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Aesthetic Considerations in Mathematics

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Abstract

Drawing on some of the principles of humanistic mathematics first outlined by Alvin White, this paper seeks to examine the way in which value judgments are implicated in the growth of the mathematics discipline. After a short overview of some of the roles ascribed to the mathematical aesthetic historically, I turn to more contemporary positioning of the aesthetic in order to develop a framework that offers insight into the particular values, assumptions and desires that constrain what is done in mathematics, how it is done and why. My goal, at least in part, is to bring together under one umbrella some of the recent work that is being done in the cultural ethnography and cognitive history of mathematics and, in so doing, provide a stronger rationale for the importance and relevance of aesthetic considerations in the history and philosophy of mathematics. Finally, I use this framework to promote the idea of a mathematics critic, who could do for mathematics what art critics do for the arts, namely, to not only evaluate and explain art, but to work toward establishing its accessibility and credibility.

In the edited collection *Essays in Humanistic Mathematics*, Alvin M. White [58] describes the humanistic dimension of mathematics as including both: "an appreciation of the role of intuition in understanding and creating concepts" and "An understanding of the value judgments implied in the growth of any discipline [...] what is investigated, how it is investigated, or why it is investigated" (vii). For me, both these aspects relate strongly to the mathematical aesthetic in two ways. First, in writing about the importance of the value judgments made in the discipline, White points to the axiological dimension of the philosophy of mathematics, which include both aesthetic and ethical questions of what is beautiful or ugly, good or bad, and why. Second, in pointing to the role of intuition in understanding and creating concepts, I interpret White as being concerned with the informal and non-propositional ways in which mathematical ideas derive their meaning, which are deeply embodied – and thus aesthetic in the etymological sense of the Ancient Greek word αἰσθητικὸς ("of sense perception"), which in turn comes from αἰσθάνομαι ("I feel").

Aesthetic considerations have not featured strongly in the Western philosophy of mathematics tradition, which tends to focus more on issues of epistemology and ontology, and which tends to treat the aesthetic as epiphenomenal, overly vague, or even frivolous. By casting the aesthetic in this way, few undertake to study it seriously, which – as I will argue –allows the aesthetic to operate in somewhat covert ways within the mathematical community, and even the broader areas of society impacted by the discipline. Despite its marginal position in the philosophy of mathematics, and its largely unexamined use in the discourse of the mathematics community, I wish to offer a compelling evidence for its fundamental role in defining the very nature of the discipline, in guiding its growth, and in mediating its teaching.

I will begin with a short overview of the various ways in which the construct of aesthetics has been taken up in relation to mathematics. Then, drawing on more contemporary positioning of the aesthetic – especially in relation to the epistemic – I will offer a framework for thinking about the aesthetic in mathematics that is consistent with its use in other disciplines, especially the arts, one that offers productive ways of gaining insight into the particular values, assumptions and desires that constrain what is done in mathematics, how it is done and why. My goal, at least in part, is to bring together under one umbrella some of the recent work that is being done in the cultural ethnography and cognitive history of mathematics and, hopefully, to show that this work it worth doing, not least for the fact that the values, assumptions and desires that have shaped mathematics today have a tremendous affect on society. In the final section I will look more closely at how aesthetic considerations are taken up in the literature about and around mathematicians, and, in particular, I will examine the possibility of the mathematics critic who might play a role similar to that played by arts critics.

1. Mathematicians' conception of aesthetics

Though the use of words such as "beautiful" and "elegant" to talk about the mathematical strikes many non-mathematicians as rather odd, this discourse is prevalent in the mathematical community and even at the community's outer edges in textbooks and popularizing non-fiction books. A substantial part of the early writing on the mathematical aesthetic focused on exploring the aesthetic merits of mathematical products, especially proofs and theorems. Mathematicians such as G.H. Hardy attempted to offer criteria by which one could judge aesthetic merit, taking a rather objective view that aesthetic merit was independent of the observer and intrinsic to the mathematical product itself – a view that was mainstream in the philosophy of arts even at the turn of the last century (in the work, for example, of Roger Fry and Cleve Bell). In *A Mathematician's Apology*, Hardy [17] emphasizes criteria such as depth and significance, as well as "purely aesthetic qualities" such as unexpectedness, inevitability and economy. He famously claimed that "the mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way" and that "there is no permanent place in this world for ugly mathematics." (p.85). In a similar vein, in terms of objective criteria of finished products, King [24], in an attempt to distinguish "good" mathematics from "bad" (to save the world from "bad" mathematics?) proposed two definitive criteria: the principle of minimal completeness and the principle of maximal applicability.

The pursuit of objective criteria did not receive a great deal of attention in the mathematics community and, by the early 1990s – when serious philosophers of art had long ago abandoned this viewpoint – David Wells ran a survey in the Mathematics Intelligencer showing that the aesthetic metric of mathematical theorems was highly subjective. In the large number of responses he received, from eminent mathematicians around the world, who were asked to rate the beauty of twenty-two theorems, Wells [57] found that many factors were at play in evaluating the aesthetic merit of these theorems: field of interest; preferences for certain mathematical entities such as problems, proofs or theorems; past experiences or associations with particular theorems; even mood. He also points out that aesthetic judgments change over time: this was particularly evident in the rating of Euler's formula,[1] which was historically considered "the most beautiful formula of mathemat-

[1] Euler's formula ($V - E + F = 2$) has, in a sense, turned into an object in modern mathematics, namely, the Euler characteristic (of a surface). This object might have had more aesthetic merit for the mathematicians surveyed; nonetheless, as I will discuss later, the transition to object itself may be seen as part of ways of doing mathematics that are worth aesthetic consideration.

ics" (p.38), but is now, according to Wells's respondents at least, considered too obvious even to elicit an aesthetic response.

The inferences made by Wells correspond to a contextualist view of aesthetic appreciation and are summed up by this respondent: "beauty, even in mathematics, depends upon historical and cultural contexts, and therefore tends to elude numerical interpretation" (p.39). Indeed, the history of mathematics shows how new mathematical ideas often inspire negative aesthetic responses before they become accepted within the community, and even judged in positive aesthetic terms. For example, Charles Hermite, recoils with "dread" and "horror" from non-differentiable but continuous functions, writing: "*Je me détourne avec effroi et horreur de cette plaie lamentable des fonctions continues qui n'ont pas de dérivés*" (I turn in dread and horror from this lamentable sore of continuous functions that do not have derivatives) [3, p.318]. Similar trajectories are well-known in response to, for example, the work of Cantor and to the development of non-Euclidean geometry. Initial negative aesthetic judgments ("ugly" or "bad") reveal themselves in the very mathematical words themselves, such as *irrational* numbers, *complex* numbers, the *monster group*, or *annihilators*.

Acknowledging the fact that different mathematicians will find different results beautiful or elegant, depending on historical, cultural and personal contexts does not address the question of whether aesthetic judgments play a role in the development of mathematical knowledge. But this more functional view of the aesthetic was developed very strongly by the French mathematician Henri Poincaré [39], who examined the process of mathematical discovery, including its intuitive aspects. Poincaré drew particular attention to the aesthetic responses that allow a mathematician subconsciously to distinguish beautiful ideas that will lead to productive results from ugly ones that will lead nowheresuggesting that in some sense beautiful ideas are precisely those that lead to productive results. Poincaré tried to show that mathematical invention depends upon the often subconscious choice and selection of "beautiful" combinations of ideas, those best able to "charm this special sensibility that all mathematicians know" (p.2048).

In his book titled *The Psychology of Invention in the Mathematical Field*, Jacques Hadamard [15] proposes the first expansion of Poincaré's aesthetic heuristic theory, additionally claiming that aesthetic sensibilities often guide a mathematician's general choices about which line of investigation to pursue. He writes specifically about the "sense of beauty" (p.130) informing the mathematician that "such a direction of investigation is worth following; we feel that the question *in itself* deserves interest" (p.127; *italics in original*). Hadamard also adds to Poincaré's ideas on the role of the mathematical preconscious in mathematical thinking, locating the period in which it is most operative – the *incubation* period – within a larger theory of mathematical inquiry. In contrast to mathematicians such as Hardy, who were focused on the question of aesthetic merit, both Poincaré and Hadamard were more interested in the pragmatic concerns of how aesthetic responses shape and propel mathematical investigation.

Following in the tradition of Poincaré and Hadamard, Morris Kline [25] points out that aesthetic concerns not only guide the direction of an investigation, but motivate the search for new proofs of theorems already correctly established but lacking in aesthetic appeal – by means of their ability to "woo and charm the intellect" (p.470) of the mathematician. Kline takes this aesthetic motivation as a definitive sign of the artistic nature of mathematics. Similarly, in his attempt to define mathematics as the "classification and study of all possible patterns" (p.12), Warwick Sawyer [45] implies that the heuristic value of mathematical beauty stems from mathematicians sensitivity to pattern and originates in their belief that "*where there is pattern there is significance*" (p.36; *italics in original*). Sawyer goes on to explain the heuristic value of this trust in pattern:

> If in a mathematical work of any kind we find that a certain striking pattern recurs, it is always suggested that we should investigate *why* it occurs. It is bound to have some meaning, which we can grasp as an idea rather than as a collection of symbols. (p.36; *italics in original*)

Sawyer might well have explained Poincaré's special aesthetic sensibility as a sensibility toward pattern, viewed broadly as any regularity that can be recognised by the mind. For him, the mathematician is not only able to recognise regularities and symmetries, but is also attuned to look for and respond to them with further investigation.

Several decades later, the philosopher Harold Osborne [35] observes: "the reliance on the heuristic value of mathematical beauty in scientific theory has become something of a commonplace" (p.291). This indicates the extent to which scientists had placed their trust in Poincaré's notion of the mathematical aesthetic sensibility as a kind of muse who, if listened to carefully, would both guide and inspire creativity. In fact, scientists have been much more prolific than mathematicians in cataloguing and inspecting the effect of this trust on the development of scientific theories (see, for example, [8, 11, 14, 31, 56]).

Drawing on pragmatic concern for the role of the aesthetic in mathematics and on interview with contemporary mathematicians, I (see [46, 47]) have identified three distinct roles the aesthetic plays in mathematical inquiry – where I use the term "aesthetic" to designate judgements of value. The most recognized and public of the three roles of the aesthetic is the *evaluative*; it concerns the aesthetic nature of mathematical entities and is involved in judgments about the beauty, elegance, and significance of entities such as proofs and theorems. The *generative* role of the aesthetic is a guiding one and involves nonpropositional modes of reasoning used in the process of inquiry and that is often expressed through affective responses to felt patterns. Lastly, the *motivational* role refers to the aesthetic responses that attract mathematicians to (or repel them from) certain problems or certain fields of mathematics.[2]

While some may acknowledge the validity of these three roles, they may still take issue with Wolfgang Krull's claim that aesthetic concerns may even surpass epistemic ones for mathematicians:

> Mathematicians are not concerned merely with finding and proving theorems, they also want to arrange and assemble the theorems so that they appear not only correct but evident and compelling. Such a goal, I feel, is aesthetic rather than epistemological. [26, p.49]

Krull's statement invites some consideration of the difference between the aesthetic and the epistemic. Scholars such as Dewey [12] have called into question the strict distinction between the two domains; indeed, in his *Art as Experience*, where he tries to carve out a place for the aesthetic as a theme in human experience (and not just as an esoteric practice of high art), Dewey seeks to integrate the aesthetic with the cognitive and the affective as insep-

[2]More recently, Jullien [23] has proposed a more semiotic approach to the issue of the mathematical aesthetic, drawing on Nelson Goodman's work. Instead of focusing on the aesthetic merit of mathematical products, or on the aesthetic dimension of mathematical discovery, Jullien shifts attention to the question of aesthetic functioning. Parallel to the change in questioning involved in the arts – from "what is art?" to "when is it art?" – Jullien asks when mathematics functions aesthetically, drawing together both cognitive and emotional components.

arable aspects of experience. In a somewhat similar vein, Michael Polanyi's [40] groundbreaking "post-critical" examination of the non-explicit dimension of scientific knowledge, argues that mathematical knowledge cannot be divorced from personal and communal values and commitments. Other scholars such as Robert Root-Bernstein [41] and Judith Wechsler [56] have also emphasized the cognitive dimension of the aesthetic by framing it as a particular way of knowing that is both used and useful in the sciences as well as the arts.

Other scholars wish to retain a sharper distinction between the aesthetic and the epistemic. For example, Todd [53] retains a strict distinction between the two and argues that all purported aesthetic judgments in mathematics are, in fact, "masked epistemic assessments" (p.61). Todd thereby questions the actual role of the aesthetic in mathematics, claiming that nothing is ever accepted as truth on purely aesthetic grounds. Evidently, the question of whether or not the aesthetic plays an important role in mathematics depends on how one defines and distinguishes the epistemic and aesthetic. However, one problem with Todd's argument involves his strict focus on the status of mathematical results as objective entities void of cultural and historical dimensions. In other words, what counts as "true" and the ways in which such truths are communicated in the discipline is contingent on the norms and values of the discipline, which cannot be developed through cognitive or epistemic means alone.

2. Aesthetic values in the mathematics discourse

Instead of focusing on the distinction between the cognitive and the aesthetic as ways of knowing, Pimm [37] proposes a framing of the aesthetic dimension of mathematics in a way that resembles the aesthetic dimension of art. For Pimm, aesthetic considerations "concern what to attend to (the problems, elements, objects), how to attend to them (the means, principles, techniques, methods) and why they are worth attending to (in pursuit of the beautiful, the good, the right, the useful, the ideal, the perfect or, simply, the true)" (p.160). The reader will likely notice strong parallels with White's conception of the humanistic dimension of mathematics. Pimm's framing offers an avenue of inquiry that shifts attention from the modes of mathematical inquiry of individual mathematicians, as in Sinclair's tripartite model, to the practices of the community as a whole, including how truths are named, manipulated and negotiated.

One potential drawback of Pimm's framing is that it becomes overly broad. However, this can be rectified by parametrizing its scope to what the historian of mathematics Leo Corry [9, 10] calls the "images" of mathematics. In contrast to the "body" of mathematics, which includes "questions directly related to the subject matter of any given mathematical discipline: theorems, proofs, techniques, open problem," the images of mathematics "refer to, and help elucidating, questions arising from the body of knowledge but which in general are not part of, and cannot be settled within, the body of knowledge itself" (p.135). Thus, while the body of mathematics might concern itself with describing a technique used in the course of a proof, the images of mathematics refer to the motivations, choices and values related to the use of certain techniques. While the body of mathematics concerns itself with defining objects, the image of mathematics questions which objects are defined and which are not. Corry contends that mathematicians do not customarily write about their images. However, I would argue that historians and philosophers of mathematics do attend to the images of mathematics; the perennial philosophical question of whether mathematical objects exist cannot be settled within the body of mathematics, but certainly refers to the body and may help elucidate aspects of mathematical practice.

While some historical and philosophical inquiries do not have an axiological focus, I see the images of mathematics as being a layer of mathematical knowledge that very frequently involve aesthetic considerations. So, while a historical inquiry into the origins of the calculus may have little to do with aesthetics, it can provide insight into the values and preferences that were operative in the 17th century mathematics that led to Leibniz's algebraic approach winning over Newton's geometric one. I will be interested in examining the images of mathematics as they relate to aesthetic considerations, which I will explore in terms of Pimm's three categories. I take the humanist view that mathematics changes in time and place, and that discussions of the images of mathematics will necessarily shift. Instead of looking at the discussions in a chronological way, I will be categorizing them according to Pimm's framing. My aim is not to provide a portrait of the images of mathematics today, but rather, to show how discussions of the images of mathematics (from different philosophical perspectives, historical periods, and personal experiences) involve aesthetic considerations. In so doing, I want to broaden the notion of aesthetics beyond simple assertions about the prettiness, beauty or elegance or particular artifacts that individual mathematicians may (or may not) agree with.

2.1. WHY do mathematicians attend?

I'd like to begin with the question of why mathematicians prove theorems that have already been proven: in other words, *why* are existing truths worth attending to? The question relates to Krull's assertion, in the quotation at the end of Section 1, that mathematicians are not concerned only with finding, having or creating knowledge, but also with the way in which that knowledge may be made more or less evident and compelling.

A century prior to Krull, the mathematician C.F. Gauss also discussed the practice of seeking new proofs of existing results in the context of having produced six different proofs of the law of quadratic reciprocity:

> As soon as a new result is discovered by induction, one must consider as the first requirement the finding of a proof by any possible means. But after such good fortune, one must not in higher arithmetic consider the investigation closed or view the search for other proofs as a superfluous luxury. For sometimes one does not at first come upon the most beautiful and simplest proof, and then it is just the insight into the wonderful concatenation of truth in higher arithmetic that is the chief attraction for study and often leads to the discovery of new truths. For these reasons the finding of new proofs for known truths is often at least as important as the discovery itself. (Gauss, 1863, pp.159–160; in [30, p.299])

Gauss seems to be defending the motivations of his own mathematical pursuit by insisting that they are not a "superfluous luxury." His defense involves championing the worthiness of "most beautiful and simplest" proofs. The aesthetic dimension here is easy to see since the "why" question is in pursuit of the beautiful. Gauss then goes on to describe the "chief attraction for study" as being the "insight into the wonderful concatenation of truth." For Gauss, it is the relation between truths that propels the study of higher arithmetic.

The contemporary mathematician Gian-Carlo Rota [42] offers a somewhat different reason for finding new proofs, namely, that when a new theorem is discovered, its proof is usually overly complicated. Rota believes that simpler proofs of a theorem, which may take decades or even centuries to find, gradually bring out the significance of a new discovery. For Rota, these are the "definitive" proofs of theorems. Rota's answer to the "why" question is that mathematicians find new proofs in order to find the "definitive" proof that achieves understanding. As with Gauss, Rota acknowledges that new proofs are not sought for the sake of "truth," which the first proof of a theorem would establish.

The philosopher of mathematics Carlo Cellucci [7] criticizes Rota's view of the definitive proof on several counts: first, sometimes new proofs do not lead to improved understanding of the old proofs since they are based on different ideas; second, new proofs are still created even when a theorem has been well understood. Indeed, the question of when a proof is well understood begs the question of well understood for whom. Instead, Cellucci argues that the main reason for new proofs is their heuristic value, which "consists in the discovery of hypotheses which not only allow one to prove the result, but also establish connections between different areas of mathematics, where such connections may lead to new discoveries."

As a phenomenologist, Rota may not have found Cellucci's criticism compelling: after all, Rota is interested in his own experience of doing mathematics and, thus, his own reasons for seeing the value in new proofs. But we may distinguish between the personal experience of mathematicians (finding the definitive proof, or seeing the concatenation of truths) and the sanctioned practice of the mathematics discipline. Personal pleasures may well motivate the work of the individual mathematician, but why would a new proof of a known result be published in a journal?

Consider for example Apostol's [2] geometric proof of the irrationality of the square root of two, which was published in the Notes section of the *American Mathematical Monthly*, and is presented as a "remarkably simple proof [...] that is a variation of the classical Greek geometric proof" (p.841). The proof is certainly not definitive, and thus fails Rota's criterion for seeking new proofs. The mathematician Jonathan Borwein [5] says that the proof is "lovely because it offers new insight into a result that was first proven over two thousand years ago. It is also verges on being a 'proof without words'" (p.44). Borwein thus offers "new insight" as the criterion for the proof's aesthetic appeal, in addition to the proof's ability to stand on its own, as it were, without too many words. In terms of the former, the "new insight" doesn't seem to overlap exactly with Cellucci's sense of heuristic value, in part because the tools for assembling this proof have been available for over two millennia (which, for me, lends the proof a surprising quality). I find the latter part of Borwein's statement similar to Gauss's idea of the concatenation of truths in the sense that a geometric argument (and diagram) is being used to make an arithmetic claim, which can be seen as a semiotic concatenation.

While Borwein offers criteria for why the new proof is lovely, the only official comment on it, which acts as a de facto explanation for why it is being published, is that it is "remarkably simple." Of course, there is a wide gulf between this stark assessment of the proof and the kinds of emotional experiences that can be glimpsed in the writings of mathematicians such as Le Lionnais [28] and Poincaré. Indeed, Rota [42] draws attention to the way in which aesthetic descriptors used by mathematicians, which are often interpreted to be judgments of "good" mathematics, may in fact represent veiled ways of communicating the emotional dimension of their mathematical experiences:

> Mathematical beauty is the expression mathematicians have invented in order to obliquely admit the phenomenon of enlightenment while avoiding acknowledgement of the fuzziness of this phenomenon. [...] This copout is one step in a cherished activity of mathematicians, that of building a perfect world immune to the messiness of the ordinary world, a world where what we think *should* be true turns out to *be* true, a world that is free from the disappointments, the ambiguities, the failures of that other world in which we live. (pp.132–133; *italics in original*)

Rota draws attention to the particular types of pleasures motivating the mathematician – and they are not just that of solving a very difficult problem. The pleasures are connected to the way in which these solutions are "perfect," "immune to the messiness of the ordinary world," free of ambiguity and disappointment.

Rota seems to be pointing to the way in which the body of mathematics admits discussions of "good mathematics" as long as they remain objective and that in order to elucidate the question of what makes good mathematics (or what makes it worth publishing a new proof of an existing result) one must consider the emotional dimension of mathematical experience. In this sense, Apostol (or the editors of the *American Mathematical Monthly*) are counting on the fact that readers of the new proof will also experience the phenomenon of enlightenment (which Gauss also alluded to). This goal seems not very different from the goal of the artist, which is to offer for the viewer a transformative experience. The difference is that the artist, in general, is willing to admit it, and thus does not need the kind of "copout" to which Rota refers. Even Cellucci can be seen as offering a copout, staying close as he does to the rather clinical term "heuristic value," which manages to skirt the emotional.

In this section, I have focused on a very specific practice in mathematics, that of finding new proofs for existing results. I chose it because the goals for such a practice are clearly not purely epistemological (in the sense that the truth of the theorem has already been established). This practice can be analysed both in terms of the variety of aesthetic responses that individual mathematical have had (which do not always agree) and in terms of the function of such a practice in the community. While the function may very well be, as Cellucci argues, primarily intended to add heuristic value, Rota's "copout" suggests that it is not so clearly utilitarian.

The question of why mathematicians re-prove existing results could be broadened to that of why mathematicians prove at all. Rota's comments would apply just as well. However, while Rota focuses on one's own understanding, the contemporary mathematician William Thurston [52] draws attention to the value of socially-shared understanding:

> We are not trying to meet some abstract production quota of definitions, theorems and proofs. The measure of our success is whether what we do enables people to understand and think more clearly and effectively about mathematics. (p.3)

Using the proof of the Four Colour Theorem and its ensuing controversy as an example, Thurston insists that both the veracity of the theorem and the correctness of the proof are secondary to the "continuing desire for *human understanding* of a proof" (p.2). In his paper, Thurston goes on to argue that mathematicians should attend more to "the *communication* of insights (p.8, *emphasis in original*), and he proposes that this can in part be accomplished by a younger generation of mathematicians who "reinject diverse modes of thought in mathematics" and to "invent names and hit on unifying definitions that replace technical circumlocutions and give good handles for insights" (p.7). And here Thurston shifts consideration from the question of *why* to the question of *how* to attend, which is the subject of the next section. Before proceeding though, it is worth considering whether Thurston's answer implicates the aesthetic at all. Todd might argue that it rests squarely in the epistemic, since it is about knowledge. However, given the affective tone of Thurston's message and the focus on "insight," which involves non-propositional forms of understanding, I do see an aesthetic dimension to his response related to the quality of the understanding itself.

2.2. HOW do mathematicians attend?

There are many ways of addressing the question of *how* mathematicians attend. One, which focuses on the methodological, would consider the process by which mathematicians devise and prove theorems. Lakatos' work, in his *Proofs and Refutations* [27], is an investigation of how mathematicians attend describing as it does the methods (such as monster-barring and concept-stretching) by which theorems, proofs and definitions evolve over time to become sanctioned mathematical products. Although Lakatos has been criticised for the quasi-empirical philosophy he advances in the book, his description and identification of methods stand as rather robust and, indeed, given the rarity of his project as a whole, they have yet to be seriously challenged. More recently, Borwein and Bailey [6] extend Lakatos' project to include the experimental dimension of mathematics made so important by the advent of digital technologies. In their *Mathematics by experiment: plausible reasoning in the 21st century*, the computer is central to an experimental methodology as a way "to generate understanding and insight; to generate and confirm or confront conjectures; and generally to make the mathematics more tangible, lively and fun for both the professional researcher and the novice" (p.vii).

The growing use of the computer in mathematics offers an unusually public and persuasive opportunity to examine the nature of the methods and means used to conduct mathematics research. In some senses the experimental methods that are characteristic of computer use in mathematics may have long been a part of mathematical practice, as noted by Borwein and Bailey [6], who point to mathematicians such as Euler and Gauss as examples. But the use of computers to prove the Four Colour conjecture is somewhat different, and has generated an enormous amount of discussion around what counts as a proof in mathematics and, more importantly, what methods can be sanctioned in the community for arriving at those proofs. Over the past decades, in addition to computer-generated proofs, computer-aided reasoning and computer-driven theorem-proving, the computer is being used to find new proofs for existing theorems (thus occupying the sacred ground of creativity and aesthetic satisfaction described in the Gauss quotation above) as well as generating proofs or disproofs of open questions. These latter developments challenge the traditional symbolic method of deduction that has long dominated the mathematics community. It is rather evident that such changes in method will result in changes to the discipline itself (including what problems can be posed and solved) but, drawing on a materially-framed historical account of mathematics, Brian Rotman [44] argues that in changing what the mathematician can do with the dynamic imagery and haptic interaction of digital technologies – which is very different from what they can do on paper with symbols – the computer will change the nature and constitution of the mathematician. The computer can be thought of as insisting that the mathematician attend to the visual and the kinesthetic presentations and re-presentations of mathematical behaviour.

The reader is undoubtedly aware of these computer-related developments in mathematics, but my point here is to argue that the question of which methods a particular mathematician might use or whether such methods are sanctioned in the community is indeed an aesthetic one. When confined to the individual level, it may be brushed off as a personal preference or style, but, in cases such as computational methods, the shift might be compared to the move from painting to photography, where the question of how to capture reality challenges deep-seated values that implicate ontological and epistemological assumptions as well. In the case of mathematics, the issue of machinic presence continues to challenge the community, as evidenced by the involvement of the computer in the proof of the Four Colour Theorem more than two decades ago and the more recent Kepler sphere-packing conjecture.

In terms of the former, the original Appel and Haken computer-based proof in 1976, which generated much discussion (see, for example, [21, 22]), was later improved and revised by Robertson, Sanders, Seymour and Thomas in 1996. Although their method required checking about one-third of the maps that the Appel-Haken proof needed to be checked, making it more efficient, it also involved a more efficient algorithm. The computer is left to undertake the work that could, in theory, be done by hand, but that is much more practical to do by machine. The level of trust that mathematicians have in the result is evidenced by its foundational role in graph theory, where many results depend on the Four Colour conjecture being true. In terms of method then, the evolution of the proof shows a tendency toward shifting the trivial, repetitive procedures to the computer and the more conceptual, intuitive work to the mathematician.

This can be seen clearly when comparing the proof of the Four Colour conjecture with that of the Kepler sphere-packing conjecture, recently communicated by Hales. In this proof, the computer does not simply get put to work on a discrete and finite number of computations; instead it must take on much more complex operations involving continuously changing variables in high dimensional non-convex spaces. In a very unusual editorial comment, the *Annals of Mathematics* stated that there were portions of the proof that its reviewers would not be able to check, despite extensive work and effort over four years, and that the reviewers were only 99% certain that it is correct ([20, 50]). Part of the challenge around verification relates to issues around the allocation of computer memory (not knowing a priori how much to allocate) and to numerical precision – the decisions and interpretations required by the mathematician increased the entanglement between human and computer logic. In contrast with the Four Colour Theorem proof, which could technically be done by hand, this proof could not. Hales has launched the Flyspeck project, which aims to create a computer-based automatic verification of every step of the proof, essentially reducing, once again, the role of the computer to trivial, repetitive work that could be checked by hand. In other words, in Pickering's [36] terms, mathematicians do not wish to accord any agency to the computer (or to any material objects). According to Rotman's perspective, such a position ignores the fact that "mathematics has been engaged in a two-way co-evolutionary traffic with machines since its inception" (p.58), citing the way in which machines such as the wheel-and-axle have given rise to abstract mathematical concepts such as cyclicity and angular motion. If mathematicians have been able to pretend that these machines have always been mere disposable tools, Rotman argues that the digital computer will make such claims increasingly unrealistic and will force the mathematician to recognize the agency of the computer and its effect on the very way she thinks. With Poincaré's insistence on the ego, on "le moi" of mathematical invention, I suspect that the sharing of agency will be a deeply divisive process.

The publication of the sphere-packing proof in the *Annals* did not contain any of the computer code or printout that were used by Hales to achieve his results; these were published elsewhere in a computer science journal. The way in which the work was published provides insight into the question of *how* mathematical results are communicated and, more particularly, *how* mathematical texts are written. Not surprisingly, even before the sphere-packing proof, the use of the computer in mathematics has occasioned breaches in the conventional style of mathematical writing. Consider these words of advice proffered by Ewing's [13] editorial for *The Mathematical Intelligencer*, which first established the aesthetic considerations involved in the reporting of mathematical results: "Like scientists, we should continue to perform experiments, by any means available. But like artists, sculptors, and composers, we must exercise judgment about what should be placed on public display" (p.4).

Ewing is responding to the "growing tendency to breach customary etiquette" in mathematical reports (both written and in speech) in which computations done by the computer are described. Reporting on techniques used in an experiment is essential for a scientist but not, asserts Ewing, for a mathematician, who must instead wade through false starts, mistakes and revisions toward a "polished product, the correct statement with a clear proof." To elucidate his preferred style, Ewing cites von Neumann "if the deductions are lengthy or complicated, there should be some simple, general principle involved, which "explains" the complications and detours, reduces the apparent arbitrariness to a few simple guiding motivations" [54, p.196]. Ewing elaborates on his aversion to the profusion of computer-generated calculations: "if such work had been done "by hand," no one would have dared to discuss it in public." He is not alone; in *A Manual for Authors of Mathematical Papers* [1] even more direct advice is given: "Omit any computation which is routine (i.e., does not depend on unexpected tricks). Merely indicate the starting point, describe the procedure, and state the outcome." (p.2)

In addition to the theme of computation, Ewing's sense of mathematical style opts for the clean and polished. In *How to Write Mathematics* [16], Halmos adds to cleanliness a desire for efficiency and cumulativeness.

> The discoverer of an idea, who may of course be the same as its expositor, stumbled on it helter-skelter, inefficiently, almost at random. If there were no way to trim, to consolidate, and to rearrange the discovery, every student would have to recapitulate it, there would be no advantage to be gained from standing on the shoulders of giants. (p.23)

Omitting calculations, detours, and false starts in order to gain clarity, simplicity and efficiency, which will enable the discipline to move forward: this is the modern mathematical style. In the *A Manual for Authors of Mathematical Papers* [1], we find further elaboration of how this can be achieved:

> It is good research practice to analyze an argument by breaking it into a succession of lemmas, each stated with maximum generality. It is usually bad practice to try to publish such an analysis, since it is likely to be long and un-interesting. The reader wants to see the path – not examine it with a microscope. (p.2)

Once again, the reader is unlikely to find these stylistic dictates surprising. But just because they seem natural and expected, it does not mean that they are somehow immanent or that they are a necessarily by-product of mathematical practice. Indeed, one can easily appreciate the contingency of the contemporary style by comparing it with historical writings of mathematicians as diverse as Cavalieri, Descartes, Gauss and Hamilton. To make the point convincingly though, I'd like to draw on the recent work of Reviel Netz [34], who offers a detailed examination of the mathematical writing style of a group of Ancient Greek mathematicians. In his book *Ludic Proof: Greek Mathematics and the Alexandrian Aesthetic*, Netz argues that written mathematics in the time of Archimedes (from about 250 to 150 BC) had a distinct style that differed markedly from both that of other Ancient Greek periods (including that of Euclid) as well as that of contemporary writing.

Before proceeding, since I am going to compare the Alexandrian style with the contemporary one, it is important to point out that Archimedes communicated his mathematical through personal letters, and not through journal articles. One may argue that the mode of communication marks the essential difference between the two styles I want to compare. Nowadays, mathematicians are also permitted to communicate through letters (or emails) and their style of writing in these cases differs drastically from that of their more formal writing. However, I think it is still worth comparing the two styles in part because it may suggest ways in which the academic journal medium affects the style of communication and in part because in the Alexandrian letters actually communicate complete results through theorems and proofs that have a strong family resemblance to the current form of mathematical communication.

The elements that Netz proposes – and that are markedly different than the style proffered by Ewing – are as follows: narrative surprise, mosaic structure and generic experiment, and a certain "carnivalesque" atmosphere (Netz borrows this colourful adjective from Bakhtin (as cited in [34]), for whom carnivalesque describes a literary mode in which humour and chaos are used to subvert and liberate assumptions associated with a dominant style). These elements are manifest in Netz's reading of Archimedes' *Spiral Lines*, which is devoted to the proof that the area under the segment of the spiral equals one-third the area of the corresponding circular sector. One might note the surprising fact that the ratio between two curvilinear areas is an integer one, which might impel one to call the result beautiful. But Netz is less concerned with evaluating the result in terms of its aesthetic qualities than he is in analysing the particular style in which Archimedes relays the result. Some of these stylistic elements, as Netz points out, are circumscribed by the fact that Archimedes offers his results in the form of a letter, so that his writing has a very specified audience than do contemporary mathematicians.

Narrative surprise can be seen in the very introduction to the problem, which arises abruptly after Archimedes has already discussed two other problems in such a way that the reader expects these to be the central work undertaken in the letter. When he finally does talk about the spiral, he introduces it as “a special kind of problem, having nothing in common with those mentioned above” (p.3). Netz shows that the abrupt transition evident in the beginning of the letter is characteristic of the Archimedian style, and reoccurs through the letter, even as the proofs are given. The “special” character of the problem is also suggestive of the particularly exotic flavour of the writing. Archimedes then swiftly and clearly (unlike his elaboration of the previous propositions, which are dense and opaque) achieves his second goal in Proposition 18 to show that a certain straight line defined by a spiral is found to be equal to a circumference of a circle. Then, after introducing a new conceptual tool in Proposition 21 (without explaining its function), Archimedes reaches Proposition 24, where, as Netz writes “the treatise as a whole makes sense” and the enunciation of the result is given “in economic, crystal-clear terms – the first simple, non-mystifying enunciation we have had for a long while” (p.10). The meaning of the previous propositions finally comes to light.

Other aspects of the Archimedean style highlighted by Netz relate to the way in which proofs are sequenced in the text. For example, Archimedes does not elaborate the proofs according to the set of goals he establishes at the beginning, and he doesn’t even define the spiral until halfway through the treatise. Before getting there, we have a “surprising sequence going from physics through abstract, general geometrical observations, via the geometry of circles and tangents, and finally, leading on to a sui generis study of arithmo-geometry – none of these being relevant to any of the others” (p.9). Netz sees the extensive use of calculations and of physics (the spiral requires the motion of two lines) as a breaking of genre-boundaries and the ungoverned sequence of seemingly unrelated material as leading to a style of surprise and mosaic structure that contrasts greatly with the linear axiomatic presentation found in contemporary mathematics. In addition, in contemporary mathematics, efforts are made to signpost the general structure of the argument so that the reader knows how different tools – and, especially, different lemmas – will be used. This pedagogical style seems completely absent in the Archimedean treatise.

If Netz's discussion of style (in terms of mathematical writing) fits squarely into the category of *how* to attend, it also relates to other aesthetic considerations of the *why* and the *what*. For example, Netz believes that Archimedes intentionally chose an obscure and "jumpy" presentation so as to "inspire a reader with the shocking delight of discovery, in Proposition 24, how things fit together; so as to have them stumble, with a gasp, into the final, very rich results of Proposition 27" (p.14). The Archimedean writing style might thus be described less in terms of being in pursuit of the true or the good, as being designed to produce a highly satisfying emotional reaction – much in the same way we except a good poem or play to do. Further, Netz points to the way in which we can attend to this Archimedean treatise in terms of the what, namely, the novel and somewhat exotic focus on the spiral, which he is the first to study, and which involves boundary crossings not customary in Euclidean geometry, where time and motion are strictly forbidden (see [37]).

In these stylistic elements – and Netz provides an extensive number of examples of mathematical writing by Archimedes and his contemporaries – a mathematical style emerges that contrasts markedly with the contemporary one. I have already hinted at some of the differences, but it would be misleading to neglect one difference that Netz elaborates at length, namely, the way in which Archimedes' mathematical writing style was influenced by, and in turn influenced, the Hellenistic literary style in poetry. It would be difficult to make a similar kind of argument today (unless one wants to consider the works of groups such as Oulipo), but in articulating the central tensions of both literary and mathematical cultures, Netz provides insight in the way in which a common style might have been possible in the past – and, indeed, might be possible in the future.

> Literature brings together diversity and unity: the tension between diversity and unity is thus a constant of literary history. Mathematics brings together certainty and surprise: the tension between certainty and surprise is thus a constant of mathematical history. Some literary cultures emphasize diversity, others emphasize unity. Some mathematical cultures emphasize certainty, other emphasize surprise. (p.235).

I have drawn on Netz's work substantially, because it stands as a unique example of a historical study of aesthetic considerations in mathematical writing. Further, it contrasts significantly with the style evident in contemporary mathematics. If the modern mathematician wants a clear path organized in bite-sized lemmas, Archimedes and his contemporaries maximize detours, withhold information about direction and purpose, and refuse to bury (or elucidate) numerical results. While Netz calls the modern style pedagogical, presumably in its intent to inform and clarify, one could also argue that, in hiding and omitting, it does little to develop mathematical understanding. Indeed, Henderson and Taimina [19] explain how contemporary published papers may do little to satisfy questions such as "Why is it true?", "Where did it come from?", "How did you see it?" These questions indicate a search for perspicuity and meaning. Thurston reports a similar experience in which his logically correct, tasteful proofs established results but blocked meaning amongst his colleagues. The "ludic" style does not seem geared toward this type of intuition-bearing communication, either. However, it may in fact be more insightful in the sense of being less deceptive. If the modern mathematician must deceive to produce clean theorems (pretending the results were clearly, sequentially attained – perhaps even always true), the "ludic" mathematician at least refuses to erase himself or his audience: surprised must be designed with a reader in mind.

2.3. WHAT do mathematicians attend to?

As evidenced by the work of Andrew Wiles on Fermat's Last Theorem, problems (and especially ones that have stood unsolved for so long) form a central locus of attention in mathematics. Historically, the construction problems related to the Delian oracle can be seen as shaping to a large extent the word of geometry. Netz [33] provides an astute and revealing account of the central and generative role played in the history of mathematics by this problem of Archimedes: how do you cut a sphere, with a plane going through one of the latitude lines, so that the volume of the bigger part has a certain given ratio to the volume of the smaller part. Originally a geometric problem, solved in various ways over the course of the centuries through to Late Antiquity, the problem is turned into an algebraic equation by Omar Khayyam in the Middle Ages. Netz thus traces out a transitioning mathematical practice from a focus on problems toward a focus on systematic approaches, which, in modern mathematics, becomes a practice focused on theories and theorems.

The practice of mathematics based on problems persists though: one could argue that during the last four hundred years of the previous millennium, the problems that focused mathematical work were ones put to mathematics from the natural sciences. For example, the problem of describing the motion of a string fixed at one end and swinging at the other, which emerged from the study of mechanical systems, gave rise to d'Alembert's work in differential equations and eigentheory (see [18]). With Hilbert's famous twenty-three "problems," offered in 1900, which has a strong influence on the subsequent development of mathematics, one finds a mixture of open problems in the style of Classical Greek mathematics (such as the 18th: Kepler Conjecture and the 8th: Riemann hypothesis) while others focus more on a theory-driven approach (such as the 6th: Axiomatize all of physics or the 15th: Provide a rigorous foundation of Schubert's enumerative calculus). And much more recently, the creation of the Clay Institute's Millennium Prize problems in 2000 shows how open problems in mathematics are an organizing principle for the growth of the discipline. Why such focus on open problems?

The proof of Fermat's Last Theorem provides some insight into the centrality of problems in mathematics. Commentators on Wiles' tremendous achievement suggest that its importance relates not only to the fact that the theorem had remained unproved for so long, despite its misleading simplicity, but to the extensive, and unanticipated and deep connections it forged between different areas of mathematics. Mathematicians have talked about liking problems that are simply stated but surprisingly deep or complex, using Goldbach's conjecture as a paradigmatic example (see [47]). And this is certainly true of Fermat's Last Theorem, though the full weight of complexities would not become apparent until Wiles had toiled for seven years inventing new techniques and bringing together previously distinct areas of mathematics.

That problems are attended to in mathematics, that they take such a central role in shaping the discipline, has an aesthetic dimension. But this leads to the question of why certain problems are worth attending to. Here we come to what Tymoczko [55] sees as the crux of the aesthetic function in mathematics: of the infinitely many true propositions one could state in mathematics, only a very few attract attention. Unlike physics, there is no reality against which mathematical truths must be measured: the choice is *wide* open. Indeed, this freedom of choice incites many mathematicians to draw parallels between the arts and mathematics as disciplines (see [49]).

But it may be more important to note that just as in the arts, where certain themes draw repeated attention (the nude figure, the vase of flowers, the meaning of love, death, sorrow, the multiplicity of perspective), mathematicians operate in a historically mediated community where certain problems are worth coming back to. Further, as in the proof of Fermat's Last Theorem, open problems in mathematics have a tendency to be both generative and unifying; they lead to new techniques and ideas while also developing connections between previously distinct fields.

I will not attempt to be comprehensive in considering the *what* question, but I would like to consider a rather different, and perhaps more subtle example that Rotman [44] alludes to in his discussion of the effect of digital technologies in mathematical practice. In particular, he examines the well-known P=NP question, which concerns the extent to which a problem (like the Travelling Salesman Problem) can be solved in a feasible number of computational steps, and which can be framed as follows: if one can *verify* a possible solution to a given problem in polynomial time, can one also *find* a solution to that problem in polynomial time? The use of polynomial time as the measure of feasibility of finding a solution is an arbitrary one (the question could be posed in terms of any measure of time), but one that is consistent with the mathematician's attention to the theoretical as well as the infinitistic. Rotman points to a different approach to the P=NP question that attends much more to the pragmatic concerns of the computer scientist (who is interested in the actual run-time of a decision process). In contrast with the mathematician's asymptotic definition of polynomial functions being "smaller than" exponential ones, Rotman argues for an alternative framing of run-time that focuses attention on the finite world of physical reality which is necessarily bounded.

The mathematical approach to the question of feasibility thus belies a deep commitment to infinity as an object of attention (and, of course, following on Gattegno's frequent observation that "mathematics is shot through with infinity," one could say the same thing about most problems and questions in mathematics, both contemporary and historical). But Rotman also argues that a different mathematical framework for thinking about feasibility would "entail overcoming a large and difficult obstacle: how to effect a conceptual escape from the great attraction of the classical integers" (p.75). The fact that the integers are treated as idealized objects whose existence are "given" and "true" may seem obvious and inescapable, but one can appreciate the contingency of these objects of (great) attention by questioning the extent to which such numbers really are "natural" and whether there may be alternative understandings of counting (as does [43]). My goal here is not so much to settle the best approach to the P=NP question but, rather, to use Rotman's problematising of the mathematical approach to point to the way that constructs such as infinity and natural numbers are not the given elements of mathematics but, rather, the ones chosen (at this time) for further processing and inferencing.

One last comment on mathematical objects is worth making: a large amount of mathematical activity is devoted to creating and proving the existence of objects. In fact, the question of *how* to attend in mathematics involves incessant transformations of processes into objects (counting turns into a number, dividing two numbers turns into a ratio, approaching a value turns into a limit, mapping inputs to outputs turns into a function). That the object becomes the ultimate focus of attention is worth signaling, especially since an object is almost always the detemporalized, demobilized, and even dehumanized (in the sense that processes must be carried out by someone, but objects can carry on idealized existences) culmination of a process. Where Rotman emphasizes the prevalence of the infinite in mathematics, one can also point to the prevalence of the static in mathematicsto the tendency of attending to objects one can hold still.[3] But as Pimm [37] points out, since mathematical objects have no physical reality, intermediary symbols (whether written or drawn) become the objects of mathematical practice:

> In the absence of a "true" object, like a cuckoo's egg hatching in a nest, they subvert, supplant and replace, becoming instead the object of attention. In each instance, consequently, both the algebraic letter and the geometric diagram then revert to being icons in the traditional, religious sense: that is, to recall Graham-Dixon's words, "signposts to the next world, placed in this one". (p.181).

[3] Tahta [51] develops a more psychodynamic perspective on the relation between mathematics and its objects in terms of psychological need humans have to have, hold, keep and fetishise objects. He cites the mathematician Philip Maher [29] who observes: "If we accept the view that one's mathematical reality is an instantiation of one's potential space that occurs when one is doing mathematics then the objects in this psychological space – the mathematical objects one plays with [...] – function as transitional objects. From this perspective there is little psychological difference between, say, a teddy bear and a self-adjoint operator [...]" (p.137).

Pimm considers the roles of both the algebraic letter and the geometric diagram in the practice of mathematics, particularly in terms of their changing status as objects of attention. The problematic status of the diagram in the Bourbaki tradition, compared with its central cognitive role for the Ancient Greeks (for whom the diagram – and not the text – was the mathematical object, see [32]), exemplifies well the contingent nature of the mathematical object, and its relation to historically-sensitive values and preferences.

In this section, I have been trying to draw attention to the objects of mathematical attention. In arguing that these objects involve choices, preferences and values – and that they change over time and context – I have pointed to the specific aesthetic considerations they entail. These kinds of aesthetic considerations are prevalent and public in the world of art criticism, which make these statements – what to attend to, how to attend and why attend – function as real questions in this domain, questions of empirical nature and interpretative responses. In the next section I would like to consider briefly the question of why such statements may strike many as obvious or epiphenomenal parts of the mathematical discourse.

3. Aesthetics, criticism and audience

In an essay in White's volume [58], Tymoczko [55] explores the "possibility of regarding some of [mathematics'] products as objects of aesthetic enjoyment" (p.67). Following the lead of Borel [4], Tymoczko attempts to engage in aesthetic criticism about these products, trying to show that it is not just its relation to science that provides mathematics its source of value judgments. Tymoczko's interest in criticism is initially inspired by the conviction that mathematics is an art (see [48] on the development of this tradition), but is also driven by a perceived

> need to ground value judgments in mathematics [...] those positive judgments like important, elegance, relevance, promise) and those negative judgments (like in consequentiality, triviality, crudity, sterility) that are necessary to characterize a discipline and to shape its progress. (p.68)

He argues that one component of this need relates to an issue I have already mentioned, namely, the fact that selection is necessary in a discipline where not only are there infinitely many true results, but where thousands of new theorems are proved every year, and only a handful needing to be passed on. Mathematicians need to make this choice. A second component – on which he spends far less time – relates to the way in which aesthetic criticism might help shape mathematics by providing, like it does in the arts, constraints and directions for the disciplines development. Of course, if Tymoczko is right, then this sort of criticism must be taking place already (if only in the decisions made by referees and journal editors). A good art critic, he writes, draws our attention to certain features of the work, provides us with a way of seeing or hearing the work, so as to enhance our appreciation. When he uses "us" here, he refers to the audience of the work – but who might that audience be in mathematics? Is there an audience for mathematical criticism? If not, artist, audience and critic are doomed to a closed, incestuous set of people, rendering criticism superfluous (see also [38] for further discussion of the audience of mathematics). There is little distinction between the audience (those how do/make/write) and the critics (those who watch/ingest/read).

The criticism that Tymoczko advocates does happen, albeit in a covert and private form, as I mentioned before, in the context of academic publication and granting agencies. Reviewers and editors make the decisions about what will get published in books and in journals, and this influences the problems and fields that receive attention amongst mathematicians and their students, which shapes the courses given at the graduate level, and eventually also at the undergraduate level (see [59] for a discussion of how this very process led to a near disappearance of geometry in the curriculum of the latter part of the twentieth century). Such a system might be frustrating for young mathematicians, if they are not properly enculturated by mentors. But it can lead to important problems in the discipline itself, illustrated through the story of Thurston, who writes about the harm he did to a whole branch of mathematics – and to his colleagues – by moving so quickly towards results that eclipsed comprehension. Thurston was left to act as his own critic, first realizing that his work was inaccessible to everyone, and then engaging in the critical work of finding ways to help his colleagues understand and appreciate his techniques and results.

As stated in the introduction, White's vision of humanistic mathematics called for an understanding of the value judgments implied in the growth of any discipline. This vision, I believe, explicitly acknowledges the historical, cultural and material contingencies of mathematics, as well as the subjective and social dimensions of mathematical practice. My exploration of the aesthetic considerations in mathematics aimed to focus attention on the different kinds of value judgments – often only tacitly made or practiced – that animate mathematics practice in terms of what is investigated, how it is investigated, or why it is investigated. By pointing to specific examples of the way in which value judgments steer and frame the discipline, I have tried to move the notion of the mathematical aesthetic beyond appraisals of beauty or elegance, and around questions of whether aesthetic claims have epistemic value. The recent work of historians, philosophers and sociologists, whom I have quoted in this paper, contributes in important ways to illuminating and explaining the changing practices and values of the mathematical community. Increased critical work in this direction will provide even more insight into the present state of mathematics as well as suggest possibilities for a future mathematics that can be less covert in its positioning toward non-mathematicians.

References

[1] AMS (1990). *A manual for authors of mathematical papers*. American Mathematical Society, Providence, RI.

[2] Apostol, T. (2000). Irrationality of the square root of two – a geometric proof. *American Mathematical Monthly*, 107(9):241–242.

[3] Bailland, B. and Bourget, H., editors (1905). *Correspondance d'Hermite et de Stieltjer*. Gauthier-Villars, Paris.

[4] Borel, A. (1983). Mathematics: art and science. *The Mathematical Intelligencer*, 5(4):9–17.

[5] Borwein, J. (2006). Aesthetics for the working mathematician. In Sinclair, N., Pimm, D., and Higginson, W., editors, *Mathematics and the aesthetic: New approaches to an ancient affinity*, pages 21–40. Springer, New York.

[6] Borwein, J. and Bailey, D. (2003). *Mathematics by Experiment: Plausible Reasoning in the 21st Century*. A. K. Peters, Ltd., Natick, MA.

[7] Cellucci, C. (2009). Indiscrete variations on Gian-Carlo Rota's themes. In Damiani, E., D'Antona, O., Marra, V., and Palombi, F., editors, *From Combinatorics to Philosophy*, pages 211–227. Springer, Berlin.

[8] Chanrdasekhar, S. (1987). *Truth and Beauty: Aesthetics and Motivations in Science*. University of Chicago Press, Chicago, IL.

[9] Corry, L. (2001). Mathematical structures from Hilbert to Bourbaki: the evolution of an image of mathematics. In Bottazzini, U. and Dalmedico, A. D., editors, *Changing images of mathematics: from the French Revolution to the N—ew Millennium*, pages 167–185. Routledge, London, UK.

[10] Corry, L. (2006). Axiomatics, empiricism, and Anschauung in Hilberts conception of geometry: between arithmetic and general relativity. In J. Ferreirós, J. and Gray, J., editors, *The architecture of modern mathematics: essays in history and philosophy*, pages 167–185. Oxford University Press, Oxford, UK.

[11] Curtin, D., editor (1982). *The Aesthetic Dimension of Science*. Philosophical Library, New York, NY.

[12] Dewey, J., editor (1934). *Art as Experience*. Perigree, New York, NY.

[13] Ewing, J. (1984). A breach of etiquette. *The Mathematical Intelligencer*, 6(4):3–4.

[14] Farmelo, G., editor (2002). *It Must Be Beautiful: Great Equations of Modern Science*. Granta Books, London.

[15] Hadamard, J. (1945). *The mathematician's mind: The psychology of invention in the mathematical field*. Princeton University Press, Princeton, NJ.

[16] Halmos, P. (1973). How to write mathematics. In Steenrod, N. E., Halmos, P. R., Schiffer, M. M., and Dieudonne, J. A., editors, *How to write mathematics*, pages 19–48. American Mathematical Society, Providence, RI.

[17] Hardy, G. H. (1940). *A mathematician's apology*. Cambridge University Press, Cambridge, UK.

[18] Hawkins, T. (1975). Cauchy and the spectral theory of matrices. *Historia Mathematica*, 2:1–29.

[19] Henderson, D. and Taimina, D. (2006). Experiencing meanings in geometry. In Sinclair, N., Pimm, D., and Higginson, W., editors, *Mathematics and the aesthetic: New approaches to an ancient affinity*, pages 58–83. Springer, New York.

[20] Holden, C. (2003). Stacking up the evidence. *Science*, 299(5612):1512.

[21] Jaffe, A. and Quinn, F. (1993). Theoretical mathematics: toward a cultural synthesis of mathematics and theoretical physics. *Bulletin (new series) of the American Mathematical Society*, 29(1):1–13.

[22] Jaffe, A. and Quinn, F. (1994). Response to comments on theoretical mathematics. *Bulletin (new series) of the American Mathematical Society*, 30(2):208–211.

[23] Jullien, C. (2008). *Esthétique et Mathématiques: une exploration goodmanienne*. Presses Universitaires de Rennes, Rennes.

[24] King, J. (1992). *The art of mathematics*. Plenum Press, New York.

[25] Kline, M. (1953). *Mathematics in Western Culture*. George Allen and Unwin., London.

[26] Krull, W. (1930/1987). The aesthetic viewpoint in mathematics. *The Mathematical Intelligencer*, 9(1):48–52.

[27] Lakatos, I. (1976). *Proofs and refutations: the logic of mathematical discovery*. Cambridge University Press, Cambridge, UK.

[28] Lionnais, F. L. (1948/1986). La beauté en mathématiques. In Lionnais, F. L., editor, *Les grands courants de la pensée mathématique*, pages 437–465. Editions Rivages, ParisY.

[29] Maher, P. (1994). Potential space and mathematical reality. In Ernest, P., editor, *Constructing Mathematical Knowledge: Epistemology and Mathematical Education*, pages 134–140. Falmer Press, London.

[30] May, K. (1972). Gauss, Carl Friedrich. In Gillespie, C., editor, *Dictionary of Scientific Biography*, pages 298–315. Charles Scribner's Sons, New York, NY.

[31] McAllister, J. (1996). *Beauty and Revolution in Science*. Cornell University Press, Ithaca, NY.

[32] Netz, R. (1998). Greek mathematical diagrams: their use and their meaning. *For the Learning of Mathematics*, 18(3):33–39.

[33] Netz, R. (2004). *The transformation of mathematics in the early Mediterranean world: From problems to equations*. Cambridge University Press, New York.

[34] Netz, R. (2009). *Ludic Proof: Greek Mathematics and the Alexandrian Aesthetic*. Cambridge University Press, New York.

[35] Osborne, H. (1984). Mathematical beauty and physical science. *British Journal of Aesthetics*, 24(4):291–300.

[36] Pickering, A. (1995). *The Mangle of Practice: Time, Agency, and Science*. University of Chicago Press, Chicago, IL.

[37] Pimm, D. (2006). Drawing on the image in mathematics and art. In Sinclair, N., Pimm, D., and Higginson, W., editors, *Mathematics and the aesthetic: New approaches to an ancient affinity*, pages 160–189. Springer, New York.

[38] Pimm, D. and Sinclair, N. (2009). Audience, style and criticism. *For the Learning of Mathematics*, 29(2):23–27.

[39] Poincaré, H. (1908/1956). Mathematical creation. In Newman, J., editor, *The world of mathematics*, volume 4, pages 2041–2050. Simon and Schuster, New York.

[40] Polanyi, M. (1958). *Personal knowledge: Towards a post-critical philosophy*. Harper & Row, New York.

[41] Root-Bernstein, R. (2002). Aesthetic cognition. *International Studies in the Philosophy of Science*, 16:61–77.

[42] Rota, G.-C. (1997). *Indiscrete thoughts*. Birkhauser, Boston.

[43] Rotman, B. (1993). *Ad Infinitum ... : the Ghost in Turing's Machines*. Stanford University Press, Stanford, CA.

[44] Rotman, B. (2008). *Becoming Beside Ourselves: The Alphabet, Ghosts, and Distributed Human Beings*. Duke University Press, Durham.

[45] Sawyer, W. (1955). *Prelude to Mathematics*. Pelican: Penguin, Harmondsworth, Middlesex.

[46] Sinclair, N. (2004). The roles of the aesthetic in mathematical inquiry. *Mathematical Thinking and Learning*, 6(3):261–284.

[47] Sinclair, N. (2006). The aesthetic sensibilities of mathematicians. In Sinclair, N., Pimm, D., and Higginson, W., editors, *Mathematics and the aesthetic: New approaches to an ancient affinity*, pages 87–104. Springer, New York.

[48] Sinclair, N. and Pimm, D. (2006). A historical gaze at the mathematical aesthetic. In Sinclair, N., Pimm, D., and Higginson, W., editors, *Mathematics and the aesthetic: New approaches to an ancient affinity*, pages 1–19. Springer, New York.

[49] Sullivan, J. (1925/1956). Mathematics as an art. In Newman, J., editor, *The World of Mathematics*, pages 2015–2021. Simon and Schuster, New York.

[50] Szpiro, G. (2003). Does the proof stack up? *Nature*, 424(6944):12–13.

[51] Tahta, D. (2006). Sensible objects. In Sinclair, N., Pimm, D., and Higginson, W., editors, *Mathematics and the aesthetic: New approaches to an ancient affinity*, pages 190–222. Springer, New York.

[52] Thurston, W. (1995). On proof and progress in mathematics. *For the Learning of Mathematics*, 15(1):29–37.

[53] Todd, C. (2008). Unmasking the truth beneath the beauty: Why the supposed aesthetic judgments made in science may not be aesthetic at all. *International Studies in the Philosophy of Science*, 22(1):61–79.

[54] von Neumann, J. (1947/1956). The mathematician. In Newman, J., editor, *The World of Mathematics*, pages 2053–2065. Simon and Schuster, New York.

[55] von Neumann, J. (1993). Value judgments in mathematics: can we treat mathematics as an art? In White, A. M., editor, *Essays in humanistic mathematics*, pages 62–77. The Mathematical Association of America, Washington, DC.

[56] Wechsler, J. (1978). *On aesthetics in science.* MIT Press, Cambridge, MA.

[57] Wells, D. (1990). Are these the most beautiful? *The Mathematical Intelligencer*, 12(3):37–41.

[58] White, A. M. (1993). *Essays in humanistic mathematics*. The Mathematical Association of America, Washington, DC.

[59] Whiteley, W. (1999). The decline and rise of geometry in 20th century North America. In McLoughlin, J. G., editor, *Canadian Mathematics Study Group Conference Proceedings*, pages 7–30, St Johns, NF: Memorial University of Newfoundland.