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Extending List Colorings of Planar Graphs

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Abstract

In the study of list colorings of graphs, we assume each vertex of a graph has a specified list of colors from which it may be colored. For planar graphs, it is known that there is a coloring for any list assignment where each list contains five colors. If we have some vertices that are precolored, can we extend this to a coloring of the entire graph? We explore distance constraints when we allow the lists to contain an extra color. For lists of length five, we fix W as a subset of $V(G)$ such that all vertices in W have been assigned colors from their respective lists. We give a new, simplified proof where there are a small number of precolored vertices on the same face. We also explore cases where $W = \{u, v\}$ and G has a separating C_3 or C_4 between u and v.

Contents

List of Figures

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Chapter 1

Background

In this chapter, I give background on the subject of list coloring. This includes the motivation for my topic, the relevant definitions and terminology, and known results from previous papers.

1.1 Introduction

An intriguing area of graph theory involves coloring problems. A vertex coloring problem is one in which we try to assign colors to the vertices of a graph so that no two adjacent vertices share a color. However, most vertex coloring problems have been well-studied, so there are few general questions that remain open and approachable.

Graph choosability, or list coloring, is less studied and is the topic of this thesis. In list colorings, each vertex is assigned a list of colors. Informally, a list coloring is one in which each vertex is assigned a color from its list so that no two adjacent vertices have the same color.If we preassign colors to some vertices, we can then ask if there is a way to color the other vertices. This question is explored in this thesis. Because of the nice topological properties of planar graphs, and the information that is known about their list colorability, this thesis will focus on extending colorings on planar graphs.

1.2 Definitions

For a general overview of graph theory, see Diestel (2005). In addition, Bollobás (1998) specifically addresses list coloring.

Definition 1. *A* graph *is an ordered pair (V,E) where V is a set whose elements are called* vertices *and* $E \subseteq \{\{u,v\}|u,v \in V\}$ *is a set whose elements are called* edges*.*

Sometimes a graph is specified simply by G. In this case, I refer to the vertex set as $V(G)$ and the edge set as $E(G)$.

I say that an edge $e = \{u,v\}$, denoted uv, is *incident* to a vertex if the vertex is contained in e. In this case, I say the vertex is an *endpoint* of e. Two vertices u and v are *adjacent*, denoted $u \sim v$, if there is an edge with both vertices as endpoints. The vertices adjacent to v are the *neighbors* of v. The set of neighbors of v is denoted $N(v)$.

A *walk* in a graph is a sequence of vertices $x_1, \ldots, x_n$ such that x_i is adjacent to x_{i+1} for $i = 1, \ldots, n-1$. A *path* is such a sequence with $x_i \neq x_j$ for $i \neq j$. A walk or path is *closed* if $x_1 = x_n$. A closed path is called a *cycle*. A graph G is *connected* if for all $u, v \in V(G)$, there exists a path between u and v. In addition, the *distance* between two vertices u and v is the length of the shortest path between them. This distance is denoted $d(u,v)$.

Definition 2. *A graph is a* tree *if it is connected and has no cycles.*

Definition 3. *A graph H is a* subgraph *of G, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.*

A particular type of subgraph is an *induced subgraph*. Given a subset of vertices S of a graph G, the subgraph induced on S, denoted $G[S]$, is $(S, \{uv \in E(G) : u, v \in S\})$.

Because I want to focus my attention on colorings with precolored vertices, I discuss what it means to extend a coloring of a graph.

Definition 4. *Given a graph G, a k-*coloring *of G is a map $f : V(G) \to [k]$, where we use $[k]$ to denote $\{1, 2, \ldots, k\}$. A coloring is* proper *if for all $v \in V(G), f(v) \neq f(u)$ when $uv \in E(G)$. The minimum k for which there is a proper coloring is the* chromatic number*, and is denoted $\chi(G)$.*

Definition 5. *Let $W \subseteq V(G)$ and $f : W \to [k]$. We can* extend *the coloring of W to a k-coloring of G if there is a map $g : V(G) - W \to [k]$ so that $f \cup g$ creates a proper coloring of G.*

If we let W as in Definition 5, we refer to the vertices in W as the *precolored* vertices.

As I generalize to list colorings, instead of having a single list of colors available to all vertices, each vertex has its own list of available colors, which is a subset of the full list of colors.

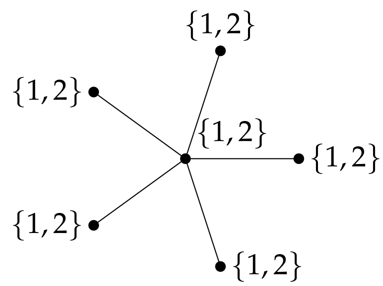


Figure 1.1 A graph with a list assignment where the lists of colors assigned to vertices are all the same. This graph is 2-choosable.

Definition 6. *Let G be a graph with vertices $\{v_1, \ldots, v_n\}$. A* list assignment *is a family of sets, $\Phi = \{\Phi(v_1), \Phi(v_2), \ldots, \Phi(v_n)\}$, where each $\Phi(v_i)$ is a set of positive integers. A* list coloring *is a map $\varphi : V(G) \to \cup_i \Phi(v_i)$ so that $\varphi(v_i) \in \Phi(v_i)$ for all $v_i \in V(G)$. A proper list coloring is one in which $\varphi(v_i) \neq \varphi(v_j)$ for $v_i v_j \in E(G)$.*

For a given vertex, $v_i \in V(G)$, $\Phi(v_i)$ will refer to the list Φ_i that is assigned to v_i.

In the context of vertex coloring, we often seek to minimize the number of colors needed to properly color the graph. Likewise, with list colorings, we seek to minimize the required lengths of the lists.

Definition 7. *A graph G is k-*choosable *if there is a proper list coloring of G for every list assignment where $|\Phi(v_i)| = k$ for all i. The minimum k for which this condition holds is called the* list colorability *or* choosability*, and is denoted $\chi_\ell(G)$.*

Figure 1.1 shows an example of a list assignment where the lists are all the same. In this case, a list coloring corresponds to a 2-coloring. In fact, this star graph admits a coloring for any assignment of lists of length two. For a proof of this fact, consider an arbitrary list assignment with $|\Phi(v)| = 2$ for all $v \in V(G)$. First color the central vertex. Remove this color from the lists of the other vertices. Because every other vertex is only adjacent to the central vertex, their lists still have at least one color that can be selected. Because this algorithm produces a coloring for any list assignment where all the lists have length two, this shows that the star graph is Figure 1.1 is 2-choosable.

In fact, in general, trees are 2-choosable. To see this fact, we order the vertices. Start by choosing an arbitrary vertex and call it v_1. Then order the vertices so that v_i is adjacent to v_j for some v_j with $j < i$. Then we can color the vertices in order. Each vertex $i \geq 2$ has a neighbor that is assigned a color, but because it has two colors available, it can be properly colored. As trees contain no cycles, no vertex has two neighbors that are assigned colors before it.

Definition 8. *Suppose we have a graph G and list assignment Φ for G. Let $W \subseteq V(G)$ and $\varphi : W \to \cup_i \Phi$ so that $\varphi(v_i) \in \Phi(v_i)$. We can* extend *the list coloring of W if there is a map $\varphi' : V(G) - W \to \cup_i \Phi$ so that $\varphi \cup \varphi'$ creates a proper list coloring of G.*

The outline of the proof showing that trees are 2-choosable also shows that any one vertex can be precolored. However, with lists of length two, it is not guaranteed that two vertices can be precolored. Given a path of odd length where each vertex has the list $\{1, 2\}$, the endpoints must have the same color. Hence, we cannot precolor them with different colors and extend the coloring.

Definition 9. *An* embedding *of a graph G is a map onto $\mathbb{R}^2$ such that the following conditions hold:*

- *The vertices are distinct.*
- *Edges are paths between the vertices.*
- *Edges only intersect at vertices.*

A graph is *planar* if it can be embedded in the plane.

Definition 10. *A* plane graph *is a fixed embedding of a graph where the arcs representing the edges do not contact other points of the embedding except at the endpoints.*

Both of the graphs in Figure 1.2 are identical as abstract graphs. However, they differ in their embedding, thus illustrating the difference between a plane graph and a planar graph.

Definition 11. *A* face *f of a plane graph G is one of the connected components of $\mathbb{R}^2 - G$, the regions left in the plane that are not covered by the plane graph. The* boundary *of a face f are the edges and vertices that separate f from the other faces.*

Definition 12. *A graph is* triangulated *if every face has three edges on its boundary. A graph is a* near-triangulation *if the boundary of every face except the outer face has three edges on its boundary.*

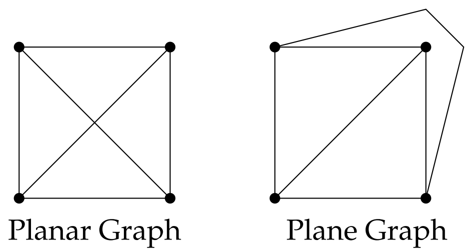


Figure 1.2 Two different embeddings of a planar graph. The embedding where edges do not cross is a plane graph.

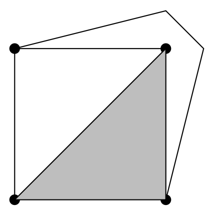


Figure 1.3 A drawing of a plane graph with a face shaded in gray. This graph is also triangulated.

Figure 1.3 shows a plane graph. In the graph, a face shaded in gray. This particular plane graph is triangulated.

Definition 13. *The* complete graph K_n on n vertices *is the graph with* $E = \{\{u, v\} | u, v \in V, u \neq v\}$.

Definition 14. *The* complete bipartite graph $K_{n,m}$ *is a graph whose vertex set can be partitioned into two independent sets, one with size n, and one with size m, where every vertex in the first set is adjacent to every vertex in the second set.*

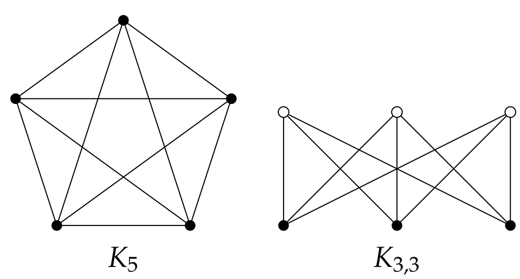


Figure 1.4 The complete graph K_5 and the complete bipartite graph $K_{3,3}$. The two independent sets of vertices in $K_{3,3}$ are indicated by black and white vertices.

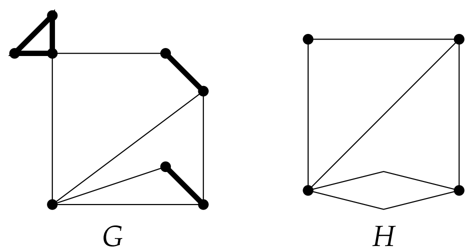


Figure 1.5 Examples of contractions of edges. The five heavy edges in G are contracted to form the graph H.

Figure 1.4 shows examples of a complete graph and a complete bipartite graph. Note that the two independent sets of vertices in $K_{3,3}$ are indicated by black and white vertices.

Definition 15. *The* contraction *of $e = uv$, denoted $G \backslash e$, replaces u and v with a single vertex v_e that is adjacent to all of the neighbors of u and v with multiple edges used when u and v have common neighbors in G.*

Definition 16. *A* minor *of a graph G is any graph that can be obtained by a series of contractions of the edges of any subgraph of G.*

Figure 1.5 shows an example of contractions along the heavy edges. The five edges indicated edges in G are contracted to form the graph H. Thus the three vertices in the upper left of G form a single vertex in H. Additionally, the two vertices in the upper right and two vertices in the lower right of G form on vertex each in H.

Definition 17. *A* subdivision of an edge *replaces an edge uv with the path u, x, v where x is a degree-two vertex not originally in the vertex set. If we allow the edges of a graph H to subdivide, perhaps repeatedly, then we call the resulting graph a* subdivision of *H.*

Definition 18. *If a subdivision of H is a subgraph of G, then H is a* topological minor *of G.*

In Figure 1.6, the graph H is a subdivision of the graph G. Thus G is an example of a topological minor of the graph H.

1.3 Choosability of Planar Graphs

Graph coloring is an active area of research. A review of results in graph coloring can be found in Kratochvíl, Tuza, and Voigt (1999). A survey focused specifically on constrained colorings can be found in Tuza (1997).

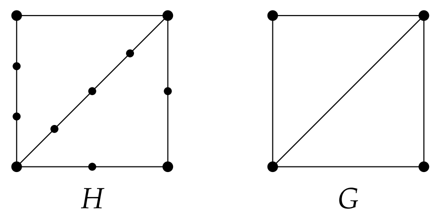


Figure 1.6 The graph H is a subdivision of the graph G. Likewise, G is a topological minor of H.

1.4 5-Choosability

For planar graphs, a well-known result of Appel and Haken states that all planar graphs are four-colorable. When it comes to choosability, the bounds are also known.

Theorem 1.1. *(Voigt (1993)) There exist planar graphs that are not 4-choosable.*

Voigt (1993) is dedicated to the construction of such a graph. The graph constructed in this paper has 238 vertices.

Theorem 1.2. *(Thomassen (1994)) All planar graphs are 5-choosable.*

A full proof can be found in Thomassen (1994). The proof relies on induction. It takes a near-triangulated graph and assumes that the lists of vertices on the boundary of the outer face have length three, whereas the internal vertices have lists of length five. By additionally precoloring two adjacent vertices, Thomassen was able to show that these conditions can be maintained when a vertex is removed in such a way that the removed vertex has an available color. The proved result is then stronger than 5-choosability, but it implies 5-choosability of planar graphs.

Together, Theorem 1.1 and Theorem 1.2 show that the bound is sharp for planar graphs in general, although there are classes of planar graphs with smaller choosability. For example, I argued that trees are 2-choosable.

Before the 5-choosability of planar graphs was proved, Voigt (1996) gave a definition that is generally stronger than choosability but is equivalent to choosability for planar graphs.

Definition 19. *A graph is* free k-choosable *if for every list assignment, Φ, with $|\Phi(v)| = k$ for all $v \in V(G)$, there exists a Φ-coloring with $\varphi(v) = f$, for every vertex $v \in V(G)$, and for every color $f \in \Phi(v)$.*

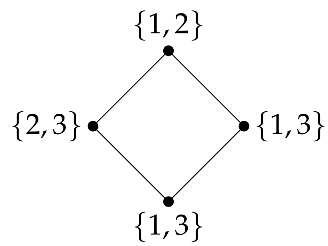


Figure 1.7 A list assignment that shows that the cycle C_4 is not free 2-choosable.

Figure 1.7, which comes from Voigt (1996), gives an example of a 2-choosable graph. However, for the given list assignment, precoloring the right vertex 1 does not allow for a proper list coloring. Therefore this graph is not *free* 2-choosable.

This definition is then stronger than k-choosability, as every free k-choosable graph is k-choosable. Voigt (1996) shows that these two definitions are equivalent in the case of 5-choosability for planar graphs. The proof takes advantage of the known examples of planar graphs that are not 4-choosable. While free k-choosability was not used in the eventual proof of 5-choosability of planar graphs, it does show that there are examples of graphs where it is not possible to precolor even one vertex. This possible lack of a coloring is a factor that needs to be considered when examining families of planar graphs that have choosability less than five. One such class is that of bipartite planar graphs, which are 3-choosable.

In extending Thomassen's proof, Böhme, Mohar, and Stiebitz (1999) look at extending colorings where the vertices are on the same face. They build on the idea of 5-list coloring a near-triangulated graph where the precolored vertices are on the boundary of the outer face but are not adjacent. They also characterize when precolorings of the vertices on the boundary of the outer face can be extended: when the boundary of the outer face has no more than six vertices. These proofs are given in detail in Section 2.2. In Section 2.2, I also apply the second result to shed further light on extending precolorings in the case where the vertices share a face but the face may have more than six vertices.

1.5 Distance Constraints for Vertex Coloring

For vertex colorings, Albertson (1998) has a number of results about precoloring vertices of a certain distance.

Theorem 1.3. *(Albertson (1998)) Suppose G is any planar graph and $W \subseteq V(G)$ is such that the distance between any two vertices in W is at least 4. Any 5-coloring of the vertices of W can be extended to a 5-coloring of G.*

Proof Sketch. Let γ be a 4-coloring of G using the colors $\{1, 2, 3, 4\}$. Assume φ is a coloring of the vertices of W using colors $\{1, 2, 3, 4, 5\}$. If $v \in W$ and $\varphi(v)$ disagrees with $\gamma(v)$, switch $\gamma(v)$ to $\varphi(v)$ and the neighbors of v that were colored $\varphi(v)$ to color 5, creating a valid 5-coloring of G that extends the coloring given by φ. $\square$

Theorem 1.4. *(Albertson (1998)) Suppose G is any planar graph and $W \subseteq V(G)$ is such that the distance between any two vertices in W is at least 3. Any 6-coloring of W can be extended to a 6-coloring of G.*

Proof Sketch. Fix a coloring of W. Let $G' = G - W$. For each vertex v in G', note that v has at most one neighbor in W. Thus a list of five colors from $\{1, 2, 3, 4, 5, 6\}$ can be constructed that excludes the color assigned to the neighbor of v in W, if it has one. Then G' is colorable by the 5-choosability of planar graphs. The coloring of G', combined with the coloring of W, gives a coloring for G. $\square$

Theorem 1.5. *(Albertson (1998)) Suppose G is r-choosable and $W \subseteq V(G)$ is such that the distance between any two vertices in W is at least 3. Any $(r+1)$-coloring of W can be extended to an $(r+1)$-coloring of G.*

The proof idea is the same as that of Theorem 1.4. These proofs all take advantage of the existence of additional colors available either due to the distance between precolored vertices or because the k-coloring being achieved is such that $k > \chi(G)$. Proof techniques similar to these can be applied to list coloring to achieve a distance-three bound similar to Theorem 1.4. This idea is explored in Section 2.1.

Chapter 2

Loosening Constraints Based on Distance and Faces

One approach to extending list colorings of planar graphs is to build on or generalize known results. By weakening the assumptions, we thereby obtain a stronger result. In Section 2.1, I extend the distance constraints from vertex colorings to list colorings using similar techniques as Albertson (1998). In the subsequent section, I discuss a few results relating to precolored vertices on the same face when the face has few vertices. I then use these ideas and a new construction to prove results for precoloring vertices on the same face even when they do not consist of all the vertices on the face.

2.1 Generalizing Albertson (1998)

We can extend a list coloring of a graph where we have an additional color available (for example, $\chi_\ell(G)+1$ colors) and the precolored vertices satisfy a distance constraint. While the proofs of these results are similar to techniques used by Albertson (1998), these results are not mentioned specifically in his paper. However, the authors allude to them in Albertson and Moore (1999). Nevertheless, because of their relevancy, the results are explicitly stated and proved in this section.

Theorem 2.1. *Suppose G is a planar graph with a list assignment Φ such that $|\Phi(v)| \geq 6$ for all $v \in V(G)$. Let $W \subseteq V(G)$ such that the distance between any two vertices in W is at least three. Then any precoloring of the vertices of W can be extended to a Φ-coloring of G.*

Proof. Let φ be a Φ-coloring in W. Let $G' = G - W$, and Φ' be a list assignment for G' with $\Phi'(v) = \Phi(v) - \{\varphi(w) : w \in N_G(v) \cap W\}$. Then, because the distance between any two vertices in W is at least three, $|\Phi'(v)| \geq 5$ for all $v \in V(G')$. Hence, we can create a Φ'-coloring, φ', of G'. Because $\varphi'(v) \in \Phi(v)$ for all $v \in V(G)$ we can create a coloring φ for all of the vertices of G by letting $\varphi(v) = \varphi'(v)$ for $v \in V(G')$. Then φ is a proper list coloring because each of the vertices of G' was assigned a color different from all of its neighbors in G' by assumption, and from all of its neighbors in W by construction. Because no two vertices in W are neighbors, we have the desired Φ-coloring of G. □

Theorem 2.2. *Suppose G is r-choosable and $W \subseteq V(G)$ such that the distance between any two vertices in W is at least 3. Let Φ be any assignment of lists to G with $|\Phi(v)| \geq r + 1$ for all $v \in V(G)$. Then for any precoloring, φ, of the vertices of W, we can find a Φ-coloring of G that extends φ.*

Proof. Let φ be a Φ-coloring in W. Let $G' = G - W$, and Φ' be a list assignment for G' with $\Phi'(v) = \Phi(v) - \{\varphi(w) : w \in N_G(v) \cap W\}$. Then, because the distance between any two vertices in W is at least three, $|\Phi'(v)| \geq r$ for all $v \in V(G')$. Because G is r-choosable and G' is a subgraph of G, it is also r-choosable. Hence, we can create a Φ'-coloring, φ', of G'. Because $\varphi'(v) \in \Phi(v)$ for all $v \in V(G)$, we can create a coloring φ for all of the vertices of G by letting $\varphi(v) = \varphi'(v)$ for $v \in V(G')$. Then φ is a proper list coloring because each of the vertices of G' was assigned a color different from all of its neighbors in G' by assumption, and from all of its neighbors in W by construction. Because no two vertices in W are neighbors, we have the desired Φ-coloring of G. □

Note that Theorem 2.2 is clearly a generalization of Theorem 2.1. Even though Theorem 2.2 is more general, I include both because Theorem 2.1 pertains to the focus of this thesis on planar graphs. In addition, Theorem 2.2 shows that if a tree is assigned lists with three colors, then any precoloring where the chosen vertices are at least distance three apart can be extended to a proper list coloring of the entire tree.

2.2 On the Same Face

Böhme, Mohar, and Stiebitz (1999) extend list colorings when the precolored vertices are on the same face. I discuss their two results in this section and then give results that I have proved using their theorems.

Theorem 2.3. *(Böhme et al. (1999))Let G be a plane graph, and let W be the set of vertices on the boundary of the outer face of G. Let $P = (v_1, \dots, v_k)$ be a path on the boundary of the outer face. Assume that Φ is a list for G satisfying $|\Phi(v)| \geq 5$ if $v \in V(G) - W$, $|\Phi(v)| \geq 4$ if $v \in V(P) - \{v_1, v_k\}$, $|\Phi(v)| \geq 2$ if $v \in \{v_1, v_k\}$, and $|\Phi(v)| \geq 3$ if $v \in W - V(P)$. Then G is Φ-colorable.*

The proof of Theorem 2.3 extends Thomassen's proof of 5-choosability so that the vertices with the shortest lists do not have to be neighbors. The proof proceeds in a similar manner, and uses induction on the number of vertices. It also assumes the path on the boundary of the outer face has length at least two.

Theorem 2.4. *(Böhme et al. (1999))Let G be a plane graph with outer cycle C of length $p \leq 6$. Assume that Φ is a list for G satisfying $|\Phi(v)| \geq 5$ for all $v \in V(G)$ and φ is a Φ-coloring of $G[V(C)]$. Then φ can be extended to a Φ-coloring of G unless $p \geq 5$, the notation may be chosen such that $C = (v_1, \dots, v_p), \varphi(v_i) = \alpha_i$ for $1 \leq i \leq p$, and one of the following conditions holds:*

a. There is a vertex w inside C such that w is adjacent to $v_1, \dots, v_5$ and $\Phi(w) = \{\alpha_1, \dots, \alpha_5\}$.

b. $p = 6$ and there is an edge w_0w_1 inside C such that, for $i = 0, 1$ the vertex w_i is adjacent to $v_{3i+1}, v_{3i+2}, v_{3i+3}, v_{3i+4}$ and $\Phi(w_i) = \{\alpha_{3i+1}, \alpha_{3i+2}, \alpha_{3i+3}, \alpha_{3i+4}, \beta\}$.

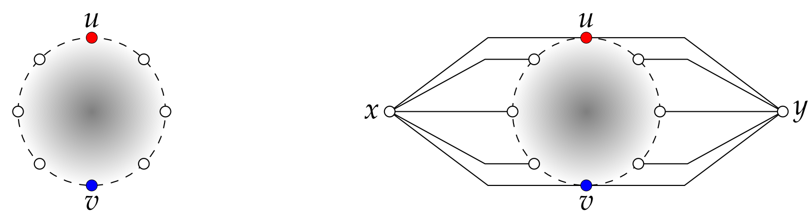
c. $p = 6$ and there is a triangle (w_0, w_1, w_2) inside C such that, for $i = 0, 1, 2$, the vertex w_i is adjacent to $v_{2i+1}, v_{2i+2}, v_{2i+3}$ and $\Phi(w_i) = \{\alpha_{2i+1}, \alpha_{2i+2}, \alpha_{2i+3}, \beta, \gamma\}$.

where all indices are computed modulo p.

Theorem 2.4 is proved via an inductive proof. It assumes that a graph is not in one of the so-called fatal cases, and then eliminates possibilities in the inductive step to show that this property is maintained.

From these theorems I obtain the remaining four theorems in this section. The first was my original application of this idea; however, use of Theorem 2.4 allows for a collection of results for precoloring vertices on the same cycle even when the cycle is not small.

The next set of theorems examine precoloring vertices on the same face. Each theorem will allow for only a small number of vertices to be precolored, but the size of the face does not matter. Theorem 2.5 allows us to precolor two vertices on the same face, given the the other vertices are assigned lists of length at least five.



a. The graph G. **b.** The graph G'.

Figure 2.1 Illustration of proof of Theorem 2.5 for extending a coloring with two precolored vertices. Not all vertices are shown, and dashed edges may represent paths.

Theorem 2.5. *Suppose that G is a plane graph. Assume that Φ is a list for G such that $|\Phi(v)| \geq 5$ for all $v \in V(G)$. Let u and v be two vertices on the same face that has the boundary C. Let $\varphi(u)$ and $\varphi(v)$ be given. Then φ can be extended to a Φ-coloring of G.*

Figure 2.1 shows the graphs referenced in the proof of Theorem 2.5. Note that the graph G is an example of a graph in the statement of Theorem 2.5. In G, the vertices u and v on the outer face have been precolored.

Proof. First suppose that $uv \in E(G)$. Then, by Thomassen's proof of 5-choosability of planar graphs, the result holds.

Next suppose that $uv \notin E(G)$. Then decompose C into two paths, P_1 and P_2, so that $V(P_1) \cup V(P_2) = V(C)$ and $V(P_1) \cap V(P_2) = \{v, u\}$. We then construct a plane graph G' by adding new vertices x and y to $V(G)$ and adding edges xz for all $z \in V(P_1)$ and yz for all $z \in V(P_2)$ to $E(G)$. Examples of G and G' are given in Figure 2.1a and Figure 2.1b respectively. Then u, x, v, and y form the boundary of the outer face of G'. Extend Φ so that $\Phi(x)$ and $\Phi(y)$ do not share elements with $\bigcup_{z \in V(G)} \Phi(z)$. Let $\varphi(x) = \varphi(y)$ be chosen. Then we can extend φ by Theorem 2.4. When we remove x and y to recover G, we retain a proper Φ-coloring of G. $\square$

Theorem 2.6 increases the number of precolored vertices from two to three without adding any additional restrictions.

Theorem 2.6. *Let G be a plane graph with list Φ with $|\Phi(v)| \geq 5$ for all $v \in V(G)$. Let v_1, v_2, v_3 be specified vertices on the boundary of the outer face. Let $\varphi(v_i) = \alpha_i$ be given. Then φ can be extended to a Φ-coloring of G.*

Proof. Let P_1, P_2, and P_3 be the (possibly empty) sets of non–precolored vertices on the outer cycle between v_1 and v_2, v_2 and v_3, and v_1 and v_3, respectively.

Let $G' = (V(G) \cup \{u_1, u_2, u_3\}, E(G) \cup \{u_i v | v \in V(P_i)\})$ and assign G' a list Φ' so that $\Phi'(v) = \Phi(v)$ for $v \in V(G)$ and $\Phi'(u_i)$ do not share elements with $\bigcup_{v \in V(G)} \Phi(v)$ for $i = 1, 2, 3$. If we let $\varphi'(v)$ agree with $\varphi(v)$ on v_1, v_2, v_3, and let $\varphi'(u_i) \in \Phi'(u_i)$, for $i = 1, 2, 3$, then we know that we cannot be in one of the exception cases of Theorem 2.4 and hence φ' can be extended to a Φ'-coloring of G'. Because this coloring agrees with φ on the vertices of G, and assigns an appropriate color in Φ to the other vertices of G we have a Φ-coloring of G by restricting φ' to G. □

Theorem 2.7 increases the number of precolored vertices to four. However, in this case, the precolored vertices must form sets so that there are at most two distinct paths along the outer face between the sets of precolored vertices.

Theorem 2.7. *Let G be a plane graph with list Φ with $|\Phi(v)| \geq 5$ for all $v \in V(G)$. Let v_1, v_2, v_3, v_4 be specified vertices on the boundary of the outer face so that one of the following is true:*

1. $v_1v_2, v_2v_3, v_3v_4 \in E(G)$
2. $v_1v_2, v_3v_4 \in E(G)$ *and* $v_2v_3, v_4v_1 \notin E(G)$
3. $v_1v_2, v_2v_3 \in E(G)$ *and* $v_1v_4, v_3v_4 \notin E(G)$

Let $\varphi(v_i) = \alpha_i$ be given. Then φ can be extended to a Φ-coloring of G.

Proof. Suppose we are in case 1 and the vertices v_1, v_2, v_3, v_4 form the entire outer cycle. Then, by Theorem 2.4, we can extend the coloring. If not, let P_1 denote the path consisting of the other vertices on the outer cycle and let P_2 be empty. If we are in case 2, let P_1 be the non–precolored vertices on the outer cycle between v_1 and v_4, and let P_2 be the non–precolored vertices on the outer cycle between v_2 and v_3. If we are in case 3, let P_1 be the non–precolored vertices on the outer cycle between v_1 and v_4, and let P_2 be the non–precolored vertices on the outer cycle between v_3 and v_4.

Construct $G' = (V(G) \cup \{x, y\}, E(G) \cup \{xv | v \in V(P_1)\} \cup \{yv | v \in V(P_2)\})$ with a list Φ' such that $\Phi'(v) = \Phi(v)$ for $v \in V(G)$ and $\Phi'(x)$ and $\Phi'(y)$ do not share elements with $\bigcup_{v \in V(G)} \Phi(v)$. If we let $\varphi'(v_i) = \varphi(v_i)$ for $i = 1, 2, 3, 4$, and let $\varphi'(v) \in \Phi'(v)$ for $v \in \{x, y\}$, then we know that we cannot be in one of the exception cases of Theorem 2.4 and hence φ' can be extended to a Φ'-coloring of G'. Because this coloring agrees with φ on the shared vertices of G and assigns an appropriate color in Φ to the other vertices of G, we have a Φ-coloring of G by restricting φ' to G. Thus we obtain the desired Φ-coloring of G. □

Finally, Theorem 2.8 increases the number of precolored vertices to five. In this theorem, the precolored vertices must be consecutive along the outer face. There is also the possibility of one of the exceptions mentioned in Theorem 2.4, which must be avoided.

Theorem 2.8. *Let G be a plane graph with list Φ with $|\Phi(v)| \geq 5$ for all $v \in V(G)$. Let $v_1, \ldots, v_5$ be consecutive vertices on the boundary of the outer face, and $\varphi(v_i) = \alpha_i$ be given. Then φ can be extended to a Φ-coloring of G unless there exists a vertex $w \in V(G)$ that is adjacent to $v_1, \ldots, v_5$ such that $\Phi(w) = \{\alpha_1, \ldots, \alpha_5\}$.*

Proof. If there is a vertex w as described in the theorem, then we cannot extend φ. Suppose that there does not exist such a vertex w. If the vertices $v_1, \ldots, v_5$ compose the entire outer cycle, then, by Theorem 2.4, we can extend the coloring.

Suppose, instead, that there are other vertices on the outer cycle and let P be a path on the outer cycle with endpoints v_1 and v_5 that contains the non–precolored vertices. Then let G' be the graph $G' = (V(G) \cup \{v\}, E(G) \cup \{vx | x \in V(P)\})$ and with a list Φ' so that $\Phi'(x) = \Phi(x)$ for $x \in V(G)$ and $\Phi'(v)$ does not share elements with $\bigcup_{x \in V(G)} \Phi(x)$. If we let $\varphi'(v_i) = \varphi(v_i)$ for $i = 1, \ldots, 5$, and let $\varphi'(v) \in \Phi'(v)$, then we know by Theorem 2.4 that φ' can be extended to a Φ'-coloring of G'. Because this coloring agrees with φ on the shared vertices of G, and assigns an appropriate color in Φ to the other vertices of G, we have a Φ-coloring of G by restricting φ' to G. □

Theorems 2.5–2.8 examine extending a list coloring from a small number of vertices on the same face. As the number of precolored vertices increases, so do the restrictions on how they can be selected. Overall, we require that the sum of the number of precolored vertices and the number of paths along the outer face between precolored vertices be no more than six.

Chapter 3

Taking Advantage of Small Separating Cycles

A preprint by Axenovich, Hutchinson, and Lastrina (2010) gives several proofs related to extending list colorings. Some of their results relate to vertices on the same face. They also deal with distance constraints. Most of their results hold only when there are no C_3 or C_4 cycles separating the precolored vertices. These results are detailed in Section 3.1. With this motivation, in Section 3.2, I give results for cases when G has two precolored vertices separated by a small cycle.

3.1 Extending Precolorings Paper

Here I state the results given in Axenovich et al. (2010) and mention their relationship to the results of Chapter 2. I begin with some definitions used in their paper.

Definition 20. *In a graph G, a cycle C is a* separating cycle *is its $G - C$ has more components than G. If u and v are vertices in the same component of G, and different components of $G - C$, then C may also be referred to as a $\{u, v\}$-*separating cycle*.*

The authors also give a result for precoloring any two vertices that are not separated by a C_3 or C_4.

Theorem 3.1. *Let G be a plane graph and $u, v \in V(G)$. If G has no $\{u, v\}$-separating C_3 or C_4, then every proper precoloring of $\{u, v\}$ is extendable to a proper 5-list coloring of G.*

Theorem 3.2. *Let G be a plane graph with C the set of vertices of on the boundary of a face of G. Let $P = \{v_0, v_1, \ldots, v_{k-1}\} \subseteq C$, where the vertices of P are labeled cyclically around C. Then every proper precoloring of P is extendable to a 5-list coloring of G if one of the following conditions holds:*

1. *$G[P]$ consists of disjoint vertices and edges with pairwise distance at least 3.*
2. *$k \leq 6$ and none of the following occur:*
   - *(a) There is a vertex w inside C such that w is adjacent to $v_1, \ldots, v_5$ and $\Phi(w) = \{\alpha_1, \ldots, \alpha_5\}$.*
   - *(b) $p = 6$ and there is an edge w_0w_1 inside C such that, for $i = 0, 1$ the vertex w_i is adjacent to $v_{3i+1}, v_{3i+2}, v_{3i+3}, v_{3i+4}$ and $\Phi(w_i) = \{\alpha_{3i+1}, \alpha_{3i+2}, \alpha_{3i+3}, \alpha_{3i+4}, \beta\}$.*
   - *(c) $p = 6$ and there is a triangle (w_0, w_1, w_2) inside C such that, for $i = 0, 1, 2$, the vertex w_i is adjacent to $v_{2i+1}, v_{2i+2}, v_{2i+3}$ and $\Phi(w_i) = \{\alpha_{2i+1}, \alpha_{2i+2}, \alpha_{2i+3}, \beta, \gamma\}$.*

This second part of this result then gives a generalization of Theorem 2.2, and my results (Theorems 2.5–2.6). However, the proof techniques used in my proofs were different as they were based on adding vertices. Like the proof of Theorem 2.4, the proof of Theorem 3.2 is a proof by induction.

Theorem 3.3. *Let P be a set of vertices in a plane graph G, with the distance between vertices in P at least three, such that there are two faces F_1, F_2 where the vertices of P lie on the boundaries of F_1 and F_2. Assume G contains no P-separating C_3 or separating C_4. Then every precoloring of P is extendable to a proper 5-list coloring of G.*

3.2 No Separating Cycles

As discussed in Section 3.1, Axenovich et al. (2010) addresses the case of extending list colorings where precolored vertices appear on the same face (Theorem 3.2), or when there are no separating C_3 or C_4 in the planar graph (Theorem 3.1). In addressing the case that there is a small separating cycle, I give the following results.

In all of these results, v is a vertex inside the separating cycle and u is a vertex on the boundary of the outer face. These results start with a separating cycle that is close to the boundary containing u and then slowly move the cycle further away.

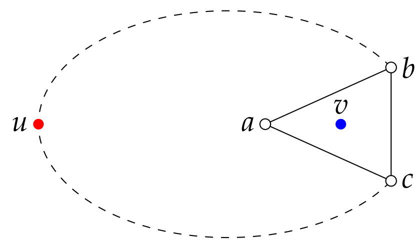


Figure 3.1 A plane graph with a $\{u, v\}$-separating cycle C of length three such that C has an edge on the boundary of the outer face.

All of the proofs implicitly assume that the vertices incident to the outer face form a cycle. If the outer face is not bounded by a cycle, edges could be added until it is a cycle without changing the vertices that are on the outer cycle. If this modified graph is colorable for the given assignment, then so is the original graph. Thus, there is no loss of generality in this assumption.

Finally, note that for all of the proofs in this section, we can actually use Theorem 1.2 (from Thomassen, 1994) to precolor any two adjacent vertices inside the separating cycle.

I begin with the case of extending a precoloring when a separating C_3 has an edge on the boundary of the outer face.

Theorem 3.4. *Suppose that G is a plane graph with a separating cycle C such that $V(C) = \{a, b, c\}$ oriented clockwise and bc is an edge on the boundary of the outer face. Let v be a vertex in the interior of C, and u a vertex distinct from a, b, c on the boundary of the outer face. Suppose that Φ is a list assignment for G with $|\Phi(x)| \geq 5$ for all $x \in V(G) - \{u, v\}$ and $|\Phi(u)| = |\Phi(v)| = 1$. Then G is Φ-colorable.*

Several sketch example graphs are for the theorems in this section. Only the vertices named in the theorem statement will be shown; however there may be other edges and vertices not shown. Dashed edges represent paths, where solid edges depict edges that exist in the graph.

Figure 3.1 shows an example of the situation described by Theorem 3.4. It shows a plane graph with a $\{u, v\}$-separating cycle $C = \{a, b, c\}$ such that C has an edge on the boundary of the outer face.

Proof. Let $\varphi(v)$ be the color in $\Phi(v)$. Call the subgraph of C and its interior H. Then φ can be extended to H using colors available in Φ. Consider the graph G' made by removing the interior of C as well as the edge bc. In G', a, b, c, and u are all on the boundary of the outer face. Let $\varphi'(x) = \varphi(x)$ for $x \in \{a, b, c\}$ and $\varphi'(u)$ be the color in $\Phi(u)$. Then, by Theorem 3.2, we

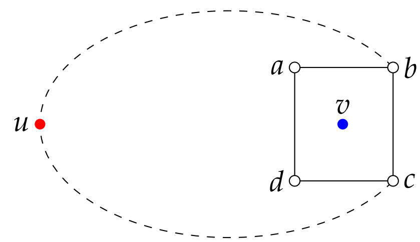


Figure 3.2 A plane graph with a $\{u, v\}$-separating cycle C of length four such that C has an edge on the boundary of the outer face.

are able to extend φ' to G'. It follows that φ' agrees with φ on all shared vertices. Thus, letting $\varphi(x) = \varphi'(x)$ for $x \in V(G')$ yields a Φ-coloring of G. □

Using a similar strategy for a separating C_4, we have the following result.

Theorem 3.5. *Suppose that G is a plane graph with a separating cycle C such that $V(C) = \{a, b, c, d\}$ oriented clockwise and bc is an edge on the boundary of the outer face. Let v be a vertex in the interior of C, and u a vertex distinct from a, b, c and d on the boundary of the outer face. Suppose that Φ is a list assignment for G with $|\Phi(x)| \geq 5$ for all $x \in V(G) - \{u, v\}$ and $|\Phi(u)| = |\Phi(v)| = 1$. Then G is Φ-colorable.*

Figure 3.2 shows an example of the situation described by Theorem 3.5. It shows a plane graph with a $\{u, v\}$-separating cycle $C = \{a, b, c, d\}$ such that C has an edge on the boundary of the outer face.

Proof. Consider the case where there is a vertex w that is adjacent to all of a, b, c, d, u. If such a w exists, then, w, b, c consists of a separating C_3 so by Theorem 3.4, there is a Φ-coloring for G.

Let $\varphi(v)$ be the color in $\Phi(v)$. Call the subgraph of C and its interior H. Then φ can be extended to H using colors available in Φ. Consider the graph G' made by removing the interior of C as well as the edge bc. In G', a, b, c, d, and u are all on the boundary of the outer face. Let $\varphi'(x) = \varphi(x)$ for $x \in \{a, b, c, d\}$ and $\varphi'(u)$ be the color in $\Phi(u)$. Then, by Theorem 3.2, φ' can be extended to G'. As φ' agrees with φ on all common vertices of H and G', letting $\varphi(x) = \varphi'(x)$ for $x \in V(G')$ yields a Φ-coloring of G. □

I move on next to consider the cases where two of the vertices of the separating cycle are on the boundary of the outer face regardless of whether or not they are adjacent.

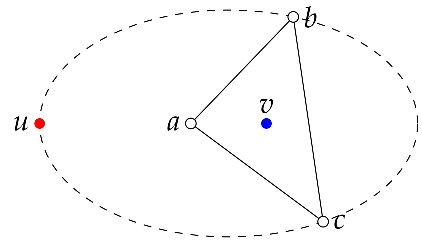


Figure 3.3 A plane graph with a $\{u, v\}$-separating cycle C of length three such that C has two vertices on the boundary of the outer face.

Theorem 3.6. *Suppose that G is a plane graph with a separating cycle C such that $V(C) = \{a, b, c\}$ oriented clockwise and b and c are on the boundary of the outer face. Let v be a vertex in the interior of C, and u a vertex distinct from a, b, and c on the boundary of the outer face. Suppose that Φ is a list assignment for G with $|\Phi(x)| \geq 5$ for all $x \in V(G) - \{u, v\}$ and $|\Phi(u)| = |\Phi(v)| = 1$. Then G is Φ-colorable.*

Figure 3.3 shows an example of the situation described by Theorem 3.6. It shows a plane graph with a $\{u, v\}$-separating cycle $C = \{a, b, c\}$ such that C has two vertices on the boundary of the outer face.

Proof. Let $\varphi(v)$ be the color in $\Phi(v)$. Call the subgraph of C and its interior H. Then φ can be extended to H using colors available in Φ. Let P be the path along the boundary of the outer face between b and c so that the interior of the cycle formed by P and the edge bc does not contain a. Call this cycle C', and consider it along with its interior. There are at most three precolored vertices all on C'. Thus, by Theorem 3.2, we can extend the coloring φ to this subgraph.

Next consider the graph G' consisting of a, b, c and the vertices that do not already have a color in φ'. In G', a, b, c and possibly u are all on the boundary of the outer face. Let $\varphi'(x) = \varphi(x)$ for $x \in \{a, b, c\}$ and, if u in $V(G')$, let $\varphi'(u)$ be the color in $\Phi(u)$. Then, by Theorem 3.2, we can extend φ' to G'. Moreover, φ' agrees with φ on all shared vertices, and together they color all the vertices. Thus letting $\varphi(x) = \varphi'(x)$ for $x \in V(G')$ yields a Φ-coloring of G. □

Theorem 3.7. *Suppose that G is a plane graph with a separating cycle C such that $V(C) = \{a, b, c, d\}$ oriented clockwise and c and d are on the boundary of the outer face. Let v be a vertex in the interior of C, and u a vertex distinct from a, b, c, and d on the boundary of the outer face. Suppose that Φ is a list assignment for G*

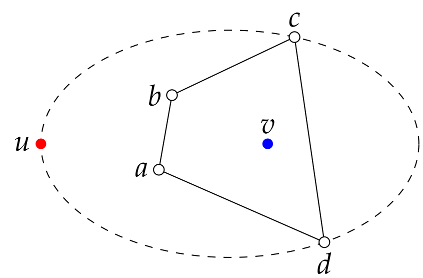


Figure 3.4 A plane graph with a $\{u, v\}$-separating cycle C of length four such that C has two vertices on the boundary of the outer face.

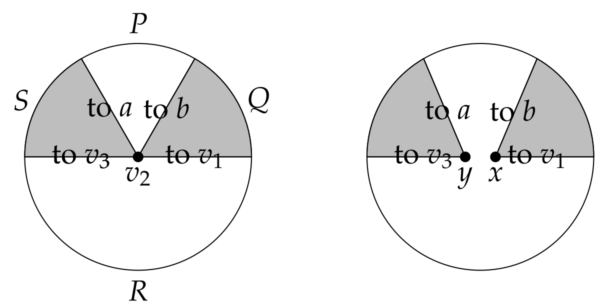
with $|\Phi(x)| \geq 5$ *for all* $x \in V(G) - \{u, v\}$ *and* $|\Phi(u)| = |\Phi(v)| = 1$. *Then* G *is* Φ*-colorable.*

Figure 3.4 shows an example of the situation described by Theorem 3.7. It shows a plane graph with a $\{u, v\}$-separating cycle $C = \{a, b, c, d\}$ such that C has two vertices on the boundary of the outer face.

Proof. Let $\varphi(v)$ be the color in $\Phi(v)$. Call the subgraph of C and its interior H. Then φ can be extended to H using colors available in Φ. Let P be the path along the boundary of the outer face between c and d so that the interior of the cycle formed by P and the edge cd does not contain a or b. Call this cycle C' and consider it along with its interior. It has at most three precolored vertices all on C'. Thus, by Theorem 3.2, we can extend the coloring φ to this subgraph.

Next consider the graph G' consisting of a, b, c, and d and the vertices that do not already have a color in φ'. In G', a, b, c, d, and possibly u are all on the boundary of the outer face. Let $\varphi'(x) = \varphi(x)$ for $x \in \{a, b, c, d\}$ and, if $u \in V(G')$, let $\varphi'(u)$ be the color in $\Phi(u)$. Then, by Theorem 3.2, we can extend φ' to G' unless $u \in V(G')$ and there is a vertex w adjacent to a, b, c, d, and u whose list is $\{\varphi'(a), \varphi'(b), \varphi'(c), \varphi'(d), \varphi'(u)\}$. However, if such a w exists, there is a C_3 with $V(C_3) = \{w, c, d\}$ so that we are in the case of Theorem 3.6, in which case there is a Φ-coloring of G. Otherwise, we get a coloring φ' of G' that agrees with φ on all shared vertices, and together they color all the vertices. Thus letting $\varphi(x) = \varphi'(x)$ for $x \in V(G')$ yields a Φ-coloring of G. $\square$

A similar result can be shown where we instead have a and c as the vertices on the boundary of the outer face.



a. The interior of the disk D in G.

b. The interior of the disk D in G'.

Figure 3.5 Illustration of the disk D centered at a vertex v_2 that is about to be split into two vertices x and y as in Lemma 3.1. Shading indicates the possibility of the presence of other edges.

Lemma 3.1 establishes a way of dividing or splitting a vertex so as to maintain planarity. It permits dividing a vertex in such a way so as to take advantage of Theorem 3.2 in a larger variety of cases.

Lemma 3.1. *Suppose that G is a plane graph with outer face bounded by a cycle C such that $V(C) = \{v_1, \ldots, v_n\}$, $n \geq 3$ ordered clockwise. Let v_2 be part of a cycle, a, b, v_2, also clockwise, where $a, b \notin V(C)$ and the interior of a, b, v_2 is empty. Then it is possible to create a plane graph, G' with outer cycle C' so that $V(C') = \{v_1, x, y, v_3, \ldots, v_n\}$ and if $v_2t \in E(G)$, then either $xt \in E(G')$ or $yt \in E(G')$ (but not both). outer face bounded by a cycle C such that $V(C) = \{v_1, \ldots, v_n\}$, $n \geq 3$ ordered clockwise. Let v_2 be part of a cycle, a, b, v_2, also clockwise, where $a, b \notin V(C)$ and the interior of a, b, v_2 is empty. Then it is possible to create a plane graph, G' with outer cycle C' so that $V(C') = \{v_1, x, y, v_3, \ldots, v_n\}$ and if $v_2t \in E(G)$, then either $xt \in E(G')$ or $yt \in E(G')$ (but not both).*

Proof. Consider a closed disk D in the plane that contains v_2 but is small enough so as to not contain any other vertex of G. Thus, D passes through all the edges incident to v_2. In particular, it passes through the edges v_2a, v_2b, v_2v_1 and v_2v_3. Label the arcs of D as follows:

- P from the intersection edge from a to the intersection edge from b,
- Q from the intersection edge from b to the intersection edge from v_1,
- R from the intersection edge from v_1 to the intersection edge from v_3,

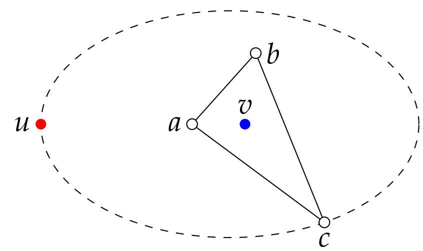


Figure 3.6 A plane graph with a $\{u, v\}$-separating cycle C of length three such that C has one vertex on the boundary that includes u.

- S from the intersection edge from v_3 to the intersection edge from a,

so that Q, R, S, T cover the boundary of D and pairwise intersection of the arcs includes at most the single point along one of the specified edges. The disk D is shown in Figure 3.5a.

Because C is an outer cycle, there are no other edges intersecting R. Likewise, because the interior of a, b, v_2 is empty, there are no other edges intersecting P. Let the set of endpoints of edges intersecting S be A', and the set of endpoints of edges intersecting Q be B'. Then let $A = A' - \{v_2\}$ and $B = B' - \{v_2\}$. Note that $v_3 \in A$ and $a \in A$; similarly, $v_1 \in B$ and $b \in B$.

Construct G' so that it has the same embedding as G outside of D. Inside D, place and x and y so that x is closer to v_1 and y is closer to v_3. Connect the vertices in A to y and the vertices in B to x so that the portions of the edges outside D are the same as they were for the edges running to v_2 in G and so that they do not cross within D. This modification is shown in Figure 3.5b. Note that the second is possible because G is a plane graph and the placement of x and y closer to v_1 and v_3, respectively, allows for these connections. We now have the desired graph G'. $\square$

After having two of the vertices of a separating cycle on the boundary of the outer face, I use the lemma show that having only one of these vertices on the boundary of the outer face is also acceptable. This result is shown for separating C_3 and C_4.

Theorem 3.8. *Suppose that G is a plane graph with a separating cycle C such that $V(C) = \{a, b, c\}$ oriented clockwise and c is on the boundary of the outer face. Let v be a vertex in the interior of C, and u a vertex distinct from a, b, c on the boundary of the outer face. Suppose that Φ is a list assignment for G with $|\Phi(x)| \geq 5$ for all $x \in V(G) - \{u, v\}$ and $|\Phi(u)| = |\Phi(v)| = 1$. Then G is Φ-colorable.*

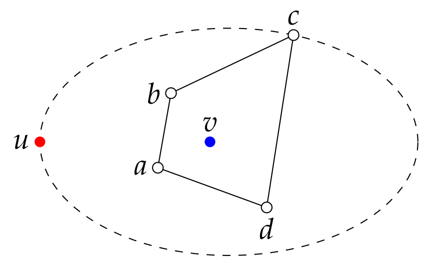


Figure 3.7 A plane graph with a $\{u, v\}$-separating cycle C of length four such that C has one vertex on the boundary that includes u.

Figure 3.6 shows an example of the situation described by Theorem 3.8. It shows a plane graph with a $\{u, v\}$-separating cycle $C = \{a, b, c\}$ such that C has one vertex on the boundary that includes u.

Proof. If either a or b is also on the boundary of the outer face, then, by Theorem 3.4 or Theorem 3.6, there is a Φ-coloring of G. Otherwise, let $\varphi(v)$ be the color in $\Phi(v)$. Call the subgraph of C and its interior H. Then φ can be extended to H using colors available in Φ. Consider the graph G' made by removing the interior of C and modifying c to c_* and c^* as described in Lemma 3.1. In G', a, b, c_*, c^*, and u are all on the boundary of the outer face. Let $\varphi'(a) = \varphi(a)$, $\varphi'(b) = \varphi(b)$, $\varphi'(c_*) = \varphi'(c^*) = \varphi(c)$, and $\varphi'(u)$ be the color in $\Phi(u)$. Then, by Theorem 3.2, φ' can be extended to G' because, even though there are five vertices on the boundary of the outer face, two of them share a color. As φ' agrees with φ on all shared vertices of H and G', letting $\varphi(x) = \varphi'(x)$ for $x \in V(G')$ yields a Φ-coloring of G. This coloring is proper because any vertex in G' that was adjacent to a vertex $x \in V(C)$ was also adjacent to a vertex with color $\varphi(x)$ in G' so that there are no conflicts created by the change in vertex c. $\square$

Theorem 3.9. *Suppose that G is a plane with a separating cycle C such that $V(C) = \{a, b, c, d\}$ oriented clockwise and c is on the boundary of the outer face. Let v be a vertex in the interior of C, and u a vertex distinct from a, b, c, d on the boundary of the outer face. Suppose that Φ is a list assignment for G with $|\Phi(x)| \geq 5$ for all $x \in V(G) - \{u, v\}$ and $|\Phi(u)| = |\Phi(v)| = 1$. Then G is Φ-colorable.*

Figure 3.7 shows an example of the situation described by Theorem 3.9. It shows a plane graph with a $\{u, v\}$-separating cycle $C = \{a, b, c, d\}$ such that C has one vertex on the boundary that includes u.

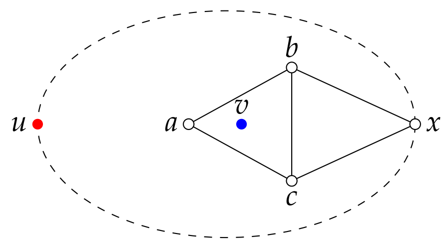


Figure 3.8 A plane graph with a $\{u, v\}$-separating cycle C of length three such that there is a vertex x on the boundary that includes u and is adjacent to two vertices of C.

Proof. If either a or b is also on the boundary of the outer face, then, by Theorem 3.5 or Theorem 3.7, there is a Φ-coloring of G. In addition, if there is a vertex that is adjacent to all of a, b, c, and d, then we are in an instance of Theorem 3.6 which means that there is a Φ-coloring of G.

Otherwise, let $\varphi(v)$ be the color in $\Phi(v)$. Call the subgraph of C and its interior H. Then φ can be extended to H using colors available in Φ. Consider the graph G' made by removing the interior of C and modifying c to c_* and c^* as described in Lemma 3.1. In G', a, b, c_*, c^*, d, and u are all on the boundary of the outer face. Let $\varphi'(a) = \varphi(a)$, $\varphi'(b) = \varphi(b)$, $\varphi'(c_*) = \varphi'(c^*) = \varphi(c)$, $\varphi'(d) = \varphi(d)$, and $\varphi'(u)$ be the color in $\Phi(u)$. Then, by Theorem 3.2, φ' can be extended to G'. We know we are not in one of the exceptional cases for Theorem 3.2 because there is no vertex adjacent to all five of a, b, d, u, and either c_* or c^*, and two of the six vertices have the same color.

As φ' agrees with φ on all the vertices shared between H and G', setting $\varphi(x) = \varphi'(x)$ for $x \in V(G')$ yields a Φ-coloring of G. This coloring is proper because any vertex in G' that was adjacent to a vertex $x \in V(C)$ was also adjacent to a vertex with color $\varphi(x)$ in G' so that there are no conflicts created by the change in vertex c. $\square$

The remaining theorems in this section focus exclusively on a separating C_3. These statements focus on moving the separating cycle away from the boundary of the boundary of the outer face. This movement is done by assuming that there is some other cycle with vertices on the boundary of the outer face that shares an edge with the separating C_3. The results then gradually increase the size of this second cycle.

The first case I address relates to having a second cycle that is also a C_3 that has one of its vertices on the boundary of the outer face.

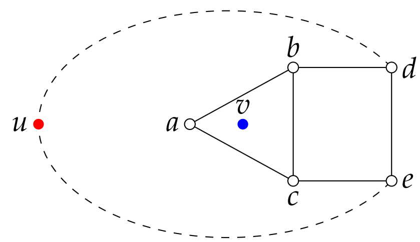


Figure 3.9 A plane graph with a $\{u, v\}$-separating cycle C of length three such that there is an edge de on the boundary of the outer face so that d and e are adjacent to distinct vertices of C.

Corollary 3.1. *Suppose that G is a plane with a separating cycle C such that $V(C) = \{a, b, c\}$ oriented clockwise and there is a vertex x on the boundary of the outer face of G that is adjacent to b and c. Let v be a vertex in the interior of C, and u a vertex distinct from a, b, c, and x on the boundary of the outer face. Suppose that Φ is a list assignment for G with $|\Phi(x)| \geq 5$ for all $x \in V(G) - \{u, v\}$ and $|\Phi(u)| = |\Phi(v)| = 1$. Then G is Φ-colorable.*

Figure 3.8 shows an example of the situation described by Theorem 3.1. It shows a plane graph with a $\{u, v\}$-separating cycle $C = \{a, b, c\}$ such that there is a vertex x on the boundary that includes u and such that x is adjacent to b and c.

Proof. Notice that a, b, x, and c form a separating C_4. As v is in the interior of C, it will also be on the interior of the cycle a, b, x, c. Because u is on the boundary of the outer face, we meet the suppositions of Theorem 3.9 so that G is Φ-colorable. $\square$

The second case is that there is a C_4 with two of its vertices on the boundary of the outer face.

Theorem 3.10. *Suppose that G is a plane graph with a separating cycle C such that $V(C) = \{a, b, c\}$ oriented clockwise and there is an edge de on the boundary of the outer face of G such that $d \sim b$ and $e \sim c$. Let v be a vertex in the interior of C, and u a vertex distinct from a, b, c, d, and e on the boundary of the outer face. Suppose that Φ is a list assignment for G with $|\Phi(x)| \geq 5$ for all $x \in V(G) - \{u, v\}$ and $|\Phi(u)| = |\Phi(v)| = 1$. Then G is Φ-colorable.*

Figure 3.9 shows an example of the situation described by Theorem 3.10. It shows a plane graph with a $\{u, v\}$-separating cycle $C = \{a, b, c\}$ such that there is an edge de on the boundary of the outer face so that d and e are adjacent to b and c, respectively.

Proof. We consider a few possibilities for reduction before proceeding with the coloring.

Suppose there is a vertex w that is adjacent to five vertices in $\{a, b, c, d, e, u\}$.

- If w is adjacent to a, b and c, then w, b, c form a larger separating cycle and we can consider it instead.
- If w is on the boundary outer face of G, then we are in the case of Theorem 3.8 and there is a Φ-coloring.
- Otherwise w must be adjacent to d and e, so that we are in the case of Theorem 3.4, which implies that there is a Φ-coloring.

Next consider the existence of and edge w_0w_1 so that we are in the second type of exception to Theorem 3.2. We then have three cases for the neighbors of w_0 and w_1:

- If w_0 is adjacent to u, d, b, a and w_1 is adjacent to a, c, e, and u, then we have a separating C_4 of G with v on the inside and are in the case of Theorem 3.9. Thus, there is a Φ-coloring of G.
- If, instead, w_0 is adjacent to e, u, d, c and w_1 is adjacent to c, a, b, and e, then d, e, w_0 form a separating C_3 in G with v in its interior. Hence, by Theorem 3.8, there is a Φ-coloring of G.
- The last rotation is that w_0 is adjacent to d, c, a, and b and w_1 to b, e, u, and d, then w_1, d, e form a separating C_3 in G with v in its interior and there is similarly a Φ-coloring of G.

Next consider the existence of a triangle (w_0, w_1, w_2) in accordance with the third type of exception to the second part of Theorem 3.2. Then we have two cases for the neighbors of w_0, w_1, and w_2:

- The first case is that w_0 is adjacent to u, d, and e; w_1 to d, b, and a; and w_2 to a, c, and e. Then w_0, d, and e form a separating C_3 in G, so that Theorem 3.8 guarantees a Φ-coloring.
- The second case has w_0 adjacent to c, e, and u; w_1 to u, d, and b; and w_2 to b, a, and c. Then w_0, w_1, d, e consist of a separating C_4 in G that has v in its interior. Thus, by Theorem 3.9, there is a Φ-coloring of G.

If none of these occur, let $\varphi(v)$ be the color in $\Phi(v)$. Call the subgraph of $\{a, b, d, e, c\}$ and its interior H. Then φ can be extended to H using colors available in Φ. Consider the graph G' made by removing the interior of H

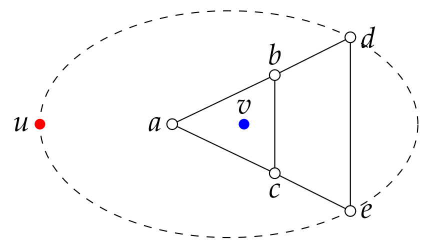


Figure 3.10 A plane graph with a $\{u, v\}$-separating cycle C of length three such that there is a pair of adjacent vertices d and e on the boundary of the outer face so that d and e are adjacent to distinct vertices of C. Furthermore, there can be a path from u to a that does not go through d or e.

as well as the edge de. In G', a, b, c, d, e, and u are all on the boundary of the outer face. Let $\varphi'(x) = \varphi(x)$ for $x \in \{a, b, c, d, e\}$ and $\varphi'(u)$ be the color in $\Phi(u)$. By Theorem 3.2, there is a Φ-coloring of G' that agrees with φ'. Then φ' agrees with φ on all shared vertices. Thus, letting $\varphi(x) = \varphi'(x)$ for $x \in V(G')$ yields a Φ-coloring of G. $\square$

Finally, I allow the second cycle to be larger, by permitting a subdivision of the edge de from Theorem 3.10. However, I do require that u not be one of the vertices along the subdivision.

Lemma 3.2. *Suppose that G is a plane with a separating cycle C such that $V(C) = \{a, b, c\}$ oriented clockwise and there is a pair of vertices d, e on the boundary of the outer face of G with $d \sim e$, $d \sim b$, and $e \sim c$. Let v be a vertex in the interior of C, and u a vertex distinct from a, b, c, d, and e on the boundary of the outer face so that there is a path from u to a that does not pass through d or e. Suppose that Φ is a list assignment for G with $|\Phi(x)| \geq 5$ for all $x \in V(G) - \{u, v\}$ and $|\Phi(u)| = |\Phi(v)| = 1$. Then G is Φ-colorable.*

Figure 3.2 shows an example of the situation described by Theorem 3.10. It shows a plane graph with a $\{u, v\}$-separating cycle $C = \{a, b, c\}$ such that there is a pair of adjacent vertices d and e on the boundary of the outer face so that d and e are adjacent to b and c respectively. In addition, there can be a path from u to a that does not go through d or e.

Proof. We consider a few possibilities for reduction before proceeding with the coloring.

Assume there is a vertex w that is adjacent to five vertices in $\{a, b, c, d, e, u\}$.

- If w is adjacent to a, b and c, then w, b, c form a larger separating cycle and we can consider it instead.

- If w is on the boundary of outer face of G, then we are in the case of Theorem 3.8 and there is a Φ-coloring.
- Otherwise w must be adjacent to d and e so that we are in the case of Theorem 3.4, which implies that there is a Φ-coloring.

Next consider the existence of and edge w_0w_1 so that we are in the second type of exception to Theorem 3.2. We then have three cases for the neighbors of w_0 and w_1:

- If w_0 is adjacent to u, d, b, and a and w_1 is adjacent to a, c, e, and u then we have a separating C_4 of G with v on the inside and are in the case of Theorem 3.9. Thus, there is a Φ-coloring of G.
- If, instead, w_0 is adjacent to d, u, e, and c and w_1 is adjacent to c, a, b, and d, then d, e, w_0 consist of a separating C_3 in G with v in its interior. Hence, by Theorem 3.8, there is a Φ-coloring of G.
- The last rotation is that w_0 is adjacent to b, a, c, and e and w_1 to e, u, d, and b then w_1, d, e consist of a separating C_3 in G with v in its interior and there is similarly a Φ-coloring of G.

Finally, consider the existence of of a triangle (w_0, w_1, w_2) in accordance with the third type of exception to the second part of Theorem 3.2. Then we have two cases for the neighbors of w_0, w_1, and w_2:

- The first case is that w_0 is adjacent to u, d, and e, w_1 to d, b, and a and w_2 to a, c, and e. Then w_0, d, and e consist of a separating C_3 in G, so that Theorem 3.8 guarantees a Φ-coloring.
- The second case has w_0 adjacent to c, e, and u, w_1 to u, d, and b and w_2 to b, a, and c. Then w_0, w_1, d, e consist of a separating C_4 in G that has v in its interior. Thus by Theorem 3.9 there is a Φ-coloring of G.

If none of these occur, let $\varphi(v)$ be the color in $\Phi(v)$, and let P be the path between d and e that does not contain u. Call the subgraph of a, b, d, e, c, $V(P)$, and its interior H. Then φ can be extended to H using colors available in Φ. Consider the graph G' made by removing the interior of H as well as the edge, de. In G', a, b, c, d, e, and u are all on the boundary of the outer face. Let $\varphi'(x) = \varphi(x)$ for $x \in \{a, b, c, d, e\}$ and $\varphi'(u)$ be the color in $\Phi(u)$. By Theorem 3.2, there is a Φ-coloring of G' that agrees with φ'. Then φ' agrees with φ on all shared vertices. Thus letting $\varphi(x) = \varphi'(x)$ for $x \in V(G')$ yields a Φ-coloring of G. □

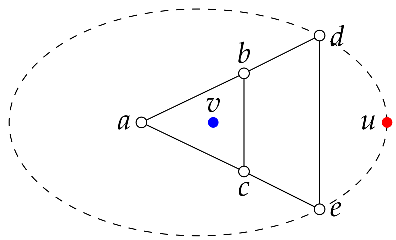


Figure 3.11 A plane graph with a $\{u, v\}$-separating cycle C of length three such that there is a pair of adjacent vertices d and e on the boundary of the outer face so that d and e are adjacent to distinct vertices of C. Furthermore, there cannot be a path from u to a that does not go through d or e.

Lemma 3.3. *Suppose that G is a plane with a separating cycle C such that $V(C) = \{a, b, c\}$ oriented clockwise and there is a pair of vertices d, e on the boundary of the outer face of G with $d \sim e$, $d \sim b$, and $e \sim c$. Let v be a vertex in the interior of C, and u a vertex distinct from a, b, c, d, and e on the boundary of the outer face so that there is* not *a path from u to a that does not pass through d or e. Suppose that Φ is a list assignment for G with $|\Phi(x)| \geq 5$ for all $x \in V(G) - \{u, v\}$ and $|\Phi(u)| = |\Phi(v)| = 1$. Then G is Φ-colorable.*

Figure 3.11 shows an example of the situation described by Theorem 3.3. It shows a plane graph with a $\{u, v\}$-separating cycle $C = \{a, b, c\}$ such that there is a pair of adjacent vertices d and e on the boundary of the outer face so that d and e are adjacent to b and c respectively. In addition, there cannot be a path from u to a that does not go through d or e.

Proof. Let $v_1, \ldots, v_k$ be a path along the outer cycle so that $v_1 = d, v_k = e$ and that $v_i = u$ for some $1 < i < k$. Then $b, v_1, \ldots, v_k, c, b$ is a cycle. Call this cycle and its interior H.

Assume, without loss of generality, that $\{a, b, c\}$ is the largest separating C_3 that contains v and the edge bc; that is, if there is a vertex w adjacent to b and c, then w should be interior to $\{a, b, c\}$.

First, let $\varphi(v)$ be the color in $\Phi(v)$. Then, by Thomassen's Theorem, φ can be extended to $\{a, b, c\}$ and its interior. Next consider, the subgraph H together with the assignment Φ. Let $\varphi(u)$ be the color in $\Phi(u)$. Recall that H has outer cycle $b, v_1, \ldots, u, \ldots, v_k, c, b$. Of these vertices φ has assigned colors to b, c and u. By Theorem 3.2, we can extend then φ to H.

Next, consider the subgraph G' of G formed by removing the interior of $a, b, v_1, \ldots, v_k, c, a$ along with v_i for all $1 < i < k$. In G', a, b, c, d, and e are all on the boundary of the outer face. Let $\varphi'(x) = \varphi(x)$ for $x \in \{a, b, c, d, e\}$.

By Theorem 3.2, there is a Φ-coloring of G' that agrees with φ' unless there is a vertex w in G' that is adjacent to each of a, b, c, d, and e. However, in that case there would be a larger separating C_3 using the edge bc, which we assumed was not the case. Thus, we extend the coloring φ' to all of G'.

Notice that φ' agrees with φ on all shared vertices to which they both assign colors. Thus, letting $\varphi(x) = \varphi'(x)$ for $x \in V(G')$ yields a Φ-coloring of G. $\square$

Theorem 3.11. *Suppose that G is a plane graph with a separating cycle C such that $V(C) = \{a, b, c\}$ oriented clockwise and there is a pair of vertices d, e on the boundary of the outer face of G with $d \sim b$ and $e \sim c$. Let v be a vertex in the interior of C, and u a vertex distinct from a, b, c, d, and e on the boundary of the outer face. Suppose that Φ is a list assignment for G with $|\Phi(x)| \geq 5$ for all $x \in V(G) - \{u, v\}$ and $|\Phi(u)| = |\Phi(v)| = 1$. Then G is Φ-colorable.*

Proof. Either there is a path from u to a that does not contain one of b, c, d, or e, or there is no path from u to a that does not contain b, c, d or e. In the first case, by Lemma 3.2, there is a Φ-coloring of G. In the second, by Lemma 3.3, there is a Φ-coloring of G. $\square$

Chapter 4

Future Directions and Conclusions

There are a number of directions that can be pursued based off the results obtained and the research done for this thesis.

4.1 Minor Free Graphs

Pruchnewski and Voigt (2009) present an algorithm that shows that list colorings can be extended from a very particular set of vertices in K_4-minor–free graphs. They then use this algorithm to show that, given a list assignment with lists of length four on an outerplanar graph (which is $K_{2,3}$- and K_4-minor–free), a precoloring of a bipartite subgraph can be extended.

One future direction of inquiry would be to analyze the assumptions of their algorithm as well as how the forbidden minors are used in the proof to see if these results can be strengthened.

4.2 Continuing with Distance Constraints

Another direction involves papers written by Albertson and Moore. In Albertson and Moore (1999), the authors discuss precoloring cliques that are a certain distance apart. They prove their results using Kempe chains. Recall that a Kempe chain is a path consisting of vertices that alternate in two colors. In Albertson (1999), he discusses extending colorings without using extra colors. This topic is of particular interest because Theorem 2.2 already addresses the case when the list assignments are larger than χ_ℓ. Finally, in Albertson and Moore (2001), they show that colorings can be extended without an extra color as long as one of the colors is infrequent. Their results, again, use Kempe chains and take advantage of the fact that these chains cannot grow much due to the constraints on the infrequent color.

Working on distance constraints may involve further analysis of these papers and looking into the possible existence of an analog to Kempe chains for list colorings.

4.3 Recent Paper

The paper by Axenovich, Hutchinson, and Lastrina (2010) involves a number of rich results, not all of which are included here. While I looked at improving the restriction about small separating cycles, the authors suggest that the distance constraints in some of their other theorem could be improved through a more detailed and careful treatment.

Another direction would be to continue to work with small separating cycles. The last three results in Section 3.2 involve only C_3. One way to push further would be to try to find analogues of these three results for a separating C_4. Another would be to push a separating C_3 even further from the face whose boundary includes u.

4.4 Conclusions

In conclusion, the work in this thesis falls into three categories. The first, covered in Section 2.1, gives explicit proofs for precoloring vertices with a distance constraint. These proofs follow the techniques found in Albertson (1998). The second, covered in Section 2.2, discusses precoloring a small number of vertices that are all on the same face. The third, covered in Section 3.2, examines some cases of precoloring two vertices, u and v, where there exist specific types of small $\{u, v\}$-separating cycles.

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