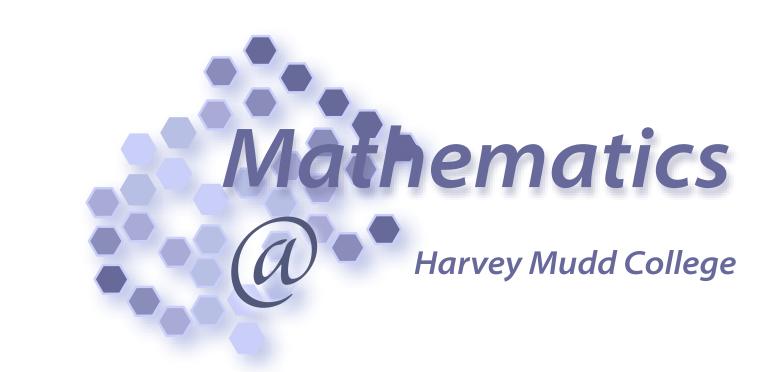


Understanding Committee Elections Using Wreath Products

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Let's Vote!

Suppose Harvey Mudd College wants to elect a committee of size n, where each member must come from a different academic department. In this election, each department is required to put forth two candidates. In this thesis, we construct an algebraic framework using the wreath product group $S_2[S_n]$ to understand the mathematical behavior of such elections for n = 2 and n = 3. With this framework we gain an understanding of how groups of voters contribute to the results of an election.

The Algebraic Framework

We view a voting procedure as a linear transformation from the *profile space* to the *results space*

$$T: P \to R$$
.

- *P* is the space of possible ways a body of voters might vote.
- *R* is the space of possible scores the committees can receive.
- *T* is determined by the voting procedure; *T* scores the votes.

The following figure helps us visualize the n=3 case. For the n=2 case, simply ignore department C.

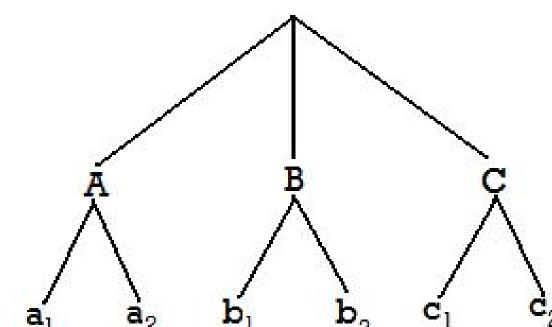


Figure 1: A visual representation of the n=3 election structure. A, B, and C represent three departments, while the lowercase letters represent the candidates from each.

We define the action of the wreath product $S_2[S_n]$ on the candidates which extends to an action on the profile space and results space. Thus we view P and R as $\mathbb{Q}S_2[S_n]$ -modules, and we view T as a module homomorphism from P to R. This perspective allows us to apply useful results from module theory and representation theory such as Schur's Lemma and Maschke's Theorem.

Results

Once we view P and R as modules, we can

- Find how they decompose into irreducible submodules.
- View any vector in *P* and *R* as the sum of components from these submodules.
- See how different components of the voter data contribute differently under various voting procedures.

Results Space Decomposition

For our n = 2 case, we label the four possible committees as follows:

$$W = \{a_1, b_1\}$$
 $X = \{a_1, b_2\}$
 $Y = \{a_2, b_1\}$ $Z = \{a_2, b_2\}.$

Suppose vectors in R are indexed lexicographically, so the vector $[5,2,3,6]^t$ indicates that committee W receives 5 points, X receives 2, etc. We find that as a $\mathbb{Q}S_2[S_2]$ -module, R decomposes into the following irreducible submodules:

$$R_1 = \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle$$
, $R_2 = \left\langle \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\rangle$, $R_3 = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\rangle$

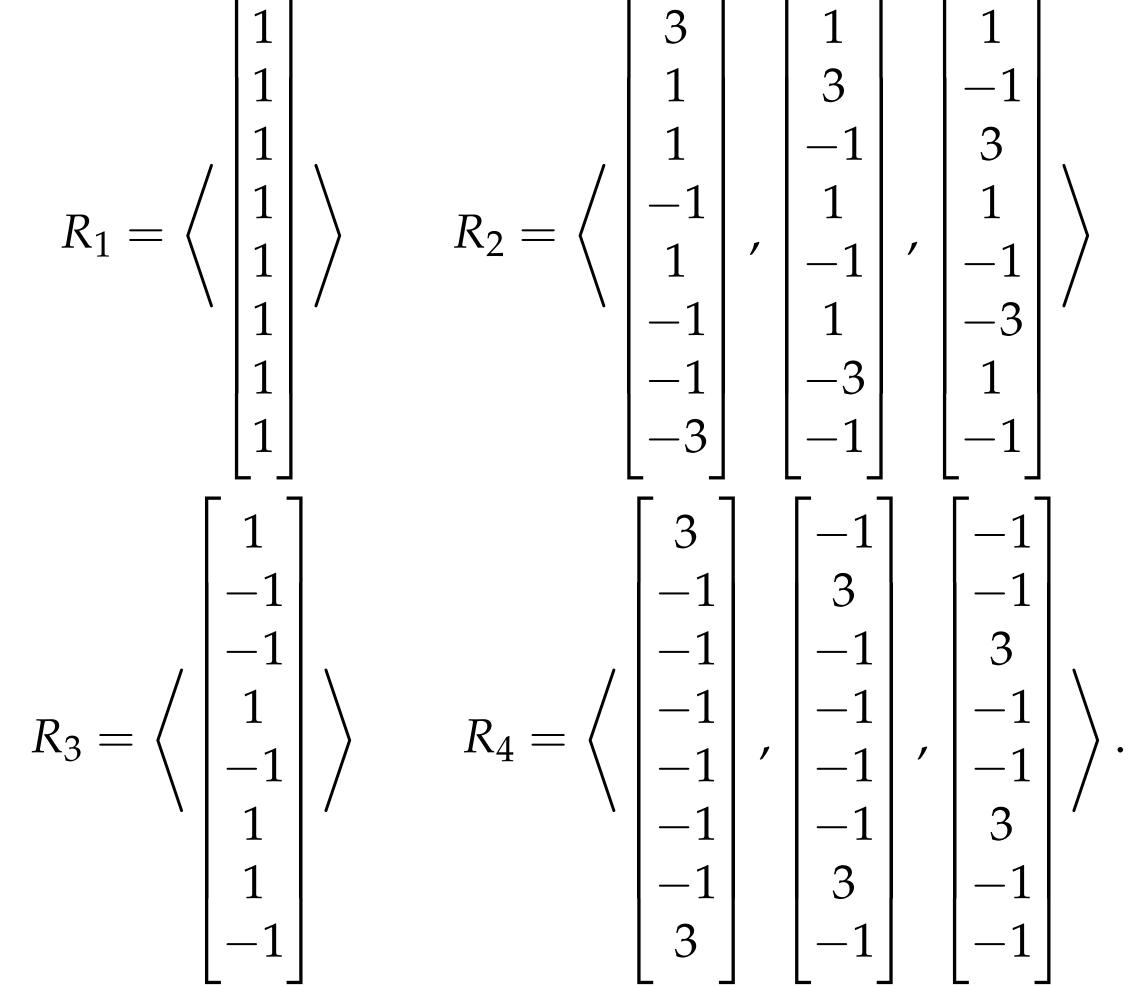
In each submodule, the vectors tell a different story about how the committees are scored based on the candidates shared between them.

For our n = 3 case, we label the eight possible committees as follows:

$$S = \{a_1, b_1, c_1\}$$
 $T = \{a_1, b_1, c_2\}$ $U = \{a_1, b_2, c_1\}$
 $V = \{a_1, b_2, c_2\}$ $W = \{a_2, b_1, c_1\}$ $X = \{a_2, b_1, c_2\}$
 $Y = \{a_2, b_2, c_1\}$ $Z = \{a_2, b_2, c_2\}.$

Suppose vectors in R are indexed lexicographically, so the vector $[5,2,3,6,1,1,9,7]^t$ indicates that committee S receives 5 points, T receives 2, U receives 3, etc. We find that as a $\mathbb{Q}S_2[S_3]$ -module, R decomposes into the

following irreducible submodules:



These vectors tell a story similar to the one told in the n = 2 case.

Voting Paradoxes

What we find is a voting paradox also commonly found in single-candidate elections. Using our module decompositions, we can easily construct the following scenario: voter data is collected and then scored with two reasonable but different scoring procedures, but the election results are completely different. These paradoxes arise because different procedures ignore or emphasize different components of the given voting data. Once the votes are submitted, the fate of the election may rest in the hands of the people choosing the voting procedure and scoring the votes rather than in the hands of the voters.

Conclusions

With these results, we can:

- Understand how well procedures reflect voters' preferences. In particular, we study preferences based on how the candidates make up the committees.
- Compare election procedures to see how they differ.
- Inform voters of how their votes actually contribute to an election.

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For Further Information

For more information on this research, you may e-mail me at slee@hmc.edu. In addition, the thesis report itself may be found at http://www.math.hmc.edu/~slee/thesis/.

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