) even vertices of degree 0. Thus, \( T_M \) has \( (a - 1)/2 + b/2 = n \) edges. Suppose for the sake of contradiction, that \( T_M \) has a cycle \( C \). Then \( C \), denoted by \( a_0 a_1 \cdots a_{n} a_0 \), must use two edges adjacent to \( a_0 \). Thus \( p_{2i} \) and \( p_{2k} \) have degree 0 in \( M \) and hence some vertex \( p_{2j+1} \) must also have degree 0 where \( p_{2j+1} \) is strictly between \( p_{2i} \) and \( p_{2k} \) on \( C \). But since \( p_{2j+1} \) has degree 0, there is no edge in \( T_M \) from \( a_j \) to \( a_{j+1} \), a contradiction. Hence no cycle \( C \) exists on \( T_M \) and so \( T_M \) is a tree.

The process is also reversible since a spanning tree \( T \) of \( A_n \) completely determines the degree \( d_k \in \{0, 1\} \) of each vertex \( p_k \) in a subgraph of \( P_{2n+1} \). Again, not all \( d_k \) are equal to 1, since \( T \) would contain the cycle \( a_0 a_1 \cdots a_{n} a_0 \). To prove that \( P_{2n+1} \) has a unique matching that satisfies this degree sequence, suppose that for some \( k, j \), \( d_k = 0 \), \( d_{k+1} = d_{k+2} = \cdots = d_{k+j} = 1 \), and \( d_{k+j+1} = 0 \). As before, if \( k = 2i \) is even and \( j \) is odd, then the tree \( T \) contains the cycle \( a_0 a_i a_{i+1} \cdots a_{i+(j+1)/2} a_0 \). If \( k = 2i - 1 \) is odd and \( j \) is odd, then \( T \) is not connected since the path \( a_i a_{i+1} \cdots a_{i+(j-1)/2} \) is discon-
nected from the rest of $T$. Thus $j$ must be even, and the matching generating $T$ is unique.

References


Department of Mathematics, Harvey Mudd College, Claremont, CA 91711, benjamin@hmc.edu, cyerger@hmc.edu