Math 197: Senior Thesis

Arithmetic on Specializable Continued Fractions

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Introduction

It is well known that simple continued fractions, that is, those of the form

$$[a_0, a_1, a_2, ...] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{...}}}$$

$$a_i \in \mathbb{Z} \;\; \forall i \geq 0, \quad a_i > 0 \;\; \forall i > 0$$

correspond nicely with real numbers, with finite simple continued fractions representing rationals and infinite simple continued fractions taking on irrational values. From this analogy, we would expect these objects to be amenable with the four basic arithmetic operations; however, there exists no obvious way to apply these operations and have the results agree with the natural arithmetic of real numbers. We begin by using an algorithm from Bill Gosper to show that arithmetic can be performed on simple continued fractions in an analgous manner to real numbers.

Continuing further, we consider a more general class of continued fractions, known as *specializable*. These objects take as partial quotients elements of $\mathbb{Z}(X)$. Formally, we define this type of continued fraction as one of the following form:

$$[a_0(X), a_1(X), a_2(X), ...] = a_0(X) + \cfrac{1}{a_1(X) + \cfrac{1}{a_2(X) + \cfrac{1}{...}}}$$

$$a_i(X) \in \mathbb{Z}(X) \;\; \forall i \geq 0, \quad deg(a_i(X)) > 0 \;\; \forall i > 0$$

It is clear that a finite specializable continued fraction represents a rational function of X, while the infinite case can be thought of as the limiting case for the sequence of rational functions representing its convergents. In this general case, we consider the following questions:

- How can Gosper's algorithm be generalized to apply to specializable continued fractions?
- When is the sum or product of specializable continued fractions also specializable?

Gosper's Method

Let $A = [a_0, a_1, a_2, ...]$, $B = [b_0, b_1, b_2, ...]$ be simple continued fractions. For any set of constants a, b, c, d, e, f, g, h we consider the following function

$$F(A, B) = \frac{aAB + bA + cB + d}{eAB + fA + gB + h}.$$

Such a function F is called *bihomographic*. Note that for any choice of A and B, $F(A, B)$ varies between the ratios $\frac{a}{e}, \frac{b}{f}, \frac{c}{g}, \frac{d}{h}$. Thus if the integral parts of these ratios are equal, that value is also the integral part of $F(A, B)$. Gosper's algorithm manipulates these four quotients to obtain this condition by using the following identity

$$[a_0, a_1, a_2, ...] = a_0 + \frac{1}{[a_1, a_2, ...]} \qquad (1)$$

Note that substituting $a_0 + \frac{1}{A}$ for A in the definition of F yields another bihomographic function, although now the symbol A is understood to represent the continued fraction $[a_1, a_2, ...]$

$$\frac{a(a_0 + \frac{1}{A})B + b(a_0 + \frac{1}{A}) + cB + d}{e(a_0 + \frac{1}{A})B + f(a_0 + \frac{1}{A}) + gB + h}$$

$$= \frac{(a_0 a + c)AB + (a_0 b + d)A + aB + b}{(a_0 e + g)AB + (a_0 f + h)A + eB + f}$$

Once a value is determined for the integral part of $F(A, B)$ (i.e. its first partial quotient), we may apply the same identity to $F(A, B) = [x_0, x_1, x_2, ...]$, noting that the next partial quotient, x_1, is equivalent to the integral part of

$$\frac{1}{F(A, B) - x_0} = \frac{eAB + fA + gB + h}{(a - x_0 e)AB + (b - x_0 f)A + (c - x_0 g)B + (d - x_0 h)}$$

The algorithm can then be applied to this function to find the next partial quotient for F. Surprisingly, we find that the arithmetic operations can all be achieved through some bihomographic function, simply by choosing the proper initial values of the constants $a, b, \ldots, h$.

Running the Algorithm

Consider the simple continued fractions $A = \coth 1 = [1, 3, 5, 7, 9, ...]$ and $B = \sqrt{6} = [2, 2, 4, 2, 4, 2, 4, ...]$. To compute $F(A, B)$ we begin with the matrix of constants associated to F:

$$\begin{matrix} b & d \\ f & h \\ a & c \\ e & g \end{matrix}$$

It is straightforward to verify that substituting from (1) for B is equivalent to multiplying the matrix $\begin{smallmatrix} a & c \\ e & g \end{smallmatrix}$ by the next partial quotient of B, and adding the result to the matrix $\begin{smallmatrix} b & d \\ f & h \end{smallmatrix}$. This substitution gives the new matrix of values for a, c, e, g while the original matrix for these values represents the new values for b, d, f, h. Similarly, substitutions for A can be realized by multiplying the matrix $\begin{smallmatrix} a & b \\ e & f \end{smallmatrix}$. by the next partial quotient of A and summing with the matrix $\begin{smallmatrix} c & d \\ g & h \end{smallmatrix}$.

Letting $F(A, B) = \frac{2AB + A}{AB + B}$, the matrix of coefficients is placed in the upper-right corner of an array, with the partial quotients for A running right to left across the top of the array and the partial quotients for B listed down the right side. The calculations for Gosper's algorithm can be adjoined down the array or to the left as needed to obtain the following result:

...	7	5	3	1		
				1	0	
				0	0	
		8	2	2	0	2
		7	2	1	1	
		20	5	5	0	2
		14	4	2	2	
						4
						...

This array is the result of substituting for B followed by substituting for A twice. Note that the four ratios $\frac{20}{14}, \frac{8}{7}, \frac{5}{4}, \frac{2}{2}$ now all have integer part 1, so this must be the first partial quotient for $F(A, B)$. Continuing the algorithm, we find that $F(A, B) = [1, 2, 1, 2, 1, 1, ...]$.

Generalizations to Specializable Continued Fractions

Since $\mathbb{Q}(X)$ is a Euclidean Domain, we may apply this same algorithm to any continued fractions whose partial quotients lie in $\mathbb{Q}(X)$. Therefore, as $\mathbb{Z}(X) \subset \mathbb{Q}(X)$, the algorithm immediately gives an answer to the question of whether $F(A, B)$ is specializable for specific A and B. Unfortunately, it gives no insight as to when the sum of any two such continued fractions is specializable, as the resultant partial quotients may in general lie in $\mathbb{Q}(X)$ but outside of $\mathbb{Z}(X)$. We can conclude, however, that any bihomographic function of continued fractions with partial quotients in $\mathbb{Q}(X)$ will yield another such continued fraction.

References

- Gosper, Bill *Continued Fraction Arithmetic* `http://perl.plover.com/classes/cftalk/INFO/gosper.html`

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For Further Information

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- Further information on my thesis as well as my full report are posted on `http://www.math.hmc.edu/~rmerriam/thesis/`

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