

Research Proposal: Domino Tilings of a Checkerboard

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1 Introduction

The modern student of mathematics is first introduced to tilings via the Fibonacci sequence. That is, if we define $f_0 = f_1 = 1$ and let

$$f_n = f_{n-1} + f_{n-2}$$

for $n \geq 2$, it can be shown that f_n counts the number of ways to tile a $1 \times n$ board with squares and dominoes. This interpretation of the Fibonacci numbers admits clever counting proofs of many Fibonacci identities.

When we consider more complex combinatorial objects, we find that simply counting the number of tilings is not quite enough, and that we must consider instead *weighted* tilings. For example, Chebyshev polynomials of the second kind count the total weights of all tilings of a $1 \times n$ board, where squares are given a weight of $2x$ and dominoes assigned a weight of -1 , and the weight of a tiling is simply the product of the weights of the tiles used [1]. This interpretation of the Chebyshev polynomials can be used to prove, for example,

$$T_n(\cos \theta) = \cos(n\theta),$$

where T_n is the n th Chebyshev polynomial of the first kind, using mostly elementary counting methods.

Of course, tilings are not just limited to $1 \times n$ boards, nor are we always permitted squares and dominoes. Domino coverings of a square board have many applications to other fields of science, such as the study of molecules arranged in a lattice. However, many identities regarding these more difficult tilings have no known combinatorial proof.

2 Proposed Research

It has been independently shown by Kasteleyn [2] and Fisher and Temperley [4] that the number of tilings of a $2m \times 2n$ checkerboard by dominoes is given by

$$M(m, n) = 4^{mn} \prod_{i=1}^m \prod_{j=1}^n \left(\cos^2 \frac{i\pi}{2m+1} + \cos^2 \frac{j\pi}{2n+1} \right).$$

Their work involves studying the pfaffian of the adjacency matrix encoding the tiling of the board.

My thesis would attempt to find a combinatorial proof of this result. Specifically, since the trigonometric terms inside the double product can be written in terms of complex exponentials, we can perhaps weight the dominoes of a tiling in some way according to

their orientation. Then, ideally, summing the weights of all tilings would produce the formula for $M(m, n)$ above.

If time permits, I will also consider domino tilings of a torus, investigated by Propp in [3]. He notes that “it might be interesting to have a combinatorial explanation for t vanishing of [some quantity] via some sort of pairing of terms”, and hopefully the approaches developed in studying the checkerboard case can be applied directly to the toroidal board.

References

- [1] Arthur T. Benjamin, Larry Ericksen, Pallavi Jayawant, and Mark Shattuck. Combinatorial trigonometry with chebyshev polynomials. Submitted, 2008.
- [2] P. W. Kasteleyn. The statistics of dimers on a lattice: I. the number of dimer arrangements on a quadratic lattice. *Physica*, 27(12):1209 – 1225, 1961.
- [3] James Propp. Dimers and dominoes. 1997.
- [4] H. N. V. Temperley and Michael E. Fisher. Dimer problem in statistical mechanics - an exact result. *Philosophical Magazine*, 6:1061 – 1063, 1961.