HARVEY MUDD C O L L E G E

Introduction

The study of regular lattices is important in solidstate physics for calculating properties of semiconductors and other materials. One important statistic is the number of possible configurations of *dimers* (molecules that take up two adjacent slots in a crystal lattice) on a grid, or, equivalently, the number of domino tilings of a checkerboard.

In 1961, Kasteleyn [2] and Fisher and Temperley [3] independently discovered that the number of ways to tile a $2m \times 2n$ checkerboard is

$$\prod_{j=1}^{m} \prod_{k=1}^{n} \left[4\cos^2 \frac{j\pi}{2m+1} + 4\cos^2 \frac{k\pi}{2n+1} \right].$$
(1)

Their proof involved the computation of a complicated Pfaffian. We attempt to give a clean, combinatorial proof of this result, and studied other identities with trigonometric terms and complex numbers. This investigation led to a novel evaluation of the sum of evenly spaced binomial coefficients.

Binomial Coefficients

Among the first combinatorial identities introduced in a discrete math course are the sum

$$\sum_{k\geq 0} \binom{n}{k} = 2^n$$

of binomial coefficients and the sum

$$\sum_{k\geq 0} \binom{n}{2k} = 2^{n-1}$$

of every other binomial coefficient. However, once we increase the spacing *r* of the binomial coefficients beyond 2, the sums suddenly become quite a bit more *complex*—in both their closed form and in their methods of evaluation.

In fact,

$$\sum_{k\geq 0} \binom{n}{rk} = \frac{1}{r} \sum_{j=1}^{r} \left(1 + \omega^j\right)^n, \qquad (2)$$

where $\omega = e^{\frac{2\pi i}{r}}$ is a primitive *r*th root of unity. The standard method uses the binomial theorem, but here we shall offer a combinatorial approach using weighted walks. This proof technique will motivate our approach towards finding a combinatorial interpretation of Equation 1.

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Math 197: Senior Thesis **Complex Combinatorial Proofs** Bob Chen

Walks on Looped Cycles

Definition 1. The *looped cycle graph* G_r on r vertices is the directed cycle on *r* vertices, plus an additional edge from each vertex to itself.

Now, any length n walk on G_r is completely described if we choose

- An initial vertex; and
- For each of the *n* steps, whether to remain at the same vertex or to advance to the next.

Hence

Lemma 2. The number of closed walks of length n on G_r is given by

$$r\sum_{k>0}\binom{n}{rk}.$$

Weighted Walks

Let us now assign a *weight* to each of the $r2^n$ possible walks on G_r . Say a walk X begins on vertex *j* and *a* of the *n* steps are forward. Then, letting ω be a primitive *r*th root of unity, we define the weight of *X* to be

$$f(X) = \omega^{aj}.$$

The cleverness of our weight function lies in that **Proposition 3.** A walk X on G_r is closed if and only if it has weight 1.

Moreover,

Proposition 4. The set of open walks can be partitioned into orbits, each with total weight 0.

In other words, summing the weight of *every* walk on G_r gives the *number* of closed walks on G_r . Therefore,

Lemma 5. The number of closed walks of length n on G_r is given by

$$\sum_{X \in W} f(X) = \sum_{j=1}^{n} \left(1 + \omega^j \right)^n,$$

where W is the set of all length n walks on G_r .

Now, because Lemmas 2 and 5 enumerate the same objects, the resultant quantities in fact must be equal. Hence,

$$Y\sum_{k\geq 0} \binom{n}{rk} = \sum_{j=1}^{n} \left(1 + \omega^{j}\right)^{n}.$$

Dividing both sides by *r* yields Equation 2.

Trigonometric Form

In the special case when *r* divides *n*, we may rewrite Equation 2 as a sum of cosines. Hence, we have

 $\sum_{k>0} \binom{n}{rk} = \frac{1}{r} \sum_{i=1}^{r} (-1)^{\frac{n}{r}} \left(2\cos\frac{\pi j}{r} \right)^{n}.$

This formulation suggests that other combinatorial identities involving cosine may be proved by first writing the cosine as a sum of roots of unity, then counting the total weight of an appropriate combinatorial object.

Dominoes and Checkerboards

We would like to use a similar approach to enumerate the number of domino tilings of a $2m \times 2n$ checkerboard. Let us rewrite each factor in the double product of Equation 1 as

 $\omega_{2m+1}^{2j} + \omega_{2m+1}^{-2j} + \omega_{2n+1}^{2k} + \omega_{2n+1}^{-2k} + 1 + 1 + 1 + 1,$

where $\omega_r = e^{\frac{\pi i}{r}}$ is an *r*th root of unity. This expression seems to take the product of *mn* independently chosen weights, each of which has 8 possible values, and we attempt to find a combinatorial proof using such a weight function.

We will use both white and black dominoes, and independently choose one of 8 possible dominoes (2 colors, 4 orientations) to cover each *doubly even* cells (the *mn* cells with both coordinates even). Next, we will assign each domino configuration a weight, and say that the weight of a doubly even tiling is the product of the weights of its constituent dominoes.

We find that

Proposition 6. Every doubly even tiling with no overlapping or black dominoes can be extended in a unique way to a complete tiling of the checkerboard. Moreover, each such tiling has weight 1.

In addition, we have a partial result which states that

Proposition 7. Every doubly even tiling that contains

- A column with exactly one vertical black domino; or
- A row with exactly one horizontal black domino

can be put into an orbit with total weight 0.



Future Work

The combinatorial proof of the 1961 result has yet to be completed. In fact, a combinatorial proof for even the m = 1 case has yet to be completed. Future work might also investigate the related problem of domino tilings of a toroidal checkerboard.

Roots of unity show up in many other binomial coefficient identities as well. Gould lists

$$\sum_{k\geq 1} \frac{1}{\binom{kr}{r}} = \sum_{k=1}^{r-1} (-\nu^k) (1-\nu^k)^{r-1} \log \frac{1-\nu^k}{-\nu^k},$$

where $\nu = e^{\frac{2\pi i}{r}}$ is a primitive *r*th root of unity, as Identity 2.24 in [1]. Perhaps with the right combinatorial interpretation, this identity will no longer appear so complex.

References

- [1] Henry W. Gould. Combinatorial Identities. Morgantown Printing and Binding Co., 1972.
- [2] P. W. Kasteleyn. The Statistics of Dimers on a Lattice: I. The Number of Dimer Arrangements on a Quadratic Lattice. *Physica*, 27(12):1209–1225, 1961.
- [3] H. N. V. Temperley and Michael E. Fisher. Dimer Problem in Statistical Mechanics — An Exact Result. Philosophical Magazine, 6:1061–1063, 1961.

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