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Four Years with Russell, Gödel, and Erdős: An Undergraduate's Reflection on His Mathematical Education

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Claremont McKenna College

Four Years with Russell, Gödel, and Erdős: An Undergraduate’s Reflection on His Mathematical Education

submitted to
Professor Robert Valenza

by
Michael Boggess

for
Senior Thesis
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Introduction

Senior Thesis at CMC is often described institutionally as the capstone of one’s undergraduate education. As such, I wanted my own to accurately capture and reflect how I’ve grown as a student and mathematician these past four years. What follows is my attempt to distil lessons I learned in mathematics outside the curriculum, written for incoming undergraduates and anyone with just a little bit of mathematical curiosity. In it, I attempt to dispel some common preconceptions about mathematics, namely that it’s uninteresting, formulaic, acultural, or completely objective, in favor of a dynamic historical and cultural perspective, with particular attention paid to the early twentieth century search to secure the foundations of mathematics and a detailed look at contemporary Hungarian mathematics. After doing so, I conclude that the scope of mathematics is not what one might expect but that it’s still absolutely worth doing and appreciating.

Enjoy!

Michael Boggess
First Dialogue

*Enter Mike, outside Dr. Rob’s office. Mike knocks tentatively on the door frame.*

**Mike:** Hello, hello! You don’t know me, and I hope you don’t mind me stopping by…but a number of people now have suggested it might be a good idea. So, here I am!

**Dr. Rob:** (hesitant) Hello?

**Mike:** Oh! I’m getting ahead of myself. My name’s Mike; I’m a student here. I’m interested in mathematics, and I need to declare soon, but I’m worried.

**Dr. Rob:** Worried?

*Dr. Rob gestures for Mike to sit. He does.*

**Mike:** About how I can continue to keep my academic interests broad. I actually enjoy GE’s - that’s partly why I like the liberal arts so much.

**Dr. Rob:** There are certain practical matters here, which I can’t really comment on. But, I can personally attest that it’s possible to maintain broad cultural interests, even in academia.

**Mike:** Sure, but academia has become so specialized now. Some mathematicians can’t even speak to other mathematicians!

**Dr. Rob:** True, and maybe a traditional academic path isn’t the one for you. It’s still very early. But you also have to consider while you’re here what might enrich your life culturally. Another upper division math course might be marginally helpful for grad school, but looking back, if I could, I’d easily trade one out for a course on Jane Austen.

**Mike:** (unconvinced) Hmmm.

**Dr. Rob:** I know, it’s one thing for me to offer advice in retrospect, with a good job, but the freedom I have in my job and my academic pursuits has come at the expense of a lack of
freedom in other aspects. I’ve taught the same two courses more or less for I don’t know how many years now. Which is to say that even within an academic career, there’s a surprising amount of flexibility in some areas and a lack of flexibility in others. Even within the disciplines themselves.

Mike: What do you mean?

Dr. Rob: Well, even within mathematics, there’s opportunity for culture and personal exploration. Hofstadter’s Gödel, Escher, Bach, for example. It’s really this amazingly personal expression of a landmark result in mathematics - Gödel’s Theorem, which in itself is quite difficult in its presentation - and more broadly about unintelligent parts combining and coalescing into an intelligent whole.

Mike: That sounds incredible! Gurdal… could you write that down, please?

Dr. Rob scribbles something on a legal pad, rips off the top sheet, and hands it to Mike.

Dr. Rob: I think this is right. Give it a look through. It might help put you at ease. There’s plenty else out there as well: Men of Mathematics immediately comes to mind.

Mike: I will. Thank you, thank you. In the meantime, if I have any more questions, would you mind if I stopped by again?

Dr. Rob: Not at all. Mike, it genuinely was my pleasure. Take a look, and we’ll talk again soon.
Chapter 1: Gödel’s Theorem and The Search for Truth

Wherein the historical and biographical context necessary to understand Gödel’s Theorem is presented and the theorem’s ramifications for mathematical truth are discussed.

Published in 1931, the development of Gödel’s Theorem actually began much earlier, amidst the backdrop of an internal crisis in mathematics, the turn of the twentieth century, and the First World War, as the culmination of a pursuit initiated by Georg Cantor and Gottlob Frege, debated by David Hilbert and Henri Poincaré, and taken up by Bertrand Russell and Alfred North Whitehead.¹ An attempt at a comprehensive historical survey of such a time and its mathematical developments would be nothing short of encyclopedic, and so, in the spirit of Logicomix,² it’s considered here from the perspective of Bertrand Russell as he engaged with the ideas of Cantor and Frege, pressed forward in his Logicist program with Whitehead in the Principia Mathematica, and witnessed the collapse of such a program with the publication of Gödel’s Theorem.

Russell first encountered the ideas of Cantor and Frege in the years succeeding his residence at Trinity College, Cambridge in the 1890’s. At that time, building on his earlier work in understanding infinite sets and different sizes of infinity, Cantor had just introduced the transfinite cardinal numbers,³ established the uncountability of the real numbers via his diagonalization argument, and stated the Continuum Hypothesis⁴ (Gowers, 778). These ideas and their potential repercussions caught the mathematical world aflame, engrossing and dividing even the mathematical elite leading up to and after the 1900 International Congress of

₁ Among others!
₂ Also the source of the comics displayed in this chapter.
₃ Each cardinal corresponds to a different size of infinity, so to speak.
₄ No set exists of cardinality strictly between that of the natural numbers and the real numbers.
Mathematicians. Given Russell’s burgeoning interest in the foundations of mathematics and Cantor’s increasing notoriety, it would have been virtually impossible for them not to engage with one another. Perhaps more surprising was Frege’s immense impact on Russell. In his *Begriffsschrift* in 1879 and his *Grundgesetze der Arithmetik* in 1893, Frege laid the framework for the development of modern logic and for the Logicist pursuit of deriving arithmetic entirely from logical principles (Gowers, 780). These works proved immensely important over time, but they were mostly neglected on publication, largely due to his unwieldy notation. Still, Russell realized their importance and would later publish them in relation to his own work seeking a proper foundation for mathematics (Gowers, 795).\(^5\)

If it was through Cantor’s and Frege’s work that Russell first got his start, it was the 1900 International Congress of Mathematicians in Paris that served as a catalyst for his professional career and the direction of mathematics as a whole. At the Congress, attended by the who’s who of mathematics, two figures loomed large: David Hilbert and Henri Poincaré, divided by Cantor’s new understanding of infinity. Hilbert viewed Cantor’s work with awe and respect, as a framework to be used and explored, so much so that in his famous list of twenty-three problems, ten of which he presented at the Congress, the Continuum Hypothesis was the first. Poincaré, on the other hand, rejected Cantor’s ideas for their counterintuitive results.

\(^5\) Specifically in his *The Principles of Mathematics* published in 1903.
Born out of this conflict and the excitement of the Congress were a number of important developments in the foundations of mathematics. Russell himself was highly productive, publishing his famous paradox in 1901, calling into question the allowability of the notion of the sets of all sets which don’t contain themselves, before entering a decade long collaboration with Whitehead to produce three volumes of the *Principia Mathematica* in 1910, 1912, and 1913 respectively (Gowers, 795). The *Principia* was their attempt to set forth a set of axioms and inference rules described using symbolic logic from which all mathematical theorems and truths could be derived. Whitehead and Russell successfully derived many mathematical results in this way, but two of the axioms\(^6\) were never properly represented as logical propositions. Although the effort was left ultimately incomplete, it laid the groundwork for further discussions in

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\(^6\) The axiom of infinity and the axiom of reducibility.
mathematical philosophy and served as a source of inspiration for Kurt Gödel in conceiving his incompleteness theorems.

![Fig 1.2: Russell’s Quest (Doxiadis, 85)](image)

Chief among those discussions was that between the schools of Logicism, Intuitionism, and Formalism. Logicism (the same Logicism mentioned earlier) was championed by Frege and Russell as a means of subordinating mathematics to logic and thus providing a firmer foundation on which it could rest (Snapper, 184). The thought was that if mathematics could be written entirely via logic, since logic was better understood, mathematics would then be much better understood. It’s also important to note that logic in the Logicist sense wasn’t first-order but meant something more along the lines of representing propositions that were true with respect to their form rather than their content. As mentioned earlier, although they put forth a heroic effort in the *Principia*, Russell and Whitehead came up two axioms short, stymieing the Logicist

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7 An example: All CMC students are students.
school and leading to the “first crisis in mathematics” (Snapper, 186). Intuitionism, meanwhile, was founded around 1908 by L.E.J. Brouwer as a response to Russell’s paradox and other unresolved issues in the foundations of mathematics (Snapper, 186). The Intuitionists sought to redefine mathematics only to allow “inductive” and “effective” mental constructions in the same sense that the natural numbers can be constructed using the idea of a successor and an intuitive notion of time (Snapper, 186). Mathematically and philosophically, Intuitionism is arguably consistent, yet it does away with many important theorems and elegant proofs. Because of this, the mathematical community has overwhelmingly rejected it, a result termed “the second crisis in mathematics” (Snapper, 188). Formalism, then, was started around 1910 by David Hilbert as a means of checking various branches of mathematics for contradictions (Snapper, 189). The process of formalization involved taking some axiomatic system, say that of Euclidean geometry, recasting it in terms of first order logic and adding additional symbols to represent the undefined terms specific to the axiomatic system. It’s interesting to note that Russell and Whitehead attempted a sort of formalization in their *Principia* but did so in order to subordinate it to logic, not to examine it for contradictions. Ultimately, Gödel’s Theorem showed sufficiently strong axiomatic systems to be unable to prove their own consistency, a death blow for Hilbert’s Formalist program and the “third crisis in mathematics” (Snapper, 191).

Out of these discussions rose the figure of Kurt Gödel. In 1931, he published two incompleteness theorems that rocked the mathematical community (Gowers, 700):

1. There are statements about the natural numbers that can be neither proved nor disproved from Peano’s axioms.

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8 In the Kantian sense.
9 In the case of Euclidean geometry, these were: “point”, “line”, and “incidence.”
2. It is impossible to prove from Peano’s axioms that they are consistent.

Gödel made use of Peano’s axioms\(^{10}\) in his proof of the incompleteness theorems, but the result doesn’t actually depend on them. In fact, in any axiomatic system strong enough to support arithmetic on the natural numbers,\(^{11}\) there must be unprovable, true statements and it must be impossible to prove such a system’s consistency from within. Gödel’s proof is long and difficult, but his primary insight was to make use of self-reference in a clever way to construct a sentence of the form “This sentence is not provable in Peano’s Axioms” (Hofstadter, “Analogies”, 267).

In one case, the sentence is indeed provable, in which case it is true, and hence also false by its own admission, leading to a contradiction. To avoid that, the sentence must not be provable, in which case it is also true, leading to the conclusion that there must exist true, unprovable statements, or in Gödel’s own words:

![Gödel Presents His Incompleteness Theorems](image)

**Fig 1.3:** Gödel Presents His Incompleteness Theorems (Doxiadis, 147)

\(^{10}\) See the Appendix.

\(^{11}\) This applies to ZFC.
Second Dialogue

Enter Mike, outside Dr. Rob’s office. Mike knocks boldly on the door frame.

Mike: Hello again, Dr. Rob! I’ve done some of the readings you suggested, and I’d like to chat, if you have the time.

Dr. Rob: I do! Good to see you again, Mike. Please come in. Gödel’s Theorem I take it?

Mike: Yes. I’m having trouble understanding the consequences. I mean it’s not that difficult to state: In any formal axiomatic system strong enough to support arithmetic on the natural numbers, there will be unknowable truths. Sure, but what does that mean for truth in general?

Dr. Rob: How far Gödel’s Theorem reaches and what exactly it implies is still in debate, but one immediate consequence is that at least in mathematics, provability does not encapsulate truth as was previously thought to be the case. In fact, if anything, the opposite is true.

Mike: Hmmm. So, then what’s a better characterization of mathematical truth? Is there one that’s commonly accepted?

Dr. Rob: That’s a difficult question, since it’s so intimately connected to the question of what mathematics is in general. In some sense, mathematical truth occupies this goldilocks area: On the one hand, we have provability pushing outwards. On the other, we have the fact that mathematical truths aren’t empirically falsifiable pushing inwards. In between these, somewhere lies mathematical truth. It’s similar to the idea of a basis for a vector space. Again, from below, there’s the notion of linear independence, of having few enough vectors for them to be distinct. From above, there’s the notion of a spanning set, of having enough vectors to cover the space of interest. Where those two ideas meet is called a basis. It’s an extremely rich, thoughtful
definition, but it took many, many years for mathematicians to get it right. This paradigm
appears again and again in mathematics.

**Mike: That’s beautiful. (pausing) But, then in that middle area, what do you make of the idea of empirical evidence within mathematics? For example: computing values, and then inductively hypothesizing theorems, as Ramanujan did.**

**Dr. Rob: That can be a powerful technique for guiding mathematical research. I mean, no one expected Fermat’s Last Theorem wouldn’t be true. We computed the values for large n and couldn’t find a counterexample. But until Wiles, there was still room for uncertainty. Everyone also expected the Continuum Hypothesis to be true, but via Cohen, that turned out very differently. It’s also interesting how mathematical intuition almost never stems from the axioms themselves. In practice, mathematicians don’t concern themselves with the axioms, but ultimately, mathematical truth requires the burden of proof to be traceable to the axioms if need be. If a theorem is one of the unfortunate unprovable ones, it can be difficult to accept. But understanding that also tells us something about mathematics.**

**Mike: …**

**Dr. Rob: I’ve been talking now for quite a while. Was that helpful? Did you have something else you wanted to ask?**

**Mike: Very helpful, but I need some time to digest it all. I’m heading to Budapest next term with the Budapest Semesters in Mathematics program, so I won’t be in to visit for a while. I’m sure we’ll have lots more to discuss when I get back!**
Dr. Rob: Oh, I’m delighted to hear that! What an excellent program. Please do stop by afterwards and let me know about your experience. In the meantime, I’m always available by email.

Mike: I certainly will. Thank you again; have a wonderful weekend.
Chapter 2: Mathematical Culture

Wherein the notion of mathematical culture is introduced through examples, stories, and a particular emphasis on Hungarian mathematics.

Typical definitions of culture include references to language, religion, and art, and much else, yet mathematics seldom makes the list. This is hardly unsurprising. Though some mathematicians might treat mathematics as language, religion, or art, it’s more often held up as a paragon of objective science that transcends culture. Typical justifications for this view often boil down to how it manages to convey seemingly universal principles about how the world works or to the accepted formal axioms, techniques, and frameworks in which mathematics is done. However, the former justification is not unique to mathematics; the latter is certainly more characteristic of mathematics but still fails to acknowledge that it’s ultimately a human endeavor, in which where it’s done, who does it, and how it’s communicated matters. Mathematics is very much cultural: a glance back in history or even towards contemporary Hungarian mathematics provides plenty of evidence of that.

Extending back thousands of years, the mathematical tradition is extraordinarily rich. Modern mathematics still owes much to the Ancient Greeks, from the paradoxes of Zeno to the methods of Archimedes to the geometric axioms of Euclid, but the paradoxes, methods, and geometric axioms themselves initially arose from examples and spatial problems important in Greek mythology and society.\textsuperscript{12} More recently in France, work in celestial mechanics and mathematical physics taken up by Lagrange, Laplace, and Fourier had much to do with the types of problems put forth by the Academy of Sciences and the reorganization of the educational system during and following the French Revolution (Bell, 153-200). Many of the Russian

\textsuperscript{12} Achilles appeared in a race with a tortoise, the altar to Apollo served as motivation for the attempts to double the cube, etc.
mathematicians in the early Twentieth century drew inspiration from various ideologies, from
state-sponsored religion to counter-culture mysticism (Graham, 84). In Britain and Germany,
Newton and Gauss alone guided the mathematical direction their various countries would take
for decades.

The non-Western world is also a source of rich examples. Traditional peoples, such as the
Angolese Tshokwe, drew sona, similar to modern graphs, in the sand to tell stories. The northern
Australian Warlpiri developed a group-like\textsuperscript{13} social structure of eight sections with strict
inter-section marriage rules (Ascher, 37; 71). Early Islamic mathematicians, including
Al-Khwarizmi, developed algorithmic notions of mathematics in sentence form prior to the
development of modern Arabic numerals. And for many years, Eastern mathematics developed
independently of the West, in local languages and characters, amongst local cultures.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig21.png}
\caption{The succession of the dead chief. Dots 1, 2, and 3 are thieves. Only dot 3 can reach the village and steal the insignia.}
\end{figure}

\textbf{Fig 2.1:} A Reproduction of a Tshokwean Sona (Ascher, 39)

\textsuperscript{13} An abstract mathematical group.
In the contemporary, post-Bourbaki\(^{14}\) world of modern academia, complete with English as the universal academic language, the claim of mathematics as acultural is made more specious. But again, even sheltered in the ivory tower, mathematics is still done by mathematicians, often still in their home countries. A prime example: Throughout the twentieth century and into the twenty-first now, Hungarian mathematics has remained distinctly Hungarian. Of course, Hungarian mathematics has been influenced by the types of problems thought important and the education system, but there’s hardly a better way to introduce that distinct Hungarian spirit than through the life and mannerisms of Paul Erdős.

Paul Erdős, born in 1913 in Budapest, Hungary, lived a life devoted to mathematics, but beyond his mathematical proofs, he also proved very quotable. In Erdős’s own words\(^{15}\) is perhaps the best expression of his unique life and character.

\[\text{Fig. 2.2: Paul Erdős}^{16}\]

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\(^{14}\) A pseudonym used by a group of French mathematicians who worked to standardize mathematics textbooks in the twentieth century.

\(^{15}\) All sourced from Hoffman’s *The Man Who Loved Only Numbers*.

\(^{16}\) From http://www.renyi.hu/~erdoscm/erdos-4x6cm.jpg.
“Another roof, another proof”:

For decades, Erdős spent his professional career moving from roof to roof, collaborating with other mathematicians. He did this for two reasons: He never learned how to properly take care of himself and he enjoyed talking about mathematics with others. Week after week, Erdős would be ushered to destinations throughout the world where he’d show up, provide some insight, and leave the host mathematician to write up the joint result. In total, he collaborated with 485 people, many of whom more than once (Hoffman, 13).

“My brain is open!”:

Though generally a genial houseguest, Erdős was exhausting. Amped up on mathematics and caffeine¹⁷, he’d often wake up in the middle of the night, then wake up his host, saying, “My brain is open!” before working on problems. Lectures, funerals - any event was an opportunity to do mathematics.

“The Book”:

When Erdős came across a particularly beautiful or elegant proof of some mathematical result, he’d say it belonged in the the Book, a fictional entity to which only God had full access. Mathematicians only ever caught glimpses.

“Supreme Fascist”:

Beyond solving problems and finding proofs from the Book, Erdős made his other mission in life to keep the Supreme Fascist’s score low - the Supreme Fascist meaning God and the score meaning the game of life. In the game, the Supreme Fascist scored two points

¹⁷ In that order.
whenever you did something bad, one point whenever you failed to do something good when you could have, and no points otherwise.

“*No doubt he was a great man*”:

Born prior to both World Wars, Erdős spent part of his life in the throngs of actual fascism. For the most part, he was able to travel freely and avoid conflict, but his cluelessness occasionally got the better of him. Entering the United States in 1954, in response to an agent with the US Immigration and Naturalization Service, he remarked with regard to Marx, “I’m not competent to judge. But no doubt he was a great man.”

“*Epsilons*”:

Still, Erdős mostly meant well, and he cared deeply for children, affectionately referring to them as Epsilons,\(^\text{18}\) and young mathematicians alike. The latter group he supported with a supply of problems,\(^\text{19}\) collaboration, and the occasional scholarship.

Erdős was a dominant influence in Hungarian and global mathematics during the twentieth century; the effects are still evident to this day. Hungary produces a disproportionate number of mathematicians in general, in particular those working in areas of interest to Erdős - combinatorics, graph theory and the like. Some of these mathematicians were collaborators of Erdős early in his career, but others simply rose up the academic ranks and then similarly started to take it upon themselves to nurture future generations. In this spirit, the education system still emphasizes conjecture and proof style learning, which is particularly useful for identifying and cultivating mathematical talent. For a country of ten million, the mathematical tradition is well and strong and seems likely to remain so.

\(^{18}\) Typically used to represent extremely small quantities.

\(^{19}\) For an example of a problem he left to posterity, see the Appendix.
My own experience studying through Budapest Semesters in Mathematics convinced me of as much. During my time there, I heard inside stories about the mathematical community, of BSM instructors who formerly collaborated with Erdős, of instructors raving about the results (and introducing them for problem sets) of their peers across town. I received a proper education in combinatorial mathematics and conjecture and proof,\textsuperscript{20} and I learned that even in mathematics, sometimes alternative forms of communication are necessary to get your point across. In the last case, I came into the program fully aware that the Hungarian language barrier would occasionally prove problematic. In my homestay with my host nagymama,\textsuperscript{21} Klara, I took to writing notes in broken Hungarian supplemented with doodles, to which she would respond with broken English and doodles of her own. Although I anticipated struggling from time to time with the mathematics, I was much more confident in my ability. I expected at least to be able to understand most problems I was given, but working with a research advisor, some weeks, I was at a loss for words, with pictures and patience the only recourse. That semester, neither English, Hungarian, nor mathematics was sufficient, but taken together, they gave me a more complete understanding of Hungarian culture.

\textsuperscript{20} Arguably BSM’s flagship course.
\textsuperscript{21} Literally big mama, actually grandmother.
Written by the author’s roommate, it says, “Hi Klara! Michael and I will be traveling this weekend (Friday afternoon to Tuesday morning) and there’s no need for breakfast. Also, in the future, after Tuesday, we will start to make our own breakfast. Thank you!”
Third Dialogue

Enter Mike, inside Dr. Rob’s office.

Dr. Rob: Welcome back, Mike! It’s good to see you again.

Mike: It’s good to see you as well! How have you been?

Dr. Rob: Quite well. Which is to say pretty standard fare this past Spring. And Budapest?

Mike: Incredible. Four of the best months of my life. From the homestay to the other people on the program to the city itself, everything was better that I imagined.

Dr. Rob: Wonderful to hear that. Did you get the chance to travel?

Mike: Infrequently during the semester, but I was able to backpack for a few weeks after the fact with my brother. My favorite places were Transylvania and Slovenia, the former for the history and the castles, the latter for the mountains and caves. They were both so otherworldly. In general, Budapest was a great place to base out of - trains could get you most anywhere.

Dr. Rob: And the classes themselves? (grinning) Should I have asked that first?

Mike: (smiling) Believe it or not, classes actually did come first. My graph theory course was quite difficult, and research kept me busy. Hungarians really love their combinatorics! I also appreciated the Hungarian culture course. Between that, a Hungarian language course, and the math courses themselves, it was like I had three windows into Hungarian culture. That ended up being a regular theme on my travel blog.

Dr. Rob: Glad you kept a record. You’ll certainly be grateful years from now, and I’m sure it helped you make better sense of things in the moment.

Mike: It definitely did, and it was fun to write. Updating it with stories and photos each week was something I did very willingly, even if the assignment was mandatory.
**Dr. Rob:** Have you thought about writing a longer reflective piece about your experience or what it taught you about mathematics?

**Mike:** Not seriously, but that’s a good idea. I think, for now at least, I’ve said all that I wanted to say. *(glancing at his phone)* I also should be going. I need to run to lunch before it closes.

**Dr. Rob:** Of course, I don’t want to hold you up. Please come by again whenever you feel so inclined.

**Mike:** Gladly. Until then.
Conclusion

The preceding pages are rooted in my personal experience, but they still say something universal about the universality of mathematics: namely, that mathematics is neither universal in its ability to realize truth nor in its ability to connect across culture. Despite the best efforts of mathematicians over the years to provide a secure axiomatic foundation, problems remain. Certain mathematical truths will remain inaccessible and mathematics itself will remain a human endeavor, complete with all the cultural and personal idiosyncrasies therein. Yet, mathematics remains very much worth doing: the process of inquiry is no less exciting, of discovery no less profound, of proof no less beautiful. I’m not much of a mathematician, but I’ve come to appreciate mathematics as much from the outside as from the inside. It’s my sincere hope that you might come to a similar appreciation and that what you’ve read here might help you along the way.
References


Snapper, Ernst. "The Three Crises in Mathematics: Logicism, Intuitionism, and Formalism."


Wigner, Eugene. "The Unreasonable Effectiveness of Mathematics in the Natural Sciences."

Appendix

An Erdős Problem

Let $1 \leq a_1 < a_2 < \cdots < a_k$ be a sequence of integers. Assume that all sums

$$\sum_{i=1}^{k} \epsilon_i a_i (\epsilon_i = 0 \text{ or } 1)$$

are distinct. Estimate or determine $\min a_k$.

Commentary:

Erdős claimed this may have been his first serious problem when he came across it in 1931 (Erdős, 467). He conjectured $a_k > c2^k$, which is to say that he thought the largest value in the sequence of integers would be of greater order than $2^k$ (where $c$ denotes a constant). This seems reasonable since there are $2^k$ different sums. This problem set the tone for the types of problems Erdős would be attracted to, namely those that can be stated easily but that are surprisingly difficult to solve or even estimate, often making use of techniques in number theory, combinatorics, and probability. Also characteristic of many Erdős problems, this problem is still open to improved approximations and bounds.

Peano Axioms

1. For all $x$, $s(x) \neq 0$.
2. For all $x$ and $y$, if $x \neq y$, then $s(x) \neq s(y)$.
3. Let $A$ be any subset of the natural numbers with the following properties: $0, s(x) \in A$ whenever $x \in A$. Then $A$ must be the set of all natural numbers.
Commentary:

Intuitively, the Peano axioms were designed to capture mathematical reasoning about the natural numbers with as few assumptions as possible, introducing only the notion of a zero, ‘0’, and of a successor function, ‘s’. The zero is what you’d expect and the successor function does what you’d expect as well, incrementing given values by one. Axiom 1 says that ‘0’ is never a successor. Axiom 2 says that distinct elements have distinct successors. Axiom 3 expresses induction, allowing a representation of all natural numbers. Together, these characterize the natural numbers, and with the introduction of addition and multiplication, can be made into Peano arithmetic, an equally simple formal system that Gödel relied on heavily in his proof of the incompleteness theorems.