2-1-2005

Counting Lattice Points in Admissible Adelic Sets

Lenny Fukshansky
Claremont McKenna College

Counting lattice points in admissible adelic sets

Lenny Fukshansky

Texas A&M University

February 2005

Classical problem

Let C be a compact domain in $\mathbb{R}^N$, $N \geq 2$, and let Λ be a lattice of full rank in $\mathbb{R}^N$. For a positive real parameter t, write

$$tC = \{t\boldsymbol{x} : \boldsymbol{x} \in C\}.$$

Question 1. *What is the the cardinality of the set $\Lambda \cap tC$ as a function of t?*

This is a fundamental problem in Diophantine approximations, and it has a very large number of applications. The basic expectation is that

$$|\Lambda \cap tC| \approx \frac{Vol(tC)}{\det(\Lambda)}.$$

More specifically, the following asymptotic estimate is recorded in Lang's "Algebraic Number Theory".

Theorem 1. *If C has "nice" (Lipschitz parametrizable) boundary, then*

$$|\Lambda \cap tC| = \frac{\mathrm{Vol}(C)}{\det(\Lambda)} t^N + O(t^{N-1}).$$

The constant in O-notation depends on Λ, N, and the Lipschitz constants.

A large amount of work has been done in the direction of producing estimates for the error term in various more specific situations. These more explicit bounds can be extremely useful for applications. For instance, there is a celebrated result of Davenport giving an explicit, although somewhat complicated bound on the error term in case of a "nice" domain C.

An important application of such results is to counting points of bounded height over number fields and in algebraic varieties.

Height functions

Let K be a number field of degree d over $\mathbb{Q}$. Let $M(K)$ be the set of places of K. For each place $v \in M(K)$ let K_v be the completion of K at v and $d_v = [K_v : \mathbb{Q}_v]$ be the local degree. For each place $v \in M(K)$ we define the absolute value $\| \ \|_v$ to be the unique absolute value on K_v that extends either the usual absolute value on $\mathbb{R}$ or $\mathbb{C}$ if $v|\infty$, or the usual p-adic absolute value on $\mathbb{Q}_p$ if $v|p$, where p is a prime. We also define the second absolute value $| \ |_v$ for each place v by $|a|_v = \|a\|_v^{d_v/d}$ for all $a \in K$. Then for each non-zero $a \in K$ the *product formula* reads

$$\prod_{v \in M(K)} |a|_v = 1. \tag{1}$$

For each $v \in M(K)$, $v \nmid \infty$, we also define the local ring of *v-adic integers* by

$$O_v = \{a \in K_v : |a|_v \leq 1\}.$$

Then the ring of algebraic integers of K is

$$O_K = \bigcap_{v \nmid \infty} O_v.$$

We extend absolute values to vectors by defining the local heights. For each $v \in M(K)$ define a local height H_v for each $\boldsymbol{x} \in K_v^N$ by

$$H_v(\boldsymbol{x}) = \begin{cases} \max_{1 \leq i \leq N} |x_i|_v & \text{if } v \nmid \infty \\ \left(\sum_{i=1}^N \|x_i\|_v^2 \right)^{d_v/2d} & \text{if } v | \infty \end{cases}$$

We define the following global height function on K^N:

$$H(\boldsymbol{x}) = \prod_{v \in M(K)} H_v(\boldsymbol{x}), \tag{2}$$

for each $\boldsymbol{x} \in K^N$. Notice that H is projectively defined, meaning that

$$H(a\boldsymbol{x}) = H(\boldsymbol{x}),$$

for every nonzero $a \in K$, $\boldsymbol{x} \in K^N$. Hence we can talk about height of projective points.

We also define height on subspaces of K^N. Let $V \subseteq K^N$ be an M-dimensional subspace, and let $\boldsymbol{x}_1, ..., \boldsymbol{x}_M$ be a basis for V. Then

$$\boldsymbol{x}_1 \wedge ... \wedge \boldsymbol{x}_M \in K^{\binom{N}{M}}$$

under the standard embedding. Define

$$H(V) = H(\boldsymbol{x}_1 \wedge ... \wedge \boldsymbol{x}_M).$$

This definition is legitimate, i.e. does not depend on the choice of the basis.

If V is subspace of K^N, write $\mathbb{P}(V)$ for the projective space over V. A fundamental property of height is the following.

Northcott's theorem: *For each subspace V of K^N, the set*

$$S_V(t) = \{\boldsymbol{x} \in \mathbb{P}(V) : H(\boldsymbol{x}) \leq t\}$$

is finite for every positive real number t.

Question 2. *What is the cardinality of $S_V(t)$ as a function of t?*

The first asymptotic answer to this question when $V = K^N$ was given by S. Schanuel in 1966/1979. We present a more general very nice version due to J. Thunder, 1993.

Theorem 2. *Let V be an M-dimensional subspace of K^N. Then*

$$|S_V(t)| = a(M,K)\frac{t^{dM}}{H(V)^d} + O(t^{dM-1}),$$

as $t \to \infty$, where the constant implicit in the O notation depends on M and K.

The constant $a(M,K)$ is explicit, but complicated. Also the bound is analogous to Lang's theorem in the classical case, not to Davenport's since it does not provide explicit upper and lower bounds, only asymptotics.

Notice that the set $S_V(t)$ is not really "nice", for instance it is not convex.

Adelic ball

Let P be any finite subset of $M(K)$, containing all archimedean places. Let

$$K_{\mathbb{A}}(P) = \prod_{v \in P} K_v \times \prod_{v \notin P} O_v.$$

Define the *ring of adeles* of K to be

$$K_{\mathbb{A}} = \bigcup_{P} K_{\mathbb{A}}(P),$$

where the union is taken over all such subsets P. We can embed K into $K_{\mathbb{A}}$ by the standard *diagonal* embedding

$$a \mapsto (a, a, ...).$$

The additive group $K_{\mathbb{A}}$ is locally compact under the topology that makes each $K_{\mathbb{A}}(P)$ into an open subset, and K under diagonal embedding is a discrete subgroup, i.e. a lattice. Therefore, an M-dimensional subspace V of K^N can be viewed as a lattice of rank M in the adelic space $K_{\mathbb{A}}^N$.

For a positive real number t, define the *adelic ball* of radius t by

$$B^N_K(t) = \prod_{v \nmid \infty} O_v \times \prod_{v \mid \infty} \{\boldsymbol{x} \in K_v^N : H_v(\boldsymbol{x}) \leq t\}.$$

This is a fundamental example of an *admissible* set in terms of the adelic geometry of numbers.

Let V be an M-dimensional subspace of K^N, and consider the set $V \cap B^N_K(t)$. It is easy to see that

$$V \cap B^N_K(t) \subset S_V(t),$$

but $V \cap B^N_K(t)$ is a "nicer" set. As an analogue of questions in the classical geometry of numbers, and for purposes of various Diophantine applications, it is interesting to estimate the cardinality of $V \cap B^N_K(t)$.

Theorem 3 (F., 2004). *Let $V \subseteq K^N$ be an M-dimensional subspace, $1 \leq M \leq N$, and let $t \geq 1$ be a real number. Then*

$$b_1(M,K)\frac{t^{dM}}{H(V)^d} \leq |B_K^N(t) \cap V|$$
$$\leq \left(b_2(M,K)\frac{t}{H(V)^d} + 1\right)(2\sqrt{2}t+1)^{dM-1}.$$

The constants $b_1(M,K)$, $b_2(M,K)$ are explicit, and the result provides actual upper and lower counting bounds, not just asymptotics. Moreover, comparing this with Thunder's result, one can see that

$$\lim_{t\to\infty} \frac{|V \cap B_K^N(t)|}{|S_V(t)|} = c(M,K),$$

i.e. is finite and depends only on K and M. This makes our result applicable in many situations that would require a Thunder-like bound with all explicit constants.