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Alfonso Castro
Harvey Mudd College

Djairo G. de Figueiredo
Universidade Estadual de Campinas, Brazil

Emer Lopera
Universidad Nacional de Colombia, Colombia

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Existence of positive solutions for a semipositone $p$-Laplacian problem

Alfonso Castro
Department of Mathematics, Harvey Mudd College, Claremont, CA 91711, USA (castro@g.hmc.edu)

Djairo G. de Figueredo
Instituto de Matemática, Estatística e Computação Científica, Universidade Estadual de Campinas, Caixa Postal 6065, Campinas, SP 13083-859, Brazil (djairo@ime.unicamp.br)

Emer Lopera
Escuela de Matemáticas, Universidad Nacional de Colombia, Sede Medellín, Apartado Aéreo 3840, Medellín, Colombia (edlopera@unal.edu.co)

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We prove the existence of positive solutions to a semipositone $p$-Laplacian problem combining mountain pass arguments, comparison principles, regularity principles and a priori estimates.

Keywords: mountain pass theorem; semipositone problem; positive solutions; $p$-Laplacian; maximum principles; a priori estimates

2010 Mathematics subject classification: Primary 35J92; 35J20; 35J60

1. Introduction

In this paper we study the existence of positive weak solutions to the problem

$$
\begin{aligned}
-\Delta_p u &= \lambda f(u) & \text{in } \Omega, \\
 u &= 0 & \text{on } \partial \Omega,
\end{aligned}
$$

(1.1)

where $\Delta_p(u) = \text{div}(|\nabla u|^{p-2}\nabla u)$ denotes the $p$-Laplacian operator, $p > 2$. $\Omega$ is an open smooth bounded domain in $\mathbb{R}^N$, $N > 2$. The function $f: \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f(0) < 0$ (semipositone). We assume that there exist $q \in (p - 1, Np/(N - p) - 1)$, $A > 0$, $B > 0$ such that

$$
\begin{aligned}
A(u^q - 1) &\leq f(u) \leq B(u^q + 1) & \text{for } u > 0, \\
f(u) &\equiv 0 & \text{for } u \leq -1.
\end{aligned}
$$

(1.2)

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We also assume an Ambrosetti–Rabinowitz type of condition, namely that there exist \( \theta > p \) and \( M \in \mathbb{R} \) such that
\[
uf(u) \geq \theta F(u) + M,
\]
where
\[
F(u) = \int_0^u f(s) \, ds.
\]
The assumption \( f(0) < 0 \) implies that \( u = 0 \) is not a subsolution to (1.1), making the finding of positive solutions rather challenging; this was pointed out in [6].

The aim of this paper is to prove the following result.

**Theorem 1.1.** There exists \( \lambda^* > 0 \) such that if \( \lambda \in (0, \lambda^*) \), then the problem (1.1) has a positive weak solution \( u_\lambda \in C^{1,\beta}(\bar{\Omega}) \) for some \( \beta \in (0,1) \).

Our results extend [1, theorem 1.1], where the case \( p = 2 \) was studied. Extending such a theorem to \( p > 2 \) is not straightforward due to the lack of regularity and linearity of \( \Delta_p \). Associated to (1.1) we have a functional, which will be defined in the next section. We show that this functional has a critical point of mountain pass type and, consequently, a weak solution of (1.1) for appropriate values of \( \lambda > 0 \). Finally, using order properties of \( -\Delta_p \), we prove that by further restricting \( \lambda \) such a solution is actually positive. For recent results on semipositone problems the reader is referred to [2,3].

**2. Preliminary results**

Let \( W^{1,p}_0(\Omega) \) denote the Banach space of functions in \( L^p(\Omega) \) with first-order partial derivatives in \( L^p(\Omega) \) and vanishing on \( \partial\Omega \). By a weak solution to (1.1) we mean an element \( u \in W^{1,p}_0(\Omega) \) such that
\[
\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx = \lambda \int_\Omega f(u) \phi \, dx
\]
for all \( \phi \in W^{1,p}_0(\Omega) \). We denote by \( \| \cdot \|_s \) the norm in the space \( L^s(\Omega) \) and by \( \| \cdot \|_{1,p} \) the norm in the Sobolev space \( W^{1,p}_0(\Omega) \).

Associated to (1.1) we have the functional \( J_\lambda: W^{1,p}_0(\Omega) \to \mathbb{R} \) defined by
\[
J_\lambda(u) := \int_\Omega \frac{|\nabla u(x)|^p}{p} \, dx - \int_\Omega \lambda F(u(x)) \, dx,
\]
where
\[
F(s) := \int_0^s f(r) \, dr.
\]

It is well known that \( J_\lambda \) is a functional of class \( C^1 \) (see [7]) and that the critical points of the functional \( J_\lambda \) are the weak solutions of (1.1). The proof of theorem 1.1 consists of two main steps:

(i) the proof of existence of one solution via the mountain pass theorem,

(ii) the proof that for proper values of \( \lambda \) the solution is indeed positive.
Lemma 2.2

and 2.1

Lemma 2.1. There exists $\lambda_1 > 0$ such that if $\lambda \in (0, \lambda_1)$, then $J_{\lambda}(c\lambda^{-r}\varphi) \leq 0$.

Proof. Let $s = c\lambda^{-r}$, with $c$ and $r$ as defined above. Hence, due to (2.4),

$$J_{\lambda}(s\varphi) = \int_{\Omega} \left\{ \frac{\|\nabla (s\varphi)\|^p}{p} - \lambda F(s\varphi) \right\} \, dx$$

$$\leq \frac{sp}{p} - \lambda A_1 \int_{\Omega} (s^{q+1}\varphi^{q+1} - 1) \, dx$$

$$= \frac{sp}{p} - A_1 s^{q+1}\|\varphi\|^q_{q+1}\lambda + \lambda A_1|\Omega|$$

$$= c^p \left( \frac{\lambda^{-rp}}{p} - \lambda A_1 c^{r+1-p}\lambda^{-r(q+1)}\|\varphi\|^q_{q+1} \right) + \lambda A_1|\Omega|. \tag{2.6}$$

Substituting (2.5) into (2.6) yields

$$J_{\lambda}(s\varphi) \leq c^p \left( \frac{\lambda^{-rp}}{p} - \frac{2}{p} \lambda^{1-r(q+1)} \right) + \lambda A_1|\Omega|$$

$$= c^p \lambda^{-rp} \left( \frac{1}{p} - \frac{2}{p} \lambda^{1+rp-r(q+1)} \right) + \lambda A_1|\Omega|$$

$$= -c^p \lambda^{-rp} \frac{1}{p} + \lambda A_1|\Omega|. \tag{2.7}$$

Taking $\lambda < \min\{1, (pA_1 c^{-p}|\Omega|)^{-1/(1+rp)}\}$, the lemma is proven.

Lemma 2.2. There exist $\tau > 0$, $c_1 > 0$, and $\lambda_2 \in (0, 1)$ such that if $\|u\|_{1,p} = \tau \lambda^{-r}$, then $J_{\lambda}(u) \geq c_1 (\tau \lambda^{-r})^p$ for all $\lambda \in (0, \lambda_2)$.

Proof. By the Sobolev embedding theorem there exists $K_1 > 0$ such that if $u \in W_0^{1,p}(\Omega)$, then $\|u\|_{q+1} \leq K_1 \|u\|_{1,p}$. Let

$$\tau = \min\{(2pK_1^{q+1}B_1)^{-r}, c\|\varphi\|_{1,p}\}. \tag{2.8}$$
If \( \|u\|_{W^{1,p}_0} = \tau \lambda^{-r} \), then
\[
J_\lambda(u) = \left( \frac{\tau \lambda^{-r}}{p} \right) - \int_\Omega \lambda F(u)
\]
\[
\geq \left( \frac{\tau \lambda^{-r}}{p} \right) - \lambda \int_\Omega B_1 |u|^{q+1} - \lambda |\Omega| B_1
\]
\[
= \left( \frac{\tau \lambda^{-r}}{p} \right) - \lambda B_1 K_1^{q+1} \|\nabla u\|_p^{q+1} - \lambda |\Omega| B_1
\]
\[
= \lambda^{-r} \left[ \frac{\tau p}{2p} - \lambda^{1+rp} |\Omega| B_1 \right]
\]
where we have used that \( \tau \leq (2p K_1^{q+1} B_1)^{-r} \) (see (2.8)). Taking \( c_1 = \tau p / (4p) \) and \( \lambda_2 = \tau p / (1+rp) (4p B_1 |\Omega|)^{-1/(1+rp)} \), the lemma is proven. \( \square \)

Next, using the mountain pass theorem we prove that (1.1) has a solution \( u_\lambda \in W^{1,p}_0(\Omega) \).

**Lemma 2.3.** Let \( \lambda_3 = \min\{\lambda_1, \lambda_2\} \). There exists \( c_2 > 0 \) such that, for each \( \lambda \in (0, \lambda_3) \), the functional \( J_\lambda \) has a critical point \( u_\lambda \) of mountain pass type that satisfies \( J_\lambda(u_\lambda) \leq c_2 \lambda^{-p^*} \).

**Proof.** First we show that \( J_\lambda \) satisfies the Palais–Smale condition.

Assume that \( \{u_n\}_n \) is a sequence in \( W^{1,p}_0(\Omega) \) such that \( \{J_\lambda(u_n)\}_n \) is bounded and \( J_\lambda'(u_n) \to 0 \). Hence, there exists \( \nu > 0 \) such that \( \|J_\lambda'(u_n), u_n\| \leq \|\nabla u_n\|_p \) for \( n \geq \nu \). Thus,
\[
-\|\nabla u_n\|_p^p - \|\nabla u_n\|_p \leq -\lambda \int_\Omega f(u_n) u_n \, dx \quad \text{for} \quad n \geq \nu.
\]
Let \( K \) be a constant such that \( |J_\lambda(u_n)| \leq K \) for all \( n = 1, 2, \ldots \). From (1.3), we obtain
\[
\frac{1}{p} \|\nabla u_n\|_p^p - \frac{\lambda}{\theta} \int_\Omega f(u_n) u_n \, dx + \frac{\lambda}{\theta} M |\Omega| \leq \frac{1}{p} \|\nabla u_n\|_p^p - \lambda \int_\Omega F(u_n) \, dx \leq K.
\]
From the last two inequalities we have
\[
\left( \frac{1}{p} - \frac{1}{\theta} \right) \|\nabla u_n\|_p^p - \frac{1}{\theta} \|\nabla u_n\|_p \leq K - \frac{\lambda}{\theta} M |\Omega|.
\]
This proves that \( \{u_n\} \) is a bounded sequence. Thus, without loss of generality, we may assume that \( \{u_n\} \) converges weakly. Let \( u \in W^{1,p}_0(\Omega) \) be its weak limit. Since \( q < Np / (N - p) \), by the Sobolev embedding theorem we may assume that \( \{u_n\} \) converges to \( u \) in \( L^q(\Omega) \). These assumptions and Hölder’s inequality imply
\[
\int_\Omega \lambda f(u_n)(u_n - u) \to 0.
\]
From (2.10) and \( \lim_{n \to +\infty} J_\lambda(u_n) = 0 \) we have

\[
\lim_{n \to +\infty} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n (\nabla u_n - \nabla u) \, dx = 0.
\] (2.11)

Using again that \( u \) is the weak limit of \( \{u_n\} \) in \( W_0^{1,p}(\Omega) \) we also have

\[
\lim_{n \to +\infty} \int_{\Omega} |\nabla u|^{p-2} \nabla u (\nabla u_n - \nabla u) \, dx = 0.
\] (2.12)

By Hölder’s inequality,

\[
\int_{\Omega} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) (\nabla u_n - \nabla u) \, dx \\
\geq \|\nabla u_n\|_p^p - \|\nabla u\|_p \|\nabla u_n\|^{p-1}_p - \|\nabla u_n\|_p \|\nabla u\|^{p-1}_p + \|\nabla u\|_p^p \\
= (\|\nabla u_n\|^{p-1}_p - \|\nabla u\|^{p-1}_p) (\|\nabla u_n\|_p - \|\nabla u\|_p) \\
\geq 0.
\] (2.13)

From (2.11)–(2.13),

\[
\lim_{n \to \infty} (\|\nabla u_n\|^{p-1}_p - \|\nabla u\|^{p-1}_p) (\|\nabla u_n\|_p - \|\nabla u\|_p) = 0,
\]

which implies that \( \lim_{n \to \infty} \|\nabla u_n\|_p = \|\nabla u\|_p \). Since \( u_n \rightharpoonup u \), \( u_n \to u \) in \( W_0^{1,p} \). This proves that \( J_\lambda \) satisfies the Palais–Smale condition.

From (2.6) we see that

\[
\max\{J_\lambda(s\varphi); s \geq 0\} \leq \frac{C^{1+pr}((q+1)^r(q-p)-p)}{D^{pr}p(q+1)^r(q+1)} \lambda^{-pr} + \lambda A_1 |\Omega| \\
:= c_2 \lambda^{-pr} + \lambda A_1 |\Omega| \leq c_2 \lambda^{-pr} + A_1 |\Omega| \lambda^{-pr} \\
:= c_2 \lambda^{-pr},
\] (2.14)

where \( C = \|\nabla \varphi\|_p^{p+1} \) and \( D = A_1 \|\varphi\|_{q+1}^{q+1} \).

With this estimate and lemma 2.2, the existence of \( u_\lambda \in W_0^{1,p}(\Omega) \) such that \( \nabla J_\lambda(u_\lambda) = 0 \) and

\[
c_1 (\tau \lambda^{-r})^p \leq J_\lambda(u_\lambda) \leq c_2 \lambda^{-pr}
\] (2.15)

follows by the mountain pass theorem. \( \square \)

**Remark 2.4.** The solution \( u_\lambda \in W_0^{1,p}(\Omega) \) is indeed in \( C^{1+\alpha}(\bar{\Omega}) \) (cf. [5]).

**Lemma 2.5.** Let \( u_\lambda \) be as in lemma 2.3. Then there is a positive constant \( M_0 \) such that

\[
M_0 \lambda^{-r} \leq \|u_\lambda\|_\infty.
\] (2.16)

**Proof.** We already know that there exists \( c_1 > 0 \) such that \( J(u_\lambda) \geq c_1 \lambda^{-rp} \). On the other hand, we have that \( F(s) \geq \min F > -\infty \) and \( f(s)s \leq B_1(|s|^{q+1} + |s|) \) for all
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$s \in \mathbb{R}$. Then there is a constant $C_1 > 0$ such that

$$\lambda \int_{\Omega} f(u_{\lambda}) u_{\lambda} \, dx = \int_{\Omega} |\nabla u_{\lambda}|^p \, dx \geq pC_1 \lambda^{-r} + p|\Omega| \lambda \min F \geq C_1 \lambda^{-rp}.$$

Thus, $\lim_{\lambda \to 0} \|u_{\lambda}\|_{\infty} = +\infty$. On the other hand, by (2.3),

$$\lambda \int_{\Omega} f(u_{\lambda}) u_{\lambda} \, dx \leq B_1 \lambda \int_{\Omega} (|u_{\lambda}|^{q+1} + |u_{\lambda}|) \, dx \leq B_1 \lambda \int_{\Omega} (\|u_{\lambda}\|^{q+1} + \|u_{\lambda}\|_{\infty}) \, dx \leq 2B_1 |\Omega| \lambda \|u_{\lambda}\|^{q+1},$$

where we have used the fact that $0 < \lambda < 1$. Finally, taking $M_0 = C_1 / 2B_1 |\Omega|$, the lemma is proven.

**Lemma 2.6.** Let $u_{\lambda}$ be as in lemma 2.3. Then there exists $c_3 > 0$ such that

$$\|u_{\lambda}\|_{1,p} \leq c_3 \lambda^{-rp}$$

for all $\lambda \in (0, \lambda_3)$.

**Proof.** By (1.3) and the definition of $u_{\lambda}$,

$$\lambda \int_{\Omega} \frac{\theta - p}{\theta} u_{\lambda} f(u_{\lambda}) \, dx \leq \lambda \int_{\Omega} (u_{\lambda} f(u_{\lambda}) - pF(u_{\lambda})) \, dx - \frac{\lambda p M |\Omega|}{\theta} \leq c_2 \lambda^{-rp} + \frac{\lambda p M |\Omega|}{\theta} \leq 2c_2 \lambda^{-rp},$$

where we have used $0 < \lambda < 1$. Now the result follows from (2.18) and the fact that $u_{\lambda}$ is a weak solution of (1.1).

**3. Proof of theorem 1.1**

We prove theorem 1.1 by contradiction. Suppose there exists a sequence $\{\lambda_j\}_j$, $1 > \lambda_j > 0$ for all $j$, converging to 0 such that the measure $m(\{x \in \Omega; \ u_{\lambda_j}(x) \leq 0\}) > 0$.

Letting $w_j = u_{\lambda_j} / \|u_{\lambda_j}\|_{\infty}$, we see that

$$-\Delta_p(w_j) = \lambda_j f(u_{\lambda_j}) \|u_{\lambda_j}\|_{\infty}^{1-p}.$$  

From lemmas 2.5 and 2.6 there is a constant $C_3$ such that

$$\|w_j\|_{1,p} \leq C_3.$$
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By [4, proposition 3.7] the sequence \( w_j \) is uniformly bounded in \( C^{1,\alpha} \) for some \( \alpha \in (0,1) \). Hence, for any \( \beta \in (0,\alpha) \), the sequence \( w_j \) has a subsequence that converges in \( C^{1,\beta}_0 \). Let us denote its limit by \( w \).

Next, using comparison principles, we prove that \( w(x) \geq 0 \).

Let \( v_0 \in W^{1,p}_0(\Omega) \) be the solution of

\[
-\Delta_p v_0 = 1 \quad \text{in } \Omega, \\
v_0 = 0 \quad \text{on } \partial \Omega.
\]  

(3.3)

Let \( K_j := \lambda_j \min \{ f(t); t \in \mathbb{R}\} \| u_{\lambda_j} \|_{L_p}^{1-p} \). Then the solution \( v_j \) of the equation

\[
-\Delta_p v_j = K_j \quad \text{in } \Omega, \\
v_j = 0 \quad \text{on } \partial \Omega
\]

is given by \( v_j = (-K_j)^{1/(p-1)} v_0 \).

Since \( \lambda_j f(u_{\lambda_j}) \| u_{\lambda_j} \|_{L_p}^{1-p} \geq K_j \), it follows by the comparison principle in [9] that \( w_j \geq v_j \). Then the fact that \( w_j(x) \to 0 \) as \( j \to 0 \) implies that \( w(x) \geq 0 \) for all \( x \in \Omega \).

Since, by hypothesis, \( q > p-1 \), we have \( s = Npr/(N-p) > 1 \). This result, together with the Sobolev embedding theorem, (1.2) and lemma 2.6, gives

\[
\int_\Omega |f(u_{\lambda_j})|^s \| u_{\lambda_j} \|_{L_\infty}^{s(1-p)} \, dx \leq B^s 2^{s-1} \int_\Omega (\| u_{\lambda_j} \|_{L_p}^{(q+1-p)s} + 1) \, dx \\
\leq C(\| u_{\lambda_j} \|_{L_p}^{Np/(N-p)} + 1) \\
\leq C(c_3 \lambda_j^{-r N p/(N-p)} + 1),
\]

(3.5)

where \( C > 0 \) is a constant independent of \( j \) and, without loss of generality, we have assumed \( \| u_{\lambda_j} \|_{L_\infty} \geq 1 \). From (3.5) and the fact that \( r N p/(s N - s p) = 1 \) we see that \( \lambda_j f(u_{\lambda_j}) \| u_{\lambda_j} \|_{L_p}^{1-p} \) is bounded in \( L^s(\Omega) \), so we may assume that it converges weakly. Let \( z \in L^s(\Omega) \) be the weak limit of such a sequence. Since \( \| u_{\lambda_j} \|_{L_p}^{1-p} \lambda_j \to 0 \) as \( j \to +\infty \) and \( f \) is bounded from below, \( z \geq 0 \). Now if \( \phi \in C_0^\infty(\Omega) \), then

\[
\int_\Omega |\nabla w|^{p-2} \langle \nabla w, \nabla \phi \rangle \, dx = \lim_{j \to \infty} \int_\Omega |\nabla w_j|^{p-2} \langle \nabla w_j, \nabla \phi \rangle \, dx \\
= \lim_{j \to \infty} \int_\Omega \| u_{\lambda_j} \|_{L_p}^{1-p} \| \nabla u_{\lambda_j} \|^{p-2} \langle \nabla u_{\lambda_j}, \nabla \phi \rangle \, dx \\
= \lim_{j \to \infty} \int_\Omega \| u_{\lambda_j} \|_{L_p}^{1-p} \lambda_j f(u_{\lambda_j}) \phi \, dx \\
= \int_\Omega z \phi \, dx.
\]

(3.6)

Therefore, \( -\Delta_p w = z \). Since \( \| w_j \|_{\infty} = 1 \), \( w \neq 0 \). By Hopf’s maximum principle for the \( p \)-Laplacian operator (see [8, theorem 5.1]), \( w > 0 \) in \( \Omega \) and

\[
\frac{\partial w}{\partial \nu}(x) < 0 \quad \text{for all } x \in \partial \Omega.
\]

Here \( \partial / \partial \nu \) denotes the outward unit normal derivative. Therefore, since \( \{ w_j \}_j \)

converges in \( C^{1,\alpha} \) to \( w \), for sufficiently large \( j \), \( w_j(x) > 0 \) for all \( x \in \Omega \). Hence,
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\[ u_{\lambda_j}(x) > 0 \text{ for all } x \in \Omega, \]
which contradicts the assumption that
\[ m(\{x; u_{\lambda_j}(x) < 0\}) > 0. \]

This contradiction proves theorem 1.1.

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