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Combinatorial Interpretations of Fibonomial Identities

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Abstract

The Fibonomial numbers are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{\prod_{i=n-k+1}^{n} F_i}{\prod_{j=1}^{k} F_j}$$

where F_i is the ith Fibonacci number, defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ with initial conditions $F_0 = 0, F_1 = 1$. In the past year, Sagan and Savage have derived a combinatorial interpretation for these Fibonomial numbers, an interpretation that relies upon tilings of a partition and its complement in a given grid.

In this thesis, I investigate previously proven theorems for the Fibonomial numbers and attempt to reinterpret and reprove them in light of this new combinatorial description. I also present combinatorial proofs for some identities I did not find elsewhere in my research and begin the process of creating a general mapping between the two different Fibonomial interpretations. Finally, I provide a discussion of potential directions for future work in this area.

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Acknowledgments

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Chapter 1

Background

The *Fibonomial coefficients*, denoted $\left[{n \atop k} \right]$, are a specific instance of the *generalized binomial coefficients*, an important set of combinatorial objects. In Sagan and Savage (2010), for the first time there were detailed two combinatorial interpretations for the Fibonomial coefficients and other generalized binomial coefficients based on sequences with ideal initial conditions. Prior to this, proofs of Fibonomial identities had consisted purely of algebraic manipulations, but this new interpretation presented an opportunity to view these previously proven identities in a new light.

1.1 Basic Combinatorial Objects

The *Fibonacci numbers* are the sequence of numbers defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ with initial conditions $F_0 = 0, F_1 = 1$. The *Lucas numbers* are defined to be the sequence of numbers given by the Fibonacci recurrence with initial conditions $L_0 = 2, L_1 = 1$.

Note that $F_{n+1} = f_n$ represents the number of tilings of a strip of length n using length 1 squares and length 2 dominos. Similarly, L_n represents the number of circular tilings of a strip of length n using length 1 squares and length 2 dominos. In this case, we count all usual tilings of a strip, plus we allow the use of a *circular domino* in our tiling; that is, a domino that crosses the starting and ending edges of the strip to cover the first and nth cells.

We define an *integer partition* of the positive integer n as a way of writing n as a sum of nonincreasing nonnegative integers. We denote a partition as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, where each λ_i is called a *part* of our partitions. Note that $\lambda_i \geq 0$ and $\lambda_i \geq \lambda_{i+1}$.

In this report, I am primarily concerned with partitions that fit inside

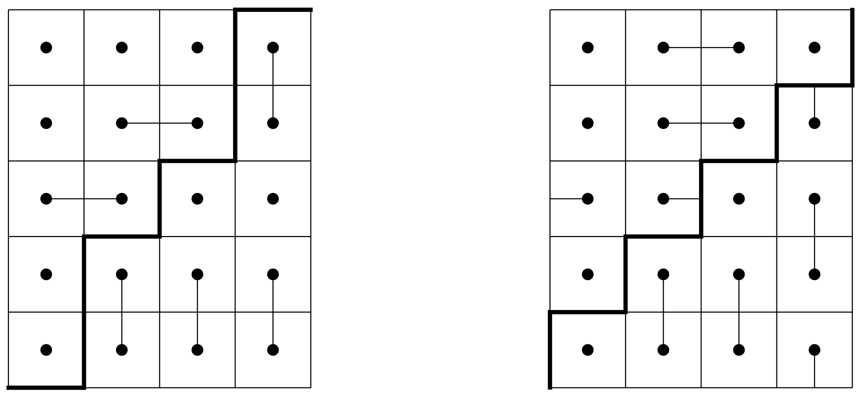

Figure 1.1 A noncircular and circular partition tiling in a 5×4 grid.

an $m \times n$ grid; that is, partitions with m parts, none of which is greater than n. We denote such a partition by $\lambda \subseteq m \times n$. Note that such a partitition is equivalent to a lattice path from the bottom left to the top right corner of our grid and that the partition defines a unique *complementary partition* λ^* such that $\lambda^* \subseteq n \times m$ (namely, this is the partition below the lattice path whose parts are determined by the lengths of the columns).

1.2 Partition Tilings

A *noncircular tiling* of $\lambda \subseteq m \times n$ is obtained by tiling each individual part of λ with squares and dominos in the same way as Fibonacci tilings. A *circular tiling* of $\lambda \subseteq m \times n$ is obtained by tiling each individual part of λ with squares and dominos, allowing for circular dominos, in the same way as Lucas tilings. Denote the set of all noncircular tilings of a particular partition λ by $\mathcal{L}_\lambda$ and the set of all circular tilings by $\mathcal{C}_\lambda$. Addititionally, define the set of all noncircular tilings of a partition λ such that the first tile of any part's tiling is a domino to be $\mathcal{L}'_\lambda$.

We define a *noncircular partition tiling* for $\lambda \subseteq m \times n$ as a tiling in $L_\lambda \times L'_{\lambda^*}$. Notice that using L'_{λ^*} forces the first tile in every column to be a domino.

Similarly, we define a *circular partition tiling* for $\lambda \subseteq m \times n$ as a tiling in $C_\lambda \times C_{\lambda^*}$. See Figure 1.1 for examples of partition tilings.

The *Fibonomial coefficient* $\left[{n \atop k} \right]$ is then defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{F_n \cdot F_{n-1} \cdots F_{n-k+1}}{F_k \cdot F_{k-1} \cdots F_1}.$$

In their recent paper, Sagan and Savage (2010) demonstrated two possible combinatorial interpretations for the Fibonomial coefficients. In the first case, we sum over all possible partitions fitting in a given grid and then over all possible noncircular partition tilings corresponding to that particular tiling; any particular noncircular partition tiling is counted exactly once. These summations gives us the following formula:

$$\begin{bmatrix} m+n \\ m \end{bmatrix} = \sum_{\lambda \subseteq m \times n} \sum_{T \in L_\lambda \times L'_{\lambda^*}} 1.$$

For the second interpretation we sum over all possible partitions fitting in a given grid and then over all possible circular partition tilings corresponding to that particular tiling. In this case, each circular partition tiling is given a weight corresponding to the number of zero parts in λ and λ^*. In particular, if we define the number of zero parts in a partition $\lambda \subseteq m \times n$ as $Z(\lambda)$, then the weight of a circular partition tiling T is

$$w(T) = 2^{Z(\lambda)+Z(\lambda^*)}.$$

For example, the circular partition tiling in Figure 1.1 has weight 2, since the partition λ has exactly one zero part. Similarly, the noncircular tiling in the same figure has weight 2 when regarded as a circular partition tiling due to λ^* having one zero part.

Using this definition of a weight function, we have that

$$2^{m+n} \begin{bmatrix} m+n \\ m \end{bmatrix} = \sum_{\lambda \subseteq m \times n} \sum_{T \in C_\lambda \times C_{\lambda^*}} w(T).$$

Chapter 2

Combinatorial Proofs for Fibonomial Identities

In this chapter, I investigate previously proven theorems for the Fibonomial numbers and attempt to reinterpret and reprove them in light of the new combinatorial descriptions. I also present combinatorial proofs for some identities I did not find elsewhere. Each proof given here uses exactly one of the two combinatorial interpretations of Fibonomial numbers; in Chapter 3 I will discuss my work to try to connect the two approaches such that a proof using one interpretation would directly imply a proof with the other.

2.1 Previously Existing Identities

The first major aspect of my work involves combinatorially proving Fibonomial identities. The first of these identities follows straightforwardly from the Sagan and Savage (2010) noncircular partition tiling interpretation because, in fact, the interpretation was built off of this recurrence.

Proposition 1. $\left[{m+k \atop k}\right] = F_{m+1}\left[{m+k-1 \atop m}\right] + F_{k-1}\left[{m+k-1 \atop k}\right]$.

Proof. We prove this proposition by combinatorial argument using the noncircular tiling interpretation. We will count the number of ways to noncircularly partition tile a $k \times m$ grid.

As Sagan and Savage (2010) demonstrated, the number of ways to noncircularly partition tile a $k \times m$ grid is $\left[{m+k \atop m}\right]$.

Now, consider any partition $\lambda \in k \times m$ and corresponding partition λ^*. Consider the top right cell of the $k \times m$ grid. This cell is included either in the partition λ or the partition λ^*.

If the top right box is in λ, then we know we know we will be tiling the entire top row, which has length m, with squares and dominoes. The number of ways to tile a full row is $f_m = F_{m+1}$. We then must partition tile the remainder of the grid that has dimensions $k-1 \times m$. We know we can tile this segment of the grid $\left[\begin{smallmatrix} m+k-1 \\ m \end{smallmatrix}\right]$ ways.

If, on the other hand, the top right box is in λ^*, we know we must tile the entire rightmost column with squares and dominos. This column has length k, but we know that the first tile we use must be a domino, so there are $f_{k-2} = F_{k-1}$ ways to tile this column. Now we want to partition tile the remainder of the grid that has dimensions $k \times m-1$. We know we can tile this section in $\left[\begin{smallmatrix} m+k-1 \\ k \end{smallmatrix}\right]$ ways.

Thus, overall, the total number of ways to partition and appropriately tile an $m \times k$ grid is

$$F_{m+1} \begin{bmatrix} m+k-1 \\ m \end{bmatrix} + F_{k-1} \begin{bmatrix} m+k-1 \\ k \end{bmatrix}.$$

Because these two quantities count the same thing, they must be equal. Thus we have

$$\begin{bmatrix} m+k \\ k \end{bmatrix} = F_{m+1} \begin{bmatrix} m+k-1 \\ m \end{bmatrix} + F_{k-1} \begin{bmatrix} m+k-1 \\ k \end{bmatrix}$$

as desired. □

We also obtain the following, similar result:

Corollary 1. $\left[\begin{smallmatrix} m+k \\ m \end{smallmatrix}\right] = F_{m-1}\left[\begin{smallmatrix} m+k-1 \\ m \end{smallmatrix}\right] + F_{k+1}\left[\begin{smallmatrix} m+k-1 \\ m-1 \end{smallmatrix}\right]$.

Proof. This result follows by a proof identical to the previous proposition; we simply use an $m \times k$ grid rather than a $k \times m$ grid. □

By similar logic using the circular partition–tiling interpretation, we can prove another Fibonomial recurrence involving the Lucas numbers.

Proposition 2. $2^{m+n}\left[\begin{smallmatrix} m+n \\ m \end{smallmatrix}\right] = 2^{m+n-1}L_m\left[\begin{smallmatrix} m+n-1 \\ m \end{smallmatrix}\right] + 2^{m+n-1}L_n\left[\begin{smallmatrix} m+n-1 \\ n \end{smallmatrix}\right]$.

Proof. This identity is often written more simply as

$$2\begin{bmatrix} m+n \\ m \end{bmatrix} = L_m \begin{bmatrix} m+n-1 \\ m \end{bmatrix} + L_n \begin{bmatrix} m+n+1 \\ n \end{bmatrix}.$$

We prove this identity by combinatorial argument using the circular partition–tiling interpretation. We will count the number of ways to partition and circularly tile an $m \times n$ grid.

As shown by Sagan and Savage (2010), the number of ways to partition and circularly tile an $m \times n$ grid is $2^{m+n}\left[{m+n \atop m}\right]$.

Now consider the top right box of our $m \times n$ grid. This cell must be part of either λ or λ^* for any given partition $\lambda \subseteq m \times n$. If the cell is part of λ, we know we are circularly tiling the entire top row of length n. The number of ways to tile a full row is L_n. We must then partition and circularly tile the remainder of our grid, which we know we can do in $2^{m+n-1}\left[{m+n-1 \atop n}\right]$ ways.

The other possibility is that the top right box is included in λ^*. In this case, we know we must circularly tile the entire right column of length m, which we can do in L_m ways. We then have $2^{m+n-1}\left[{m+n-1 \atop m}\right]$ ways to partition and circularly tile the rest of the grid. Overall, the number of ways to partition and circularly tile an $m \times n$ grid is

$$2^{m+n-1}L_n \begin{bmatrix} m+n-1 \\ n \end{bmatrix} + 2^{m+n-1}L_m \begin{bmatrix} m+n-1 \\ m \end{bmatrix}.$$

Because these two quantities count the same thing, they must be equal. Thus we have

$$2^{m+n} \begin{bmatrix} m+n \\ m \end{bmatrix} = 2^{m+n-1}L_m \begin{bmatrix} m+n-1 \\ m \end{bmatrix} + 2^{m+n-1}L_n \begin{bmatrix} m+n+1 \\ n \end{bmatrix}$$

as desired. □

With binomial coefficients, one of the most well-known identities states that $\binom{n}{k} = \binom{n}{n-k}$. I proved the following identity, a Fibonomial version of this familiar binomial coefficient theorem:

Proposition 3. $2^{m+k}\left[{m+k \atop k}\right] = 2^{m+k}\left[{m+k \atop m}\right]$.

Proof. Note that, written in a simpler form, this identity becomes

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}.$$

We prove this identity using the circular partition–tiling interpretation. Consider a partition $\lambda \subseteq m \times k$ with corresponding partition λ^* whose parts are determined by the lengths of the columns of the complement of λ in our $m \times k$ grid. Clearly, $\lambda^* \subseteq k \times m$ with corresponding partition λ, whose parts are determined by the lengths of the columns of the complement of λ^* in our $k \times m$ grid.

Thus, because we can bijectively map each circular tiling of λ and complement λ^* in an $m \times k$ grid to a circular tiling of λ^* and complement λ in an $k \times m$ grid, we can see that the number of such tilings are equal. Because the number of ways to partition and then circularly tile a partition and its complement in a $m \times k$ grid is $2^{m+k}\left[{m+k \atop m}\right]$, we have

$$2^{m+k}\begin{bmatrix} m+k \\ k \end{bmatrix} = 2^{m+k}\begin{bmatrix} m+k \\ m \end{bmatrix},$$

as desired. □

2.2 New Identities

In addition to proving these previously proven identities, I used some properties of the combinatorial interpretation of Fibonomials to derive some new identities that I did not find in my survey of previous literature.

Proposition 4. $\begin{bmatrix} m+n \\ m \end{bmatrix} = f_n^m + \sum_{i=0}^{m-2} f_n^i f_{m-i-2} \begin{bmatrix} m+n-i-1 \\ m-i \end{bmatrix}.$

Proof. We prove this identity by a combinatorial argument using the noncircular partition tiling interpretation. Consider how many ways there are to noncircularly partition tile a grid of size $m \times n$. From Sagan and Savage (2010), we know that the answer to this problem is $\left[{m+n \atop m}\right]$.

Next, consider how many "full rows" there are in our partition tiling; that is, how many parts in λ are of size n. The number of such parts can range between 0 and m. Note, however, that there cannot be exactly $m-1$ parts of size n as such a scenario implies the existence of a column tiling of length 1. Due to the domino restriction, we cannot properly tile such a column, so one cannot exist. Thus, we can have from 0 to $m-2$ or m full rows.

Consider first the case where there are m full rows. Each row is of length n and is tiled with squares and dominos. The number of ways to tile such a strip is f_n, and, because there are m rows, each of whose tilings are chosen independently, there are f_n^m ways to tile the entire grid.

Next, consider the case where there are i full rows $(0 \leq i \leq m-2)$. There are, as above, f_n^i ways to tile these i full rows. Additionally, because there are exactly i full rows, we know that the far–right column must have exactly $m-i$ cells below the lattice path (i.e., the largest part of λ^* is $m-i$). Due to the domino restriction, there are f_{m-i-2} ways of tiling this column with the bottom tile being a domino. The remainder of the grid has dimensions $(m-i) \times (n-1)$. The number of ways to partition tile a grid of this size is $\left[{m+n-i-1 \atop m-i}\right]$.

Summing over all possible numbers of full rows, we realize that the number of ways to partition tile an $m \times n$ grid is

$$f_n^m \sum_{i=0}^{m-2} f_n^i f_{m-i-2} \begin{bmatrix} m+n-i-1 \\ m-i \end{bmatrix}.$$

Because we have two answers to the same counting problem, we realize that they must be equal. Thus, we see

$$\begin{bmatrix} m+n \\ m \end{bmatrix} = f_n^m + \sum_{i=0}^{m-2} f_n^i f_{m-i-2} \begin{bmatrix} m+n-i-1 \\ m-i \end{bmatrix}$$

as desired. □

The following is a related proposition which can be proven by a similar argument, counting the number of full columns rather than full rows. Note that there can be any number of full columns, so the identity can be written compactly as a single summation.

Corollary 2. $\begin{bmatrix} m+n \\ n \end{bmatrix} = \sum_{j=0}^{m} f_{m-2}^j f_{n-j} \begin{bmatrix} m+n-j-1 \\ n-j \end{bmatrix}.$

Finally, we have a circular version of this argument. The lack of restrictions means that this identity holds whether we condition on full rows or full columns.

Proposition 5. $2^{m+n} \begin{bmatrix} m+n \\ m \end{bmatrix} = \sum_{i=0}^{m} 2^{m+n-i-1} L_n^i L_{m-i} \begin{bmatrix} m+n-i-1 \\ m-i \end{bmatrix}.$

Proof. We prove this identity by a combinatorial argument using the circular partition–tiling interpretation. Consider how many ways there are to circularly partition tile a grid of size $m \times n$. From Sagan and Savage (2010), we know that the number of ways is $2^{m+n}\left[{m+n \atop m} \right]$.

Next, consider how many "full rows" there are in our partition tiling; that is, how many parts in λ are of size n. The number of such parts can range between 0 and m.

Consider the case where there are i full rows $(0 \leq i \leq m)$. There are L_n^i ways to circularly tile these i full rows using squares and dominos. Additionally, because there are exactly i full rows, we know that the far–right column must have exactly $m - i$ cells below the lattice path (i.e., the largest part of λ^* is $m - i$); there are L_{m-i} ways of circularly tiling this column. The remainder of the grid has dimensions $(m-i) \times (n-1)$. The number of ways to circularly partition tile a grid of this size is $2^{m+n-i-1}\left[{m+n-i-1 \atop m-i}\right]$.

Summing over all possible numbers of full rows, we realize that the number of ways to partition tile an $m \times n$ grid is

$$\sum_{i=0}^{m} 2^{m+n-i-1} L_n^i L_{m-i} \begin{bmatrix} m+n-i-1 \\ m-i \end{bmatrix}.$$

Because we have two answers to the same counting problem, we realize that they must be equal. Thus, we see

$$2^{m+n} \begin{bmatrix} m+n \\ m \end{bmatrix} = \sum_{i=0}^{m} 2^{m+n-i-1} L_n^i L_{m-i} \begin{bmatrix} m+n-i-1 \\ m-i \end{bmatrix}$$

as desired. □

2.3 Current Incomplete Work

Most recently, I have been focusing on why, conceptually, the combinatorial interpretation Sagan and Savage proposed holds. To do so, I have been investigating the basic Fibonomial definition; that is,

$$\begin{bmatrix} m+n \\ m \end{bmatrix} = \frac{F_{m+n} \cdot F_{m+n-1} \cdots F_{n+1}}{F_m \cdot F_{m-1} \cdot F_1}.$$

Rearranging this definition gives us the more workable form of

$$F_m \cdot F_{m-1} \cdot F_1 \begin{bmatrix} m+n \\ m \end{bmatrix} = F_{m+n} \cdot F_{m+n-1} \cdots F_{n+1}.$$

Individually, we can identify each term of this equation with a combinatorial object involving tilings. I consulted with Jacob Scott, who suggested multiplying both sides of this equation by $F_n F_{n-1} \cdots F_1$. Doing so, we can view both sides of the expression as some form of tiling of a right triangle with both legs of length $m+n-1$. In particular, the expression

$$F_{m+n} F_{m+n-1} \cdots F_1$$

represents the number of ways to tile a strip of length $m+n-1$, a strip of length $m+n-2$, and so on down to a strip of length 0 with squares and dominos. Aligning these horizontal strips so that their right ends match up, we end up with a triangle with legs both length $m + n - 1$ (we ignore the null tiling).

Alternatively, we can view the expression

$$(F_m F_{m-1} \cdots F_1)(F_n F_{n-1} \cdots F_1) \begin{bmatrix} m+n \\ m \end{bmatrix}$$

by fixing the $m \times n$ grid of the Fibonomial tiling in the top right corner of our triangle. To the left of the ith row from the top of this Fibonomial grid, we align a horizontal strip tiling of length $m - i$. To the bottom of the jth column from the right of the Fibonomial grid, we align a vertical strip tiling of length $n - j$. Note that this arrangement also yields a triangle grid with legs of length $m + n - 1$, though here we have both horizontal and vertical tilings along with some restrictions imposed by the borders of the Fibonomial grid and the lattice path inherent in a partition tiling.

Jacob also noted that simply by taking a triangular grid of the form described above and creating a lattice path from some point on the "hypotenuse" of the triangle to the top right corner, we find that by considering columns below the path as vertical strips and rows above the lattice path as horizontal strips, we end up with stips of every length from 1 to $m + n - 1$. By tiling each of these strips with squares and dominos, we have a different arrangement of $F_{m+n} F_{m+n-1} \cdots F_1$.

If we fix the "starting point" of our lattice path at the bottom left corner of a given grid cell on the hypotenuse, the number of lattice paths from that starting point to the top right corner of the triangle grid is $\binom{m+n}{m}$ where n is the number of right steps to the far right of the grid and m is the number of up steps to the top of the grid. Then with this starting point there are exactly

$$\binom{m+n}{m} F_{m+n} F_{m+n-1} \cdots F_1$$

ways to draw a lattice path and tile the resulting horizontal and vertical strips with squares and dominos. This lattice path gives us a sort of impression of an $m \times n$ Fibonomial tiling in the grid, though not exactly since we have not enforced the domino restriction and could have tiles crossing the boundary between the grid and strip tilings.

Chapter 3

Connecting the Combinatorial Interpretations

As mentioned earlier, Sagan and Savage (2010) provide two different interpretations of the Fibonomial coefficient, noncircular partition tilings of an $m \times n$ board corresponding to $\left[{m+n \atop m} \right]$ and circular partition tilings of an $m \times n$ board corresponding to $2^{m+n}\left[{m+n \atop m} \right]$. In the case of circular partition tilings, a given tiling is assigned a particular weight, while in the case of noncircular partition tilings, each tiling is counted exactly once. By giving each noncircular tiling weight 2^{m+n}, we can create a mapping between the two interpretations by mapping some number of circular tilings with weights summing to 2^{m+n} to a particular noncirуclar tiling.

3.1 The $1 \times n$ Case

Consider first the case where $m = 1$; that is, when we have a $1 \times n$ grid. Given a particular noncircular partition tiling, we know that the partition $\lambda = (n)$ due to the domino restriction. Similarly, we could think of this as a lattice path that takes n right steps followed by an up step. We execute the following algorithm to determine which circular tilings match up with our noncircular tiling:

1. Start with the circular partition tiling identical to our circular tiling. The sum of weights of circular tilings we have counted starts at zero.

2. Examine the current partition tiling. Suppose our lattice path is i ($i > 0$) right steps, followed by an up step, followed by $n - i$ right steps.

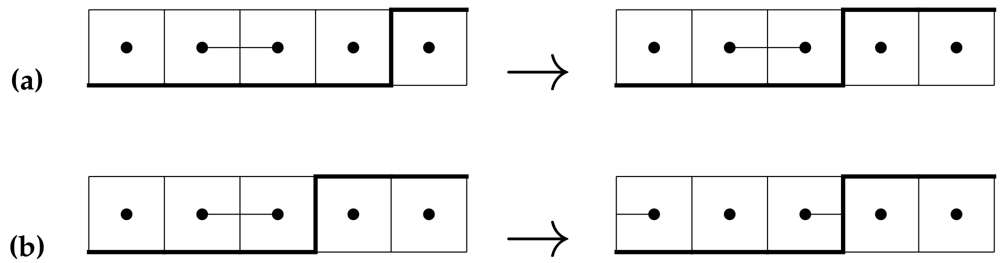

Figure 3.1 An example of the procedure our algorithm follows in steps 2a and 2b, respectively.

Add 2^i (the weight of this partition tiling) to our total weight. If $i = 0$, our first step is up and this tiling has weight 2, which we add to our total weight. Consider the tiling of λ (our horizontal tiling).

(a) If this tiling ends in a square, repeat step 2 after shifting the up step one to the left (i.e., the new lattice path is $i-1$ steps right followed by an up followed by $n-i+1$ right steps). If the first step in our lattice path is already a step up, terminate the process.

(b) If this tiling ends in a domino, cycle our tiling one to the right (this cycling should introduce a circular domino into our horizontal tiling). Note that this tiling has weight 2^i as well. Add 2^i to our total weight, and terminate the process.

See Figure 3.1 for an illustration of these processes.

Note that this process means that if the rightmost domino in our noncircular partition tiling covers cells $k-1$ and k, then the total weight w of the circular partition tilings mapped by our algorithm is

$$w = 2^k + \sum_{i=k}^{n} 2^i = 2^{n+1},$$

which is the target weight. If no dominos exist in our tiling, the weight is

$$w = 2 + \sum_{i=1}^{n} 2^i = 2^{n+1},$$

which again matches our target weight. Thus we have mapped an appropriate number of circular partition tilings to each noncircular partition tiling. It is clear that any circular tiling has been mapped to exactly one noncircular tiling, so we have a mapping for the $1 \times n$ case.

3.2 The $m \times 1$ Case

We can also create an algorithm for the case where $n = 1$; that is, the case where we have an $m \times 1$ grid. Given a particular noncircular partition tiling, we do the following:

1. If our tiling consists of only squares, any circular tiling consisting of only squares maps to this noncircular tiling. Note that, in total, these tilings have weight

$$w = 2 + \sum_{i=1}^{m} 2^i = 2^{m+1},$$

   which is the target weight. At this point, we have a complete mapping for this case.

2. If we reach this step, it means the bottom tile is a domino due to the domino restriction. Suppose our lattice path consists of k up steps, followed by a right step, followed by $m - k$ more up steps. Notice that any tile above the lattice path must be a square. We match all of the following circular partition tilings with this noncircular tiling:

   (a) The circular tiling that is identical to the noncircular tiling. This tiling has weight 2^k.

   (b) The circular tiling that is the same as the noncircular tiling with the vertical tiling cycled down by one (introducing a circular domino). This tiling also has weight 2^k because the lattice path has not changed.

   (c) Remove a square from the top and move it to the bottom cell, bumping everything else up one cell. This operation alters the lattice path to consist of $k + 1$ up steps, then a right step, then $m - k - 1$ more up steps. We can repeat this process until there are no more squares above the path (that is, the path consists of m ups followed by a right step). The total weight of all of these is $\sum_{i=k+1}^{m} 2^i$.

   Note that the total weight of these tilings is 2^{m+1}, so we have achieved the target weight. An example mapping is shown in Figure 3.2. Note that the weights in the example sum appropriately.

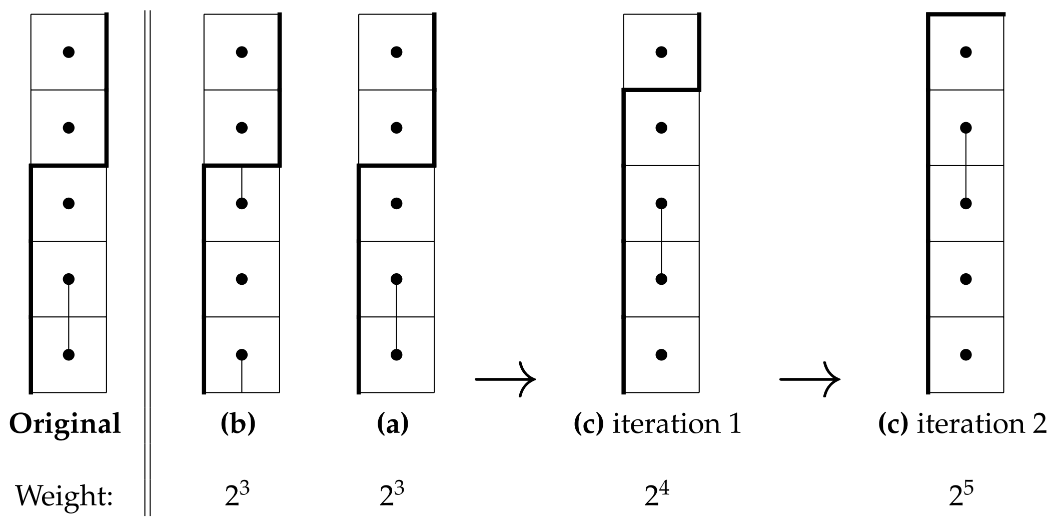

Figure 3.2 An example of the $m \times 1$ mapping in step 2.

Again, we have mapped an appropriate number of circular partition tilings to each noncircular partition tiling, and we can easily trace to where a particular circular tiling maps to, so we have created a valid mapping for the $m \times 1$ case.

3.3 The $m \times n$ Case

While we are able to apply many of the same ideas from the $1 \times n$ and $m \times 1$ cases to the general $m \times n$ case, these operations do not quite account for all possible complications introduced. As a result, the mapping for this case is still incomplete.

It does appear that we can still follow through with many of our previous rules by applying them to individual rows and columns, and it seems likely that using these mappings would apply in a final map. In particular, for any noncircular partition tiling containing a row with a horizontal tiling ending in a domino, we can cycle that horizontal tiling one cell to the right, introducing a circular domino. This is the same as case 2b the $1 \times n$ case mentioned earlier.

We can also adopt rule 2a directly from the $m \times 1$ case and rules 2b and 2c from the $m \times 1$ case by focusing on a particular column of our partition tiling. Note that in order to apply rule 2c, we must have a square in the cell immediately above the right step in our lattice path through the given column. The case that causes us difficulties is when we find part of a domino occupying the first cell above the lattice path. In the $m \times 1$ case, we never had difficulties because any cell that was part of a horizontal tiling was necessarily filled with a square.

For small cases such as 2×2, 3×2, and 2×3, I seem to have found a method to circumvent this difficulty. Unfortunately, however, with larger grids the possible layouts for tiles in the awkward position immediately above the lattice path increases, and I have yet to find a generalization for the map in the case where a domino appears in a cell immediately above the lattice path.

Chapter 4

Future Work

In the future, there are still many more Fibonomial identities to combinatorially prove. A list of potential identities follows. These identities were all pulled from issues of the *Fibonacci Quarterly*. Note that the notation (a, b) represents the greatest common divisor of a and b.

- $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]\left[\begin{smallmatrix} k \\ j \end{smallmatrix}\right] = \left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]\left[\begin{smallmatrix} n-j \\ k-j \end{smallmatrix}\right]$ Gould (1969)
- $\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j=k}^{n} \frac{F_j - F_{j-k}}{F_k} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}$ Gould (1969)
- $\sum_{j=0}^{k} (-1)^{j(j+1)/2} \begin{bmatrix} k \\ j \end{bmatrix} F_{n-j}^{k-1} = 0$ Lind (1971)
- $\sum_{j=0}^{k+1} (-1)^{j(j+1)/2} \begin{bmatrix} k+1 \\ j \end{bmatrix} \begin{bmatrix} n-j \\ k \end{bmatrix} = 0$ Lind (1971)
- $\frac{F_n}{(F_n, F_k)} \bigg| \begin{bmatrix} n \\ k \end{bmatrix}$ Gould (1974)
- $\frac{F_{n-k+1}}{(F_{n+1}, F_k)} \bigg| \begin{bmatrix} n \\ k \end{bmatrix}$ Gould (1974)
- $F_k\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = F_n\left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]$ Gould (1974)
- $F_k\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = F_{n-k+1}\left[\begin{smallmatrix} n \\ k-1 \end{smallmatrix}\right]$ Gould (1974)
- $F_{n+1}\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = F_{n-k+1}\left[\begin{smallmatrix} n+1 \\ k \end{smallmatrix}\right]$ Gould (1974)

- $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{F_n}{(F_n, F_k)} \left(\begin{bmatrix} n \\ k \end{bmatrix}, \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \right)$ Gould and Schlesinger (1995)
- $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{F_{n-k+1}}{(F_{n-k+1}, F_k)} \left(\begin{bmatrix} n \\ k \end{bmatrix}, \begin{bmatrix} n \\ k-1 \end{bmatrix} \right)$ Gould and Schlesinger (1995)
- $F_n \left(\left[{n \atop k} \right], \left[{n-1 \atop k-1} \right] \right) = \left[{n \atop k} \right] (F_n, F_k)$ Gould and Schlesinger (1995)
- $F_{n-k+1} \left(\left[{n \atop k} \right], \left[{n \atop k-1} \right] \right) = \left[{n \atop k} \right] (F_{n-k+1}, F_k)$ Gould and Schlesinger (1995)
- $\sum_{j=0}^{k} (-1)^{j(j+3)/2} \begin{bmatrix} k \\ j \end{bmatrix} F_{n+k-j}^{k+1} = F_1 \cdots F_k F_{(k+1)(n+\frac{k}{2}}$ Melham (1999)
- $\sum_{j=0}^{m-1} (-1)^{\frac{j(j+3)}{2}} \begin{bmatrix} (m+1)k+m \\ j \end{bmatrix} \begin{bmatrix} (m+1)k+m-j-1 \\ m-j-1 \end{bmatrix} F_{n+k+m-j}^{m+1}$
  $+ (-1)^{\frac{m(m+3)}{2}} F_{n-mk}^{m+1} = \left(\Pi_{j=1}^{m} F_{(m+1)k+j} \right) F_{(m+1)(n+\frac{m}{2})}$ Kilic et al. (2010)
- $\sum_{i=0}^{k+1} (-1)^{i(i+1)/2} \begin{bmatrix} k+1 \\ i \end{bmatrix} F_{n-i}^{k} = 0$ Cooper and Kennedy (1995)

The following identities are simplified versions of those proven by Hideyuki Ohtsuka with respect to more generalized binomial coefficients. His paper is forthcoming.

- $\sum_{k=1}^{n} \begin{bmatrix} 2n+1 \\ k \end{bmatrix} = \prod_{k=1}^{n} L_{2k}$ Ohtsuka et al. (forthcoming)
- $\sum_{k=1}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix} = \prod_{k=1}^{n} (L_{2k-1} + 2)$ Ohtsuka et al. (forthcoming)
- $\sum_{k=1}^{2n} i^{\pm k} \begin{bmatrix} 2n \\ k \end{bmatrix} = i^{\pm n} \prod_{k=1}^{n} L_{2k-1}$ Ohtsuka et al. (forthcoming)
- $\sum_{k=1}^{2n+1} i^{\pm k} \begin{bmatrix} 2n+1 \\ k \end{bmatrix} = (1 \pm i) i^{\pm n} \prod_{k=1}^{n} L_k^2$ Ohtsuka et al. (forthcoming)
- $\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix} = \prod_{k=1}^{n} \frac{L_{2k} F_{4k-2}}{F_{2k}}$ Ohtsuka et al. (forthcoming)

The following identities come from papers by Seiter and Trojovský:

- Let m be an odd, positive integer
  $\sum_{i=0}^{m}(-1)^{\frac{i}{2}(m+i)} \begin{bmatrix} m \\ i \end{bmatrix} = 0$ Seibert and Trojovský (2005)

- Let k be a positive integer, $\ell \leq \frac{k-1}{2}, m > k$ be nonnegative integers.
  $\sum_{i=0}^{m}(-1)^{\frac{i}{2}(2\ell+i+1)} \frac{F_{(k-i)(k-2\ell)}}{F_{k-2\ell}} \begin{bmatrix} k+1 \\ i \end{bmatrix} = 0$ Seibert and Trojovský (2005)

- Let k be a positive integer, $\ell \leq \frac{k-1}{2}, n$ and $m > k$ be nonnegative integers.
  $\sum_{i=0}^{m}(-1)^{\frac{i}{2}(2\ell+i+(-1)^k)} L_{(k-2\ell)(i+n)} \begin{bmatrix} k+1 \\ i \end{bmatrix} = 0$ Seibert and Trojovský (2005)

- $\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n-i \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \frac{F_n - F_k}{F_{n-k}}$ Trojovský (2007)

In addition to finding new proofs for these identities, it remains to conclude my work on mapping noncircular partition tilings for the general $m \times n$ grid to circular partition tilings. Following this, there are still other topics of interest to be explored. In particular, one could attempt to find a way to extend Sagan and Savage's combinatorial interpretation to hold even for sequences with nonideal initial conditions, examine what happens when F_n is replaced with f_n in the Fibonomial coefficients, and investigate whether anything of interest is counted when the domino restriction is removed from the noncircular partition tiling interpretation.

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