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From Counting to Quaternions – The Agonies and Ecstasies of the Student Repeat Those of D'Alembert and Hamilton

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Synopsis

Young learners of mathematics share a common experience with the greatest creators of mathematics: "hitting a wall," meaning, first frustration, then struggle, and finally, enlightenment and elation. We tell two intertwined stories. One story is about children, first learning to add, then later learning about negative numbers, still later about complex numbers, and finally about quaternions. Intertwined is the history of the development of number systems – from Galileo and Bombelli through D'Alembert, Euler and Gauss, up to Hamilton and Cayley.

This article is intended to show a parallel between the struggle of the pupil and the struggle of the researcher. This parallel is useful in two ways. It sheds light on the history of mathematics, and it has important consequences for the teaching of mathematics.

We start with a few sentences from an interview by Srinivasa Varadhan, the famous probabilist at the Courant Institute in New York [12, p.243]:

> **Varadhan:** If only you could do that one thing, then you would have the whole building, but this foundation is missing. So you struggle and struggle with it, sometimes for months, sometimes for years and sometimes for a lifetime! And eventually, suddenly one day you see how to fix that small piece. And then the whole structure is complete. That was the missing piece. Then that is a real revelation, and you enjoy a satisfaction which you cannot describe.
>
> **R&S:** How long does the euphoria last when you have this experience?

> **Varadhan:** It lasts until you write it up and submit it for publication. Then you go on to the next problem.

From this report of the life of the research mathematician, we now turn to an injunction for math teachers:

> "Anyone who is going to be a math teacher needs to run up against a wall and overcome it. Teachers need to help students see through the process, believe in themselves, not give up because it's frustrating, understand and come though to the other side." (Prof. Kristin Umland, speaking about her program for middle-school teachers in New Mexico.)

"Run up against a wall and overcome it! Break through frustration to understanding!" This describes the struggle by which a child learns elementary mathematics; it also describes the struggle by which a mathematician creates mathematics.

Our story will start with a 4-year-old, who is elated to discover that "$2 + 1 = 3$". Then we move up to the challenge children meet in the 4th and 5th grades, when they have to master the so-called "negative numbers," numbers which are "impossible," since "you can't have less than nothing." (Mathematicians also once thought such numbers were impossible.) Then, in the 11th grade, there come the imaginary numbers. They're called "imaginary" because for centuries they were thought to be meaningless. The discovery of their geometric meaning was one contribution of the great Carl Friedrich Gauss. Finally, we arrive at the quaternions, the major creation of William Rowan Hamilton. Both Hamilton and our young protagonists have a hard struggle, to give up the commutative law. We even include a brief mention of a strange structure beyond quaternions – the octonions! Each time, we start with confusion and frustration, then break through to elation and relief.

To develop our parallel between advanced research and early learning, we indulge in a little fiction. The imaginary twins Imre and Ludwig have already appeared in *The College Mathematics Journal.* Here I use them and their younger friend Traci to depict the emotions of learning negative and imaginary numbers and quaternions. In parallel, telling the real history of those new kinds of numbers, we go from Galileo, Cardano, Bombelli and D'Alembert, up through Stendhal, De Morgan, Gauss, Argand, Hamilton and Graves, arriving at Maxwell, Heaviside, Tait, Gibbs, and John Baez.

Imre, Ludwig and Traci are real to me, in their own way. Perhaps Huck Finn was real to Mark Twain, as Madame Bovary was real to Flaubert.

1. Counting

We start at age 4. In *Children and Number*, [9, p.145] Martin Hughes reports some mathematical experiences with 4-year-olds.

> ...understanding comes quite dramatically to Debbie (4 years 11 months). I had in front of me tins bearing the numerals "0" and "3," whereas she had the tins bearing the numerals "2" and "1."
>
> **MH:** Who has more, or do we both have the same?
>
> **Debbie:** I've got the most.
>
> **MH:** How many do you have?
>
> **Debbie:** (Pause. Opens her tins and counts.) One... two ... three. I've got three.
>
> **MH:** And how many have I got?
>
> **Debbie:** (Is about to open my tins when she exclaims in appreciation) Oh! We've both got the same! (She then points to her tins.) One...and two..makes three!
>
> The expression on Debbie's face needed to be seen to be believed. It showed that she was having what psychologists have termed an "Aha" experience, characterized by the sudden dawning of understanding. Her face lit up and she looked from the boxes away into the distance.
>
> ...I pursued this line of enquiry with my own son Owen (3 years 9 months). He was faced with exactly the same problem as Debbie and Gavin.
>
> **MH:** Who's got more, me or you?
>
> **Owen:** You? (Looks at me with a slight grin, uncertain.)
>
> **MH:** You're not sure, are you? How could we find out who's got more bricks in their tins?
>
> **Owen:** I don't know (rather doleful).
>
> **MH:** You don't know how to find out?

Owen: No... (Looks at boxes. Suddenly looks up at me very alertly.) We've both got the same!

MH: Why do you think that?

Owen: Because one (picks up the "1" and puts it back) and two more (picks up the "2" and puts it back)!

MH: Is?

Owen: Three!

As with Debbie, Owen's face suddenly lit up as he made the connection between the concrete situation and the abstract expression "One.. and two more... is three."

Now make a big move, from four-year-old Debbie and Owen to the great Galileo Galilei. In the famous *Dialogues Concerning Two New Sciences* [7], Salviati is the alter ego of Galileo himself; Simplicio is his earnest but somewhat slow-witted friend.

Salviati: I take it for granted that you know which of the numbers are squares and which are not.

Simplicio: I am quite aware that a squared number is one which results from the multiplication of another number by itself; thus 4, 9, etc. are squared numbers which come from multiplying 2, 3, etc. by themselves.

Salviati first gets Simplicio to accept what is certainly a very clear and obvious fact:

Salviati: ...if I assert that all numbers, including both squares and non-squares, are more than the squares alone, I shall speak the truth, shall I not?

Simplicio: Most certainly.

Then one more equally clear and obvious fact:

Salviati: If I should ask further how many squares there are one might reply truly that there are as many as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root, and no root more than one square.

Simplicio: Precisely so.

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9...**
1	**4**	**9**	**16**	**25**	**36**	**49**	**64**	**81...**

The perfect squares are equinumerous with the natural numbers, even though most natural numbers are not perfect squares.

But now comes the contradiction!

> **Salviati:** But if I inquire how many roots there are, it cannot be denied that there are as many as there are numbers because every number is a root of some square. This being granted we must say that there are as many squares as there are numbers because they are just as numerous as their roots, and all the numbers are roots. Yet at the outset we said there are many more numbers than squares, since the larger portion of them are not squares.

The discovery by Debbie and Owen that $2 + 1 = 3$, and the discovery by Galileo that the cardinality of all numbers equals the cardinality of the squares, are both "aha!" experiences.[1] One, by a 4-year-old, at her very beginning of abstract thought. The other, by a great scientist, at the very beginning of modern thinking about the infinite. Miles apart, intellectually. The 4-year-old and the great scientist both go from confusion to insight, and there is a shared emotion – escape, from frustration to elation.

2. Negative numbers

> "There is a story of a German merchant of the fifteenth century, which I have not succeeded in authenticating, but it is so

[1] From Salviati's contradiction Salviati-Galileo concluded that it's impossible to compare infinite sets. One can not say that one infinite set is greater than another, nor can one say that they are equal. 250 years later, Georg Cantor resolved the paradox. With infinite sets, one must distinguish between two different kinds of comparison. Inclusion or containment is one kind. One-to-one correspondence is a different kind. For finite sets these two kinds of comparison must give the same answer. But for infinite sets they can give different answers! In this way Cantor initiated the theory of sets, a new branch of mathematics.

> characteristic of the situation then existing that I cannot resist the temptation of telling it. It appears that the merchant had a son whom he desired to give an advanced commercial education. He appealed to a prominent professor of a university for advice as to where he should send his son. The reply was that if the mathematical curriculum of the young man was to be confined to adding and subtracting, he perhaps could obtain the instruction in a German university; but the art of multiplying and dividing, he continued, had been greatly developed in Italy which, in his opinion, was the only country where such advanced instruction could be obtained." (Tobias Dantzig, *Number, The Language of Science* [4])

Now we go from 4-year-olds, and from a genius of the scientific Renaissance, to my two friends, the 8th-graders and twins Ludwig and Imre. They are prodigies, in a way. But contrary, obstinate prodigies, who insist on seeing things their own way. When I catch up with them at a table in a local sweet shop, they are helping their young friend Traci. Traci is only a fourth grader.

"Look how much fun you can have with these negative numbers!" Imre is saying. "You can add, and the answer is less! You can subtract and the answer is more!"

Traci's lips were quivering a little. "How can you have less than nothing? I just don't get it. It doesn't make sense!"

Ludwig tried to help.

"How much money do you have, Traci?"

"Eighty cents."

"All right. Say you buy *Mad Magazine*, and it costs a dollar. Then you're short twenty cents, right?"

"Sure, so what?"

"Well, then that's like the opposite of having twenty cents. So we call being short of money 'negative,' and it's like the opposite of having money. Then if you find two dimes, that cancels the twenty you're short, and you come out even – zero! Twenty positive plus twenty negative, total zero."

Traci stared from Imre to Ludwig and back. "If I'm short twenty cents, it's twenty cents! How can you add what you don't have to what you do have? Adding is putting things together. Adding makes things bigger, not smaller!"

The twins groaned and shrugged.

It was my turn to be helpful.

"Actually, kids, once upon a time famous mathematicians found this confusing, too. Did you ever hear of Girolamo Cardano?"

They didn't even bother to answer. Traci probably never heard of any mathematician, let alone an Italian mathematician from the sixteenth century.

But I thought it would cheer her up, to know that Cardano, Descartes, and other mathematicians, up to Augustus De Morgan in the 1850's, were not happy, if they solved an equation, and got as the answer what we today call "a negative number." Such a solution was meaningless trash, a sign that the problem had been formulated incorrectly. "A 'correctly formulated problem' has a *meaningful* solution – a positive number or zero."

Why must little Traci "get it," in five minutes, if the professionals took something like 300 years?

"Don't you see, kids," I said with a smile, "the trouble is, there are two different kinds of numbers."

"You mean positive numbers and negative numbers?" said Ludwig.

"No. I mean, there is one kind of numbers for pennies or apples, where you can't have less than nothing. And there is another kind of numbers for going up and down a ladder, or going North and going South, where you have two opposite directions. Then two steps down cancels two steps up."

"Those are two different kinds of numbers?" asked Imre.

"Sure," I said. "The first kind are called 'natural numbers,' because they come natural. The second kind are called 'integers.' That comes from Latin. It's easy to understand, as long as you don't mix up the two kinds of numbers. Money is the first kind, if you just have coins and dollar bills. Once you have a charge account, where you can owe money, then money is like a ladder, where you can go above or below zero."

"Hey, that's some big mess!" shouted Ludwig enthusiastically. "When did you math geniuses figure that out?"

"Actually," I had to admit, "it took a long time. People had the idea that numbers are something holy, in some kind of heaven, and they thought we had to figure out the real truth about them. All the while, numbers were really just something we make up, something to use, in buying and selling, or measuring and counting. We need two different kinds of numbers, for two different situations. One for solid things, pennies or apples. Another for steps where you can go in two opposite directions."

"Well," said Ludwig, "The rules for adding and subtracting positive and negative numbers make sense. It's like moving up and down a ladder. What bugs me is, minus times minus! I know, I know, it has to be plus. Everything works out, just follow that rule. But it still bugs me. How can you get a plus out of multiplying minuses? Who said so in the first place? Is it really the truth?"

"OK. We'll work it out. But first, can I get you kids anything? A drink? A snack?"

"Never mind drinks and snacks" shouted Imre. "Just explain it to us like you said you would!"

"All right. So what is multiplication, anyhow?"

"It's a kind of counting," he said. "Like, multiplying by two, that's the same as counting by twos. If you count by twos three times, that's three times two."

"Very good!"

Pause.

"And when you tell me that, what are the twos and the three? Are they natural numbers, or integers?"

A pause for thought, and scratching of cheeks and lips. Traci answered first. "Counting numbers."

"OK! So then, multiplication makes sense for the counting numbers, the natural numbers."

"Right."

"But why would you ever want to multiply negative numbers?"

No answer for a while.

Then, Imre: "Why not?"

The others looked at him. "I mean, go ahead, do it! See what happens! Maybe it will be fun! Maybe it will be good for something!"

The other two nodded.

"OK," I said. "So we want to try and see if we can do multiplication with negative numbers. We have to see what the rules should be. But, what does it mean, to multiply negative numbers? How do you give rules for doing something, when you don't know what it means?"

"It's easy," said Ludwig. "It means reversing. Like, minus one times anything, doing that should turn the thing into its opposite. Positive into negative, negative into positive."

Traci said, "But why should it?"

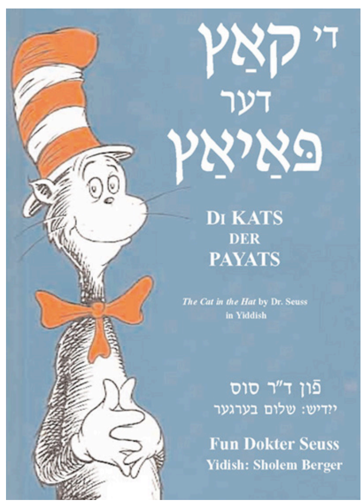
Imre said, “Reversing something isn’t multiplying it! Multiplying is just adding twice, or three times, or whatever. Multiplying by minus one? That doesn’t make sense. It doesn’t mean anything.”

“Right!” I said. “Multiplying by minus one doesn’t mean anything! So then, we can decide what we want it to mean!”

“We can?” asked Traci with wide eyes. “Who said so? What law says we have a right to do that?”

“Well, who said that ‘dog’ should mean dog? In Yiddish there’s a different word, ‘hoont.’ In Yiddish ‘hoont’ means ‘dog,’ and in Yiddish the sound ‘dog’ doesn’t mean anything. Who says what word sounds should mean? All that is something we decided, we speakers of English or French or Chinese, according to our own customs. If multiplying by negative one doesn’t mean anything to begin with, then we have a right to decide what we want it to mean.”

Silence.



I pursued my advantage. “So, you’re right, we decided that multiplying by minus one means reversing sign–turning positive to negative, and negative to positive. Once you agree to that, what is minus one times minus one?”

“Double reversal, back to where you started!” said Ludwig. “Like double negative in English grammar.”

“Exactly. And if you end up back where you started, what are you multiplying by?”

The children gratified me by a chorus: “Plus one!”

Jean Le Rond d'Alembert took his middle name from the church Saint-Jean Le Rond, where he was found new-born in 1717. When he grew up, he educated himself to be a physicist and mathematician. With Diderot, Rousseau and Condillac, he became one of the "Encyclopedists" who played a major role in forming the modern mind. He won the friendship of both Voltaire and Frederick the Great, and in 1754 D'Alembert became perpetual secretary of the *Académie Francaise*. As such, he was perhaps the most influential scientist in Europe.

His article "Negatif" in the *Encyclopédia* [3] (also see [2] was once praised for its clarity, it was used as a reference for almost a hundred years. But to us today, it is "as clear as mud." D'Alembert wrote:

> "One must admit that it is not a simple matter to accurately outline the idea of negative numbers, and that some capable people have added to the confusion by their inexact pronouncements. To say that the negative numbers are below nothing is to assert an unimaginable thing ... in computations, negative magnitudes actually stand for positive magnitudes that were guessed to be in the wrong position. The sign '$-$' before a magnitude is a reminder to eliminate and to correct an error made in the assumption... there exists no isolated negative magnitude in the real and absolute sense, abstractly; -3 communicates no idea to the mind... In a work of the nature of the present one it is impossible to develop this idea further, but it is so simple that I doubt that it can be replaced by one still clearer and more exact; and I believe that I can guarantee that its application to all problems that are solvable and include negative magnitudes will never lead to error. Be that as it may, the rules of algebraic operations with negative magnitudes are accepted by the whole world, and in general, regarded as exact."

Here is d'Alembert's explanation why a minus times a minus is a plus. (If you find this confusing, don't blame yourself, it really is confusing.)

> "When I say that one man gave another -3 thaler, this means in understandable parlance that he took from him 3 thaler. That is why the product of $-a$ and $-b$ is $+ab$: for the symbol '$-$' that precedes by assumption both a and b proves that the magnitudes a and b are conjoined and combined with others with which they

> are comparable; for if they are viewed as standing by themselves and being isolated then the '$-$' sign that precedes them would communicate nothing clearly graspable to the mind. The only reason that the '$-$' sign appears in the magnitudes $-a$ and $-b$ is that some error is hidden in the assumption of the problem or the calculation: if the problem had been well formulated, then each of the magnitudes $-a$ and $-b$ would turn up with the '$+$' sign and the product would be $+ab$; for the multiplication of $-a$ by $-b$ signifies that one subtracts the negative magnitude $-a$ b times; in terms of the previously given notion of negative magnitudes, adding or assigning a negative magnitude amounts to subtracting a positive one; for the same reason, subtracting a negative is the same as adding a positive; and so the simple and natural formulation of the problem is not to multiply $-a$ by $-b$ but $+a$ by $+b$, which yields the product $+ab$."

To put it plainly, after telling us that $-a$ by itself conveys nothing to the mind, he proceeds directly to tell us to subtract that very same thing, $-a$.

This article "repelled" the great novelist "Stendhal" (Henri Beyle, 1783-1843). As an adolescent Beyle had an "idolatrous enthusiasm for mathematics." He wrote [13]:

> "My father and grandfather owned the folio edition of the Encyclopedia of Diderot and d'Alembert that costs from seven to eight hundred francs and may have cost much more in their time. It takes a great deal for a provincial to invest that much money in books... I consulted d'Alembert's mathematical articles in the Encyclopdia but I was repelled by their arrogant tone and the absence of awe before the truth. Also I understood very little of them. How burning then was my reverence for truth! How deep was my conviction that it was the queen of the world I was to enter! The theorem minus times minus equals plus caused me a great deal of grief."

How easy to read these words of d'Alembert and Stendhal, and think, with a condescending smile, "Poor, confused, pathetic souls." And how mistaken, how arrogant that would be! We simply close our eyes to the real difficulty they confronted. And so, we fail to appreciate the difficulty for children following in the same path.

We still do recognize the difficulty of "the square root of -1." We still remember, it took hundreds of years for mathematicians to interpret that. But a similar difficulty did happen, and still happens, with the negative numbers.

In the 1980s, when Robert P. Moses, a hero of the struggle for voting rights for Blacks in the southern United States, made "math literacy" a major civil rights project, he discovered that for very many young people, negative numbers are a major obstacle (Robert P. Moses and Charles E. Cobb, Jr., *Math Literacy and Civil Rights*, [11]). The students told him that negative numbers are unreal and confusing. As they were confusing to Cardano, D'Alembert and Stendhal in centuries past.

3. Imaginary numbers

When Imre and Ludwig started learning algebra, they couldn't wait to teach it to Traci. Before long they were spending afternoons calculating squares and cubes of all sorts of expressions in x and y. Traci called it "the numbers and letters game."

The big fight came when Imre figured out how to factor the equation $x^3 - 1 = 0$. Imre discovered that

$$\begin{aligned} x^3 - 1 &= (x-1)(x^2+x+1). \\ (x-1)(x^2+x+1) &= x(x^2+x+1) - 1(x^2+x+1) \\ &= x^3 + x^2 + x - x^2 - x - 1 = x^3 - 1. \end{aligned}$$

"Look!" he shouted. "If x satisfies $x^2 + x + 1 = 0$, then it would have to satisfy $x^3 - 1 = 0$, so then x would be a cube root of 1!"

"But 1 is the cube root of 1," said Traci. "And it doesn't satisfy $x^2 + x + 1 = 0$."

"OK, this would be a different cube root of 1!"

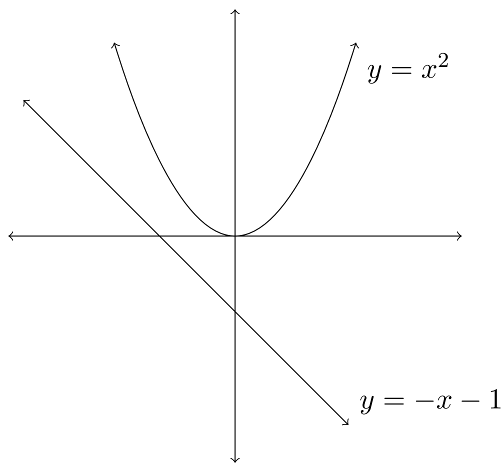
"How can there be any other cube root of 1 except 1? If you cube any number bigger than 1, the cube is too big. If you cube any number less than 1, the cube is too small. 1 has to be the only answer."

Imre started getting angry. "I'll prove there's another cube root! I'll solve $x^2 + x + 1 = 0$."

He stalked off into a corner and started scribbling.

"That's silly," said Traci. "Why waste time trying to solve a problem that can't be solved?"

"Here's another way," Ludwig said to her. "If $x^2 + x + 1 = 0$, then $x^2 = -x - 1$. Look at this!" He drew two lines for an x and a y axis, Then he graphed $y = x^2$. On the same co-ordinates he graphed the line $y = -x - 1$. The graph of the straight line stayed below the graph of the square. "There it is!" he said. "They don't even touch! y can never equal both x^2 and $-x - 1$."



The two graphs for $y = x^2$ and $y = -x - 1$ never meet.
Therefore $x^2 = -x - 1$ has no solution among the real numbers.
Therefore $x^2 + x + 1 = 0$ has no solution among the real numbers.

But Imre was back. "Look, here it is! The other cube root of 1!

$$-1/2 + 1/2\sqrt{-3/4}.\text{"}$$

"Let me see," said Traci. And she multiplied this "x" of Imre's by itself. It turned out that x^2 was almost the same as x. Only with a minus in front of the radical. Then she multiplied x^2 by x itself, to get x^3. Sure enough! All the square roots canceled each other! Out popped the number 1, as Imre claimed.

"How weird," she murmured softly.

"But just tell me one thing, Imre. What is the square root of negative 3/4?"

Long pause.

Imre sucked in his cheeks. "I don't know."

That was all. They stopped talking. They stopped writing and calculating.

The square root of a negative number? A square of any number is always positive or zero. Never negative. The square root of negative $3/4$? What in the world?

> (Leonhard Euler, the super-star of the 18th century, wrote, in his *Algebra* of 1770 [6]: "All such expressions as $\sqrt{-1}$, $\sqrt{-2}$, etc. are consequently impossible or imaginary numbers, since they represent roots of negative quantities, and of such numbers we may truly assert that they are neither nothing, nor greater than nothing, nor less than nothing, which necessarily constitutes them imaginary or impossible." The outstanding English mathematician Augustus De Morgan, in 1849, in his *Trigonometry and Double Algebra* [5], wrote, that although imaginary numbers have no existence as quantity, they are "permitted, by definition, to have an existence of another kind, into which no particular inquiry was made.")

When I showed up, Traci and the twins were happy to see me. It didn't take them long to tell me what had happened.

Ludwig and Traci stuck to their original opinion. "There is only one cube root of 1, and the equation $x^2 + x + 1 = 0$ has no solution."

Imre wouldn't give up defending $-1/2 + \sqrt{-3/4}$ as also a cube root of 1. The proof is, just cube it, you do get 1! True, he didn't know what the square root of $-3/4$ might be. Nevertheless, figures don't lie. If you cube something and get 1, it's a cube root of 1!

They expected me to make peace. But it was going to take some time and work.

I started out asking, "If $x = (-1/2 + \sqrt{-3/4})$, what is x squared?"

Traci answered right away. "I figured that out. It's $(-1/2 - \sqrt{-3/4})$."

"All right. Now if cubing x gives you 1, what do you get if you take x^2 and cube that?"

Ludwig saw it immediately. "x squared, then cubed, is the same as x cubed, then squared. So the answer is 1!"

"So," I said, "x^2 is another cube root of 1, in addition to x. And of course, there's 1 itself. How many does that make?"

"Three," they murmured quietly, all three children thinking together.

"And how many square roots does 1 have?"

"Two. Plus one and minus one."

"Two square roots, and three cube roots. All right. Now we have a puzzle here, we're confused about the cube roots of 1. There seem to be three, but two of them don't make sense. What should we do?"

They waited. Ludwig said, "What?"

"Well," said I, "what's the next number after 3?"

They didn't get it.

"Come on," I said, "You know that."

Imre spoke. "You want us to worry about the fourth roots, when we don't understand the cube roots?"

"Why not?" said I. "Do you have a better idea?"

No argument.

I went on. "Take a guess. How many fourth roots ought there to be?"

"Four!" shouted Traci.

"Very good," said I. "How many do you know already?"

"Two," they chorused. "Plus one and minus one."

"OK," said I, "Where can we find two more?"

Imre got excited. "It's easy! The fourth root would be the square root of the square root. Two square roots of plus one, and two square roots of minus one, makes four fourth roots."

"Hold on!" yelled Ludwig. "Square roots of plus one, we've already got those–plus one, and minus one. Bur square roots of minus one, what are you talking about? There aren't any!"

I smiled happily. They waited for me to make peace.

"Exactly," I said. "Now do you see why we had to look at the fourth roots in order to understand the cube roots?"

"No." On this they agreed.

"Well," said I, "You're worried about the square root of $-3/4$, in the cube root problem. Suppose you know what is the square root of -1. Would that help?"

"Yes, I guess it would," said Ludwig.

Before he could continue, Traci was shouting, "Then the square root of $-3/4$ would be the square root of -1 times the square root of $+3/4$. And we can figure out the square root of $3/4$, as close as we want to."

I nodded sagely.

But then immediately Traci returned to her skeptical mode. "But so what? You still have to tell us what is the square root of -1. And you know there isn't any. So what is the use of all this fooling around?"

I just said, "Don't be so sure."

They waited to hear more.

"You remember," I started, "Traci, when you claimed that '3 minus 5 doesn't make sense,' because you can't take 5 from 3."

She blushed. "Well, I was just a little kid then, that was last year."

"No, you were right," I answered. "With the natural numbers, you really can't take 5 from 3. But we found a different kind of numbers, the integers. With the integers, 5 from 3 makes sense, it's negative 2."

I noticed a little brightening of the eyes, a little more hopeful expressions. "Yes," I said, "It's the same thing again. With the integers, and even with the fractions, and even with the irrationals, the square root of minus one doesn't make sense. But we can make up a new kind of number, and then it will make sense."

"Look. Here are the positive numbers."

I drew a line on a sheet of paper, stretching to the right from an origin.

"And now I put in the negatives, like we did last year."

I made the line go right and left from the origin.

"Now," said I, "What's still missing?"

Of course they didn't have a clue.

"I'll help you," I said.

"What happens to numbers when you multiply by minus 1?"

"The positives become negative, and the negatives become positive."

"Yes. In this picture that I'm showing you, the right half of the line flips over and goes to the left, and the left half flips over to the right, yes?"

"Yes."

"So multiplying by minus 1 turns the line around, turns it through 180 degrees, a half turn."

"Right."

"Now, how could I make that happen in two steps? What can I do, that if I do it twice, accomplishes a half turn?" A quick answer from all three. "A quarter turn."

"Right! And if I call that quarter-turn a kind of multiplication, than doing it twice would be multiplying by its square, wouldn't it?"

"Yes."

"So, a quarter-turn squared would be a half-turn."

"Yes."

"And the half-turn is multiplication by minus 1, isn't it?

"Yes."

"So the quarter-turn squared is minus 1, right?

They sat open-mouthed.

"Then the quarter-turn must be—"

I waited. And after a few seconds, the answer came, "The square root of minus one!"

"The square root of minus one is a number, such that, when you multiply by it, it produces a quarter-turn."

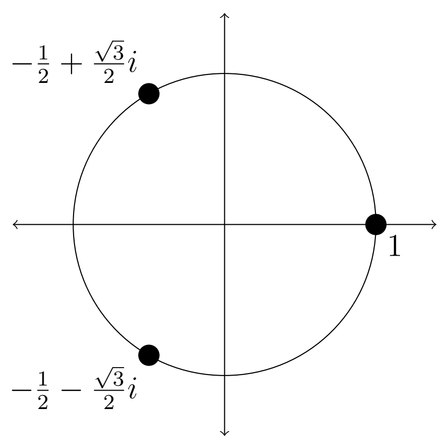
"Then, where is square root of minus one in your picture?"

"Easy. It is just the result of multiplying the number 1 by the square root of minus one. So, what do you get, if you do a quarter-turn to the line segment from zero to plus one? You get a vertical line segment, pointing upwards, one unit long."

After that, it didn't take long to locate the number $-1/2 + \sqrt{-3/4}$. We did it by first moving a distance 1/2 to the left from the origin, to take care of the $-1/2$, and then moving up, in the direction of (square root of minus 1), a distance equal to the positive square root of 3/4.

There it was, Imre's mysterious x. And x^2 was the mirror image of x, below the horizontal axis.

So there were the three cube roots of 1 staring us in the face. Three points, located at equal angles to each other, around the origin.



The three cube roots of 1.

Then we put in the four fourth roots of $+1$, at right angles to each other. One pointing to the right. That's $+1$. One pointing up. That's i. One pointing to the left. That's -1. And one pointing down. That's $-i$. Four equally spaced points around the origin.

So much more could be done, so many more questions to answer!

The cube roots of one, and the fourth roots of one, all are at a distance 1 from the origin. The set of all points at distance 1 from the origin makes a circle, it's called "the unit circle." It's not hard to show that if you multiply two points on the unit circle, you get a product that's on the unit circle. Its position on the unit circle is determined by the angle it makes with the positive x-axis (the "real axis.") In fact, if you multiply two points on the unit circle as complex numbers, their product on the unit circle has an angle with the positive x-axis which is just the sum of the angles of the two factors. In other words, multiplication of complex numbers on the unit circle is just rotation around the origin.

The path by which Imre, Ludwig and Traci came to complex numbers is not essentially different from the historical path. Complex numbers jump in your face when you solve quadratic equations. But you can reject them, you can deny that they are legitimate solutions. When Cardano and Bombelli solved cubic equations in the 16th century, their formula forced them to use complex numbers. They used the formula, but they didn't understand its meaning. Hundreds of years passed before Wessel, Argand and Gauss discovered that complex numbers could be thought of as points in the plane.

4. Quaternions and Octonions

A couple of weeks later Ludwig found me at work and was desperate to talk. The complex numbers really had him turned on.

"Every point is just a kind of number!" he warbled. "You can add and multiply points! You can move points around by multiplying or adding! It's beautiful!"

I was pleased by his enthusiasm. "Great," I said, "You're happy, I'm happy."

"Well, but there is just one thing."

"Something is lacking?"

"It's all just in the plane, in two dimensions."

"That's right."

"But actually, real space is three dimensional, isn't it?"

"Maybe even more than three. At least three, anyway."

"So why stop with just two-dimensional complex numbers? We need three-dimensional complex numbers!"

"Great idea!" said I.

"Then all this stuff you can do in two dimensions with complex numbers would work also in three! Water flowing! Gravity fields! Electromagnetic stuff. Why not? What's the hold-up?"

"You're thinking," I complimented him. "Of course, others also have had that thought."

"I guess so," he admitted a little reluctantly. "I guess it is kind of obvious. But where are three-d complex numbers? What's the hold-up?"

"Well, it's a great story. An Irishman named Hamilton gets most of the glory."

"Did he do it?"

"Yes and no."

"What is that supposed to mean?

"Well, he tried hard to do it. And he did invent something beautiful, which he called 'Quaternion.' But they weren't quite what he had in mind. Or you have in mind."

"OK, what's quaternions?"

"Let's try to figure it out ourselves right now."

"Really? Can we do that?"

"Why not?"

"I don't know. Might be really hard."

"Don't worry. It can't kill us."

"Right. Math can't kill anybody."

"Not usually."

"So how do we start?"

"Do you have any ideas?"

"Not really."

"Try thinking for ten minutes."

After three minutes his eyes brightened and he was ready to go.

"Let's take the idea that worked for two-d, and try to extend it up to three!"

"Good. Go ahead!"

"All right. At first we just had the x-axis, which was a real number axis. Then we put an imaginary axis perpendicular to it. It has a unit vector we call i, and i squared is negative. Everything else came out from that."

"Very good."

"So now we can put in a second imaginary axis, perpendicular to both ot them! Call it a j axis! We have numbers in three d! Each point in space is a super-complex number, $a + bi + cj$!"

"Not bad," said I, "Not bad at all."

"So is that it? Is that the answer?"

"There might be a little problem," I said quietly.

He didn't see it. He just looked at me.

"Well, if $a + bi + cj$ is a super complex number, does that mean you can multiply two of them together?"

"Of course," he answered. "That's the whole point. Add and multiply."

"And every point in three space would be a number, $a + bi + cj$."

"Right."

"Try it," I said. "Try squaring it."

I gave him my pen. In no time he had the "answer:"

$$a^2 - b^2 - c^2 + 2abi + 2acj + 2bcij$$

"That's no good," I said.

"No good? Why not? What's wrong?"

"It's not of the right form: it should be a real number plus a real number times i plus a real number times j. What is the ij doing there?"

He didn't talk right away. Looked at the paper, looked at me, pondered.

"OK," he said eventually. "Got to get rid of ij."

More long thought.

"It has to be some combination of i and j and one."

"Exactly," I replied.

"But which combination?" He asked himself.

"Could be anything, I guess," was his answer.

"Really?" I asked. "Just anything?"

"No, I guess not. Probably there's some right combination. But I don't see what it would be."

I was ready to give him some help.

"You're having trouble multiplying i and j, the unit vectors in the y and z directions, right? But why should multiplying those two directions be any worse than multiplying the x and y directions, or the x and z? Do the three directions in space care what letters you use for them?"

"So what?" He got defensive. "What's your point? What are you trying to prove?"

"My point is, no matter what combination of i and j and one you might use for i times j, you have a really weird situation. Multiplying the i, j pair of unit vectors would be completely different than multiplying the i and 1 or the j and 1. What sense would that make?"

"Then we're stuck!" he cried. "There's no way! It's hopeless."

"Hamilton didn't think so."

"I give up," said Ludwig. "Tell me the answer."

"Well, if it can't be done with two imaginary axes i and j, so what? We're trying to do something in three dimensions, aren't we? Three, not two!"

"Oh, you mean have three imaginary axes? i, j, and k? Is that what you mean?"

"Well, what would you rather do? Try that, or just give up?"

"So then, i times j could be k."

"Why not?"

"And then, what about j times k?"

"What do you think?"

"Would have to be i. And k times i would be j."

I said nothing, just nodded quietly.

"So what do you think?" I asked after a while. "Do you like this?"

"Yeah! It's neat. A point in three-space is $ai + bj + ck$. Another point is $a'i + b'j + c'k$. You add, and you get

$$(a + a')i + (b + b')j + (c + c')k.$$

You multiply, and you get

$$-aa' - bb' - cc' + (ab' + a'b)k + (ac' + a'c)j + (bc' + b'c)i.$$

Beautiful."

"I'm not so sure."

"What's wrong now? What's your hang-up?"

"Well for one thing, even if you start with just a triple a, b, c and another triple a', b', c', when you multiply you not only have another triple, with i, j, k, but also that plain number, $-aa' - bb' - cc'$. What is that supposed to mean?"

"I don't know. You could just throw it away."

"That might be hasty. But let that go for now. What does the rest of the answer mean?"

"It's just the answer, that's it."

"Could you divide by one of these things?" I asked.

"That sounds really hard."

"Dividing is just multiplying by the reciprocal, right? To divide by one of these quadruples, you'd have to find its reciprocal. That would be another quadruple that gives you the number 1 by multiplying. Now, how do you divide by a complex number? To multiply by the reciprocal of $a + bi$, you first multiply by its conjugate, $a - bi$. That gives you $a^2 + b^2$ – a positive number. Then you just divide by $a^2 + b^2$ to get 1. The reciprocal of $a + bi$ is $a - bi$ divided by $a^2 + b^2$."

"Right! So to divide here, we just multiply by the conjugate and then divide by the sum of the squares!"

"Does it work?"

He went back to pen and paper. "No! It doesn't work! The conjugate of $a + bi + cj + dk$ would be $a - bi - cj - dk$. But then, when you multiply, besides $a^2 + b^2 + c^2 + d^2$, you get a whole extra mess – $-2bck - 2bc\mathrm{j} - 2cdi$."

"What can we do?"

"Nothing. We're stuck again."

"The last time you said that, there was a way out after all."

"Yes, we put in the imaginary k to go with imaginary i and imaginary j."

"So what do we need now?"

"A new idea. Another new idea."

I gave him time to ponder and suffer. Then it was time for more scaffolding.

"One of the terms that we don't want is $2abk$. Where did that come from?"

"It was $2abij$, and we agreed that ij equals k."

"Let's do the multiplication over, and look carefully at what's going on."

"You have ai and bj in both factors. So multiplying, you get $aibj$ and $bjai$. That makes $2abij$."

"Why is that?"

"Because $aibj$ and $bjai$ are both the same thing, that's why."

"How do you know they are both the same thing?"

"Come on," said Ludwig. He was getting tired and annoyed. "You know why. The commutative law of multiplication, that's why."

"You mean ab equals ba."

"Right, and ij equals ji."

"Where did you get that idea?"

"I told you, the commutative law of multiplication."

"I only know about the commutative law of multiplication for ordinary numbers. i and j aren't ordinary numbers."

Ludwig is bright. He catches on pretty fast.

"Oh, so when we said $ij = k$, we didn't say anything about ji."

"I guess not."

"Great! Let's say $ji = -k$. Then $aibj$ plus $bjai$ will be $abk - abk$, zero!"

And so Ludwig and I reinvented quaternions.

> In 1865, in a letter to one of his sons, Hamilton wrote: "Every morning on my coming down to breakfast your (then) little brother William Edwin and yourself, used to ask me, 'Well, Papa, can you multiply triplets?' Whereto I was always obliged to reply, with sad shake of the head, 'No, I can only add and subtract them.' But on the 16th day of the same month I was walking along the Royal Canal, yet an under-current of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close, and a spark flashed forth, the herald (as I foresaw, immediately) of many long years to come of definitely directed thorough work. Nor could I resist the impulse to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols i, j, k; namely
>
> $$i^2 = j^2 = k^2 = ijk = -1."$$
>
> (See [8] for more on Hamilton.)

From my formulas with Ludwig we can make up a little 3-by-3 multiplication table:

	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

The entry in the left column pre-multiplies the entry in the top row. The 3-by-3 matrix of products is anti-symmetric, because the multiplication is anti-commutative. From this information you can easily get Hamilton's formula, $ijk = -1$. Try it! (Conversely, from $ijk = -1$, you easily get $ij = k$.

Just multiply both sides on the right by k, use $k^2 = -1$, and then multiply by -1. And you can get $jk = i$, by first multiplying on the left, by $-i$.)

In this 3-by-3 summary, the products are different from the factors. If we introduce $-i$, $-j$, $-k$, $+1$ and -1 as factors, we get a "group" of 8 elements. Now multiplication is "closed" – you can multiply forever, and never get anything except one of the eight members of the group. Furthermore, every element has a multiplicative inverse, so you can "divide" – either on the right or on the left!

Here's a little 8-by-8 multiplication table, called a "group table," The calculation is associative, but not commutative.

	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

To read this table, again, take a symbol from the column at the left as the pre-multiplier, and a symbol from the row at the top as the post-multiplier. The product is in the intersection of the two associated horizontal and vertical lists. Unlike an ordinary addition or multiplication table, this table is not symmetric, because this multiplication is not commutative.

However, quaternions are more than a group with a multiplication table! You can also take arbitrary combinations of them, with ordinary real numbers a, b, c, d as coefficients, expressions like $a + bi + cj + dk$, and multiply or add these expressions. The two operations satisfy the familiar distributive law, as in $3(4 + 5) = 3(4) + 3(5)$.

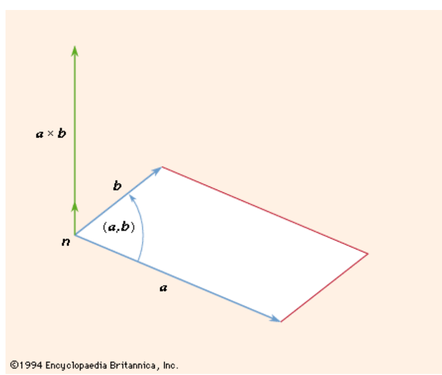
And you can divide by anything that's not equal to zero – just multiply by the conjugate, and then divide by the sum of the squares of the coefficients a, b, c, d. In fact, you can divide either on the left or on the right. But you probably will not get the same answer. Because $X^{-1}Y$ probably will be different from YX^{-1}.

The reason Hamilton got so excited and hopeful about quaternions is because of their connection with 3d geometry. A quaternion $a + bi + cj + dk$

has a "real" or "scalar" part a and a "vector part," $bi + cj + dk$. The vector part by itself comes close to doing in three dimensions what complex numbers do in two. The result of multiplying two such expressions, say $bi + cj + dk$ and $b'i + c'j + d'k$, has two parts. A scalar $-bb' - cc' - dd'$, and a new vector $(cd' - c'd)i + (db' - b'd)j + (bc' - cb')k$.

The scalar part is very important. It is the negative of the length of the projection of the first vector on the second. If we think of $X = bi + cj + dk$ as a displacement or shift in xyz-space, and $Y = b'i + c'j + d'k$ as a unit vector in 3-space, then the scalar part of their product as quaternions is the length of the displacement effected by X in the direction of Y (except for the sign, which you have to reverse).

What about the vector part of the answer? The algebra looks complicated, but the geometry is not complicated. The vector part of the quaternion product is perpendicular to both of the two factors in the multiplication; in other words, it's perpendicular or "normal" to the plane determined by those two vectors. And its magnitude is very simple also. It's the area of the parallelogram they span. Which turns out to be very meaningful in mechanics as well as geometry!



Notice that when I say the direction of the vector part of the product is perpendicular to the two factors, that doesn't give a unique answer. There are *two* unit vectors perpendicular to both X and Y, in two opposite directions. For instance, both k and $-k$ are perpendicular to both i and j.

This fact provides a deep geometric reason why the algebra of three-dimensional geometry *cannot* be commutative. The choice of k or $-k$ must be decided by the order of multiplication, ij or ji. The non-commutativity is built in to the structure of three-dimensional space.

A century later, non-commutative algebra turned out to be the heart of quantum mechanics. Quantum physicists had to use non-commuting "operators," to model Heisenberg's uncertainty principle. That principle gives an "error" in the observation of position, caused by an observation of velocity. The two physical observables, position and velocity, are represented by two mathematical operators. Position corresponds to X, the space variable; and velocity or momentum corresponds to D, differentiation with respect to that variable. Operating on a function of x, you get a different result if you first multiply by x and then differentiate, or first differentiate and then multiply. The mathematical formula that embodies Heisenberg's uncertainty principle is $XD-DX=\hbar I$. Here $\hbar$ is a very small number called "Planck's constant," and I is the identity operator. The formula says $XD-DX$ is *not zero*. In other words, X and D do not commute.

Hamilton's dream that his quaternions would do for solid geometry what complex numbers did for plane geometry had some enthusiastic advocates, called "quaternionists," including Harvard's Benjamin Peirce and Edinburgh's Peter Tait. James Clerk Maxwell wrote: "The invention of the calculus of quaternions is a step towards the knowledge of quantities related to space which can only be compared, for its importance, with the invention of triple coordinates by Descartes" [10, pp.9-10]. But most physicists were not crazy about it. Oliver Heaviside went so far as to call quaternions "a positive evil of no inconsiderable magnitude." William Thomson (Lord Kelvin) called them "an unmixed evil to those who touched them in any way, including Clerk Maxwell."

Quaternion multiplication combines in one operation two products, the scalar product and the vector product. For applications in mechanics or electromagnetism, it's more convenient to deal with the two products separately. And besides that, the quaternion scalar product has the wrong sign! Physicists rejected quaternions in favor of "vector analysis" that was developed later by Josiah Gibbs and Heaviside. Vector analysis provides two kinds of multiplication, the "scalar product" and the "vector product". Neither one permits division. In scalar multiplication, the result of multiplying two vectors isn't even a vector, it's just a number. In vector multiplication, you have the algebraically strange fact that any vector times itself is zero! Of course, this just means that two identical vectors span a "parallelogram" of zero area.

Although quaternions never became dominant in classical physics, they may become prominent in quantum physics. Some mathematical physicists are working on that possibility. I will try to explain a little about that.

Although a quaternion is a quadruple of real numbers, we can, if we like, think of it as a single "super-number," just as we think of a complex number as a single thing, even though it is made up of two real numbers. Thinking of quaternions as just a kind of number, it becomes possible to use them as coefficients in the abstract space of quantum mechanics. Hilbert space, the standard setting for quantum mechanics, consists of abstract "vectors," each of which is an infinite sequence of complex numbers. Why not quaternions instead of complex numbers? In other words, use quaternions as the components or coefficients of the vectors in quantum mechanics.

"Why bother?" you ask. Well, quantum mechanics continues to involve serious difficulties or shortcomings. So some physicists are trying to find a better foundation. Some people hope quaternions might come in handy there. A natural mathematical question is, why stop with quadruples, 4-somes? Why not quintuples, sextuples, whatever? "The very day after his fateful walk, Hamilton sent an 8-page letter describing the quaternions to his friend from college, John T. Graves. It was Graves' interest in algebra that got Hamilton thinking about complex numbers and triplets in the first place. Graves replied, 'If with your alchemy you can make three pounds of gold, why should you stop there?'" [1]

And on December 26th, Graves wrote to Hamilton describing a new eight-dimensional algebra! He called it "the octave." Nowadays we call them "the octonions." Graves showed that they are a normed division algebra; that is, every element except zero has a multiplicative inverse or "reciprocal." And he used this formula to express the product of two sums of eight perfect squares as another sum of eight perfect squares. This is the "eight square theorem."

Graves was beaten to publication by the young Arthur Cayley, whose 1845 article on Jacobi elliptic functions included a description of the same "octaves" or "octonions." They are now also called "Cayley numbers." Finally, Benjamin Peirce's son Charles Saunders Peirce and Ferdinand Frobenius independently proved that these octonions are the last step – there are no other division algebras with real coefficients. This may seem surprising, but it makes sense. To achieve each extension to a higher dimension, we had to give up some nice property of numbers. We started in one dimension, with the real numbers. They can do the four operations of arithmetic, and they also are linearly ordered. Next, in two dimensions, we bring in complex numbers. They still satisfy all the laws of arithmetic, but they no longer are ordered. Going up to 3-space with the quaternions, we had to give up the commutative law. To go up to the "octonions," we must even give up the associative law. That's the end of the line. After that, all larger "algebras" must include "divisors of zero." It's no longer possible to divide by everything except zero.

Even without the associative property, the octonions do play a role in quantum mechanics. The article by John Baez [1] is readable and informative. (And he happens to be Joan Baez's brother!)

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