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# Continued Fractions: A New Form

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# Continued Fractions: A New Form

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# Abstract

While the traditional form of continued fractions is well-documented, a new form, designed to approximate real numbers between 1 and 2, is less well-studied. This report first describes prior research into the new form, describing the form and giving an algorithm for generating approximations for a given real number. It then describes a rational function giving the rational number represented by the continued fraction made from a given tuple of integers and shows that no real number has a unique continued fraction. Next, it describes the set of real numbers that are hardest to approximate; that is, given a positive integer  $n$ , it describes the real number  $\alpha$  that maximizes the value  $|\alpha - T_n|$ , where  $T_n$  is the closest continued fraction to  $\alpha$  generated from a tuple of length  $n$ . Finally, it lays out plans for future work.



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# Chapter 1

## Introduction

This chapter gives an overview into the basic idea of continued fractions, describing both the most common form that they take and elaborating on the new form that is to be the subject of this thesis.

### 1.1 Ordinary Continued Fractions

Continued fractions are usually represented as

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

In the above model,  $a_0$  may be any integer, and higher-indexed terms must be positive integers. This form can be used to approximate any real number, and its properties have been studied extensively by Aleksandr Khinchin, among others (1992).

### 1.2 The New Form

Modified versions of the formula include a variant described by Pippenger in 1979:

$$1 + \frac{1}{-1 + t_k \left( 1 + \frac{1}{-1 + t_{k-1} (-1 + \dots)} \right)}$$

Here, each  $t_i$  must be an integer greater than or equal to 2. This model approximates real numbers between 1 and 2, and its properties are not particularly well-known.



## 2 Introduction

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Note that a continued fraction of this form is defined by a tuple of  $k$  integers. Here, I define the *order* of a continued fraction to be  $k$ ; that is, it is the number of integers used to describe the fraction.

## Chapter 2

# Algorithms

The first algorithm below describes a way to approximate real numbers with the new form of continued fractions described in the previous chapter. The second describes a method for taking a tuple of integers and determining what real number is perfectly approximated by the continued fraction constructed from it.

### 2.1 Finding an Approximation

When given a real number  $\alpha$  such that  $1 < \alpha \leq 2$ , we can generate a continued fraction approximation with the following algorithm:

First, we set

$$y_0 = \alpha.$$

Then, starting with  $i = 0$ , as long as  $y_i > 1$ , we define the following:

$$z_{i+1} = 1 + \frac{1}{-1 + y_i}.$$

Given  $z_{i+1}$ , we set it equal to

$$z_{i+1} = t_{i+1} + \rho_{i+1},$$

where  $t_{i+1}$  is an integer and  $0 \leq \rho_{i+1} < 1$ . Finally,

$$y_{i+1} = z_{i+1}/t_{i+1}.$$

Halting after finding  $y_n$  gives an approximation using  $n$  integers. We define the *order* of a continued fraction to be this value  $n$ .

## 2.2 Finding a Continued Fraction from a Tuple

**Definition 1.** The tuple  $(t_1, t_2, \dots, t_n)$  indicates the rational number that, under the above algorithm, gives  $y_i = t_i$  when  $0 < i \leq n$  and terminates on the  $n$ th step.

Given integers  $t_1, t_2, \dots, t_n$ , it is often useful to determine what continued fraction is created from these integers. Finding an explicit formula for the value to which these integers are associated will allow us to make statements about these new continued fractions.

**Theorem 2.1.** The continued fraction generated by the tuple  $(t_1, t_2, \dots, t_n)$  is equal to

$$\frac{\prod_{i=1}^n t_i}{\sum_{j=0}^n (-1)^{j+n} \prod_{k=1}^j t_k}.$$

*Proof.* We prove this theorem by induction.

*Base Case:* Let  $n = 1$ . Then, the continued fraction is simply  $1 + \frac{1}{-1+t_1}$ , which is equal to  $\frac{t_1-1}{t_1-1} + \frac{1}{t_1-1} = \frac{t_1}{t_1-1}$ . This expression matches the formula given above.

*Inductive Hypothesis:* Assume that the formula above holds whenever  $n = n_0$ . We wish to show that it also holds when  $n = n_0 + 1$ .

A continued fraction of order  $n_0 + 1$  is of the form

$$1 + \frac{1}{-1 + t_{n_0+1} \Phi(n_0)},$$

where  $\Phi(n_0)$  is a continued fraction of order  $n_0$ . By the inductive hypothesis, we know that

$$\Phi(n_0) = \frac{\prod_{i=1}^{n_0} t_i}{\sum_{j=0}^{n_0} (-1)^{j+n_0} \prod_{k=1}^j t_k}.$$

Thus, our continued fraction of order  $n_0 + 1$  is equal to

$$\begin{aligned}
 &= \frac{t_{n_0+1} \Phi(n_0)}{t_{n_0+1} \Phi(n_0) - 1} xyz \\
 &= \frac{t_{n_0+1} \frac{\prod_{i=1}^{n_0} t_i}{\sum_{j=0}^{n_0} (-1)^{j+n_0} \prod_{k=1}^j t_k}}{t_{n_0+1} \frac{\prod_{i=1}^{n_0} t_i}{\sum_{j=0}^{n_0} (-1)^{j+n_0} \prod_{k=1}^j t_k} - 1} \\
 &= \frac{t_{n_0+1} \prod_{i=1}^{n_0} t_i}{t_{n_0+1} \prod_{i=1}^{n_0} t_i - \sum_{j=0}^{n_0} (-1)^{j+n_0} \prod_{k=1}^j t_k} \\
 &= \frac{\prod_{i=1}^{n_0+1} t_i}{\sum_{j=0}^{n_0+1} (-1)^{j+n_0+1} \prod_{k=1}^j t_k},
 \end{aligned}$$

as desired. □



## Chapter 3

# Nonuniqueness

We now show that

$$(t_1, t_2, \dots, t_n) = (t_1, t_1 - 1, t_2, \dots, t_n).$$

Consider the right-hand side of the expression. It becomes equal to

$$\begin{aligned} & \frac{t_n t_{n-1} \cdots (t_1 - 1) t_1}{t_n t_{n-1} \cdots (t_1 - 1) t_1 - t_{n-1} \cdots (t_1 - 1) t_1 + \cdots \pm t_1 (t_1 - 1) \mp t_1 \pm 1} \\ &= \frac{t_n t_{n-1} \cdots (t_1 - 1) t_1}{t_n t_{n-1} \cdots (t_1 - 1) t_1 - t_{n-1} \cdots (t_1 - 1) t_1 + \cdots \pm (t_1 - 1)^2}. \end{aligned}$$

Dividing by  $(t_1 - 1)$ , we get

$$= \frac{t_n t_{n-1} \cdots t_2 t_1}{t_n t_{n-1} \cdots t_2 t_1 - t_{n-1} \cdots t_2 t_1 + \cdots \pm (t_1 - 1)},$$

which is the formula for the left-hand side.

Since any continued fraction can have an integer inserted to give another continued fraction with the same value, no rational number has a unique fraction associated with it. Note, however, that these two equivalent continued fractions are of different order; therefore, in addition, no rational number can be represented by continued fractions of only one order. The statement of nonuniqueness here does not imply that there are multiple continued fractions of minimal order associated with a given rational number.



## Chapter 4

# Bounds on the Worst-Approximated Number

Given a positive integer  $n$ , we wish to find the real number that is hardest to approximate with any continued fraction of order  $n$ . That is, we wish to find  $\alpha_n$  that maximizes the minimum of  $|\alpha_n - \Phi_n|$  over all continued fractions  $\Phi_n$  of order  $n$ .

### 4.1 The Alternating Bounds

First, note that

$$(t_1, t_2, \dots, t_n) = \frac{t_1 \prod_{i=2}^n t_i}{t_1 (\sum_{j=1}^n (-1)^{j+n} \prod_{k=2}^j t_k) + (-1)^n}.$$

Thus, we notice that, as  $t_1$  approaches  $\infty$ ,

$$(t_1, t_2, \dots, t_n) \rightarrow (t_2, \dots, t_n).$$

In addition, it approaches that value monotonically, increasing when  $n$  is even and decreasing when  $n$  is odd. Furthermore, since we know that

$$(t_1, t_2, \dots, t_n) = (t_1, t_1 - 1, t_2, \dots, t_n),$$

each  $n$ -order continued fraction where  $t_1 = t_2 + 1$  is equal to a continued fraction of order  $n - 1$ . Furthermore, the difference between continued fractions of the form  $(t_1, t_2, \dots, t_n)$  and  $(t_1 + 1, t_2, \dots, t_n)$  is maximized when  $t_1$  is minimized. Thus, we only need to consider continued fractions of the form  $(t_2 + 2, t_2, t_3, \dots, t_n) = (t_2 + 2, t_2 + 1, t_2, t_3, \dots, t_n)$ .



We have shown that, given a tuple  $(t_1, \dots, t_n)$  representing the continued fraction for a given  $\alpha \in \mathbb{Q}$ , the continued fractions  $(x, t_1, \dots, t_n)$  proceed in strictly increasing or decreasing order for increasing values of  $x$  (where  $x > t_1$ ), depending on the parity of  $n$ . We will denote by  $(x, \alpha)$  the continued fraction composed from the tuple  $(x, t_1, \dots, t_n)$ .

Let  $m_\alpha(x)$  be the difference between  $(x, \alpha)$  and  $(x + 1, \alpha)$ . Because there are no order- $(n + 1)$  continued fractions between these two values,  $m_\alpha(x)$  represents the distance between consecutive continued fractions. Evaluating this function, if  $\alpha = \frac{c}{d}$ , we have

$$\begin{aligned} m_\alpha(x) &= \left| \frac{cx}{dx + (-1)^n} - \frac{c(x+1)}{d(x+1) + (-1)^n} \right| \\ &= \frac{c}{(dx + (-1)^n)(dx + d + (-1)^n)}. \end{aligned}$$

Denote by  $M_n(x)$  the maximum value of  $m_\alpha(x)$  over all  $\alpha$  with order  $n$ . If we can show that  $M_n(x) > M_n(x + 1)$  for all  $x$  and  $n$ , then we will have demonstrated that the maximum error occurs when  $x$  is minimized with respect to  $n$ , showing that the maximum interval for an order- $(n + 1)$  continued fraction is a subinterval of the maximum interval for an order- $n$  continued fraction.

We examine two cases, based on the parity of  $n$ .

First, let  $n$  be even. Then,

$$m_\alpha(x) = \frac{c}{(dx + 1)(dx + d + 1)}.$$

If we replace each 1 with a 0, we see that this expression is strictly less than

$$\frac{c}{dx(dx + d)} = \frac{\alpha}{d} \frac{1}{x(x + 1)}.$$

If we replace each 1 with a  $d$ , we see that this expression is strictly greater than

$$\frac{c}{(dx + d)(dx + 2d)} = \frac{\alpha}{d} \frac{1}{(x + 1)(x + 2)}.$$

So, for the  $\alpha_{max}$  and its corresponding  $d_{max}$  that maximize the distance between two continued fractions of order  $n + 1$ , we have

$$M_n(x) > \frac{\alpha_{max}}{d_{max}} \frac{1}{(x + 1)(x + 2)} > M_n(x + 1),$$

implying that  $m_\alpha(x)$  is minimized by minimizing  $x$ .

Next, let  $n$  be odd. Then,

$$m_\alpha(x) = \frac{c}{(dx-1)(dx+d-1)}.$$

If we replace each  $-1$  with a  $0$ , we see that this expression is strictly greater than

$$\frac{c}{dx(dx+d)} = \frac{\alpha}{d} \frac{1}{x(x+1)}.$$

If we replace each  $-1$  with a  $-d$ , we see that this expression is strictly less than

$$\frac{c}{(dx-d)dx} = \frac{\alpha}{d} \frac{1}{x(x-1)}.$$

So, for the  $\alpha_{max}$  and its corresponding  $d_{max}$  that maximize the distance between two continued fractions of order  $n+1$ , we have

$$M_n(x) > \frac{\alpha_{max}}{d_{max}} \frac{1}{x(x+1)} > M_n(x+1),$$

implying that  $m_\alpha(x)$  is minimized by minimizing  $x$ .

In either case, we have shown that the largest interval between continued fractions of order  $n+1$  happens at the smallest possible value of  $x$ , implying that the largest such interval is a subinterval of the interval previously described for order  $n$ .

**Theorem 1.** *The greatest interval between two continued fractions of order  $n$  occurs between  $(n+1, n, \dots, 3, 2)$  and  $(n+2, n, \dots, 3, 2)$ .*

*Proof.* We proceed by induction.

*Base Case:* Let  $n = 1$ . Then, the difference between consecutive continued fractions  $(t_1)$  and  $(t_1 + 1)$  is equal to

$$\frac{1}{t_1-1} - \frac{1}{t_1} = \frac{1}{t_1(t_1-1)}.$$

This expression is maximized when  $t_1 = 2$ . Thus, the largest interval occurs between (2) and (3).

*Inductive Step:* Suppose the largest interval between two order- $n$  continued fractions occurs between  $(n+1, n, \dots, 3, 2)$  and  $(n+2, n, \dots, 3, 2)$ . We must show that the largest interval between two order- $(n+1)$  continued fractions occurs between  $(n+2, n+1, n, \dots, 3, 2)$  and  $(n+3, n+1, n, \dots, 3, 2)$ . First, observe that

$$(n+2, n, \dots, 3, 2) = (n+2, n+1, n, \dots, 3, 2).$$

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Second, given any integers  $t_2, t_3, \dots, t_n, t_{n+1}$ , we can write the continued fraction  $(t_1, t_2, t_3, \dots, t_n, t_{n+1})$  as

$$\frac{\prod_{i=1}^{n+1} t_i}{\sum_{j=0}^{n+1} (-1)^{j+n+1} \prod_{k=1}^j t_k}.$$

If we let  $c = \prod_{i=2}^{n+1} t_i$  and  $d = \sum_{j=2}^{n+1} (-1)^{j+n+1} \prod_{k=2}^j t_k$ , then the continued fraction becomes

$$\frac{t_1 \cdot c}{t_1 \cdot d + (-1)^{n+1}}.$$

Here, we notice two things. First, if we take the limit as  $t_1 \rightarrow \infty$ , this continued fraction approaches  $\frac{c}{d} = (t_2, t_3, \dots, t_n, t_{n+1})$ , and it does so monotonically. Second, the difference between two continued fractions with consecutive values of  $t_1$  is equal to

$$\begin{aligned} & \left| \frac{t_1 \cdot c}{t_1 \cdot d + (-1)^{n+1}} - \frac{(t_1 + 1) \cdot c}{(t_1 + 1) \cdot d + (-1)^{n+1}} \right| \\ &= \frac{c}{(dt_1 + (-1)^{n+1})(d(t_1 - 1) + (-1)^{n+1})}. \end{aligned}$$

Again, since  $d$  and  $t_1$  are both  $\geq 2$ , this value is minimized by the smallest possible value of  $t_1$ . If we specify that  $t_1 > t_2$ , then we get that the greatest interval between two continued fractions with fixed  $t_2, t_3, \dots, t_{n+1}$  occurs where  $t_1 = t_2 + 1$ . Thus, the endpoints of the interval are  $(t_2 + 1, t_2, \dots, t_{n+1})$  and  $(t_2 + 2, t_2, \dots, t_{n+1})$ .

If we let  $t_i = n - i + 2$  for all  $i \neq 1$ , then we get the continued fraction  $(t_1, n + 1, n, \dots, 3, 2)$  for some value of  $t_1$ . From above, we know that all of these values fall between the continued fractions  $(n + 1, n, \dots, 3, 2)$  and  $(n + 2, n, \dots, 3, 2)$ , which, by the inductive hypothesis, form the largest interval for continued fractions of order  $n$ . We also know that the largest subinterval here when considering continued fractions of order  $n + 1$  occurs where  $t_1 = t_2 + 1$  and  $t_1 = t_2 + 2$ . Thus, the interval for order  $n + 1$  has endpoints

$$(n + 2, n + 1, n, \dots, 3, 2) \text{ and } (n + 3, n + 1, n, \dots, 3, 2),$$

as desired. □

## 4.2 Convergence of Bounds

From the formula given in Chapter 2, we know that

$$(n+1, n, \dots, 3, 2) = \frac{(n+1) \times n \times \dots \times 3 \times 2}{(n+1) \times n \times \dots \times 3 \times 2 - n \times \dots \times 3 \times 2 + \dots + (-1)^{n-1} 2 + (-1)^n 1}.$$

Dividing both the numerator and denominator by  $(n+1)!$ , we get

$$\frac{1}{1 - \frac{1}{2!} + \frac{1}{3!} \dots + (-1)^n \frac{1}{n!}}.$$

Taking the limit as  $n$  approaches  $\infty$ , this expression becomes

$$\frac{1}{-\sum_{i=1}^{\infty} \frac{(-1)^i}{i!}},$$

which is equal to

$$\begin{aligned} & \frac{1}{1 - e^{-1}} \\ &= \frac{e}{e - 1}. \end{aligned}$$



## Chapter 5

# Average Error

In addition to finding the behavior for the greatest possible error among continued fractions of order  $n$ , it is also useful to find the error in the average case. We find the average error by assuming that the real numbers between 1 and 2 are assigned a uniform distribution.

If we are given an interval of length  $\delta$ , we know that the mean value of the distance from a point in that interval to one of the endpoints is equal to the mean value of the function  $f(x) = x$  on the interval  $[0, \delta/2]$ . This value is easily found to be  $\delta/4$ .

Given some positive integer  $n$ , we know (from a previous chapter) that all intervals have size  $m_\alpha(x)$ , for some  $\alpha$  with order  $n - 1$  and some integer  $x$  greater than the greatest integer in the tuple that generates  $\alpha$ . Thus, we can find the average case error for continued fractions of order  $n$  by summing over the average case on each interval, scaling by the size of that interval, giving

$$\frac{1}{4} \sum_{(\alpha, x)} m_\alpha^2.$$

### 5.1 Exact Form for Specific $n$

Let us examine the case where  $n = 1$ . Here, the only value of  $\alpha$  with order  $n - 1 = 0$  is 1; thus,  $m_\alpha(x) = \frac{1}{x(x-1)}$  by the distance formula given in the previous chapter. Thus, the average error for continued fractions of order 1 is

$$\frac{1}{4} \sum_{x=2}^{\infty} \frac{1}{x^2(x-1)^2} = \frac{\pi^2 - 9}{12} \approx 0.0725.$$

Note that the value of the summation above was found using Mathematica, although it can be found by using partial fractions.

Next, let us examine the case where  $n = 2$ . Here, each continued fraction can be expressed as the ordered pair  $(c, x)$ ; that is, each  $\alpha$  with order  $n - 1 = 1$  can be expressed as  $\frac{c}{c-1}$ . By the distance formula given in the previous chapter,

$$m_\alpha(x) = \frac{c}{((c-1)x+1)((c-1)x+c)}.$$

Thus, the average error for continued fractions of order  $n = 2$  is

$$\frac{1}{4} \sum_{c=2}^{\infty} \sum_{x=c}^{\infty} \left( \frac{c}{((c-1)x+1)((c-1)x+c)} \right)^2.$$

So far, I have been unable to find a closed form for this summation.

Finding these sums for greater values of  $n$  quickly becomes very difficult.

## 5.2 Asymptotic Behavior

In examining the worst case error, I found that, if  $\alpha$  has odd order,

$$\frac{\alpha}{d} \frac{1}{x(x-1)} > M_\alpha(x) > \frac{\alpha}{d} \frac{1}{x(x+1)},$$

and if  $\alpha$  has even order,

$$\frac{\alpha}{d} \frac{1}{x(x+1)} > M_\alpha(x) > \frac{\alpha}{d} \frac{1}{(x+2)(x+1)}.$$

In both cases,  $M_\alpha(x)$  is  $\Theta(x^{-2})$ . Thus, for any given  $\alpha$  of order  $n$ , there are an infinite number of intervals between  $\alpha = (\alpha', y)$  and  $(\alpha', y+1)$  (where  $\alpha'$  has order  $n-1$ ) that have size approximately  $\frac{d}{x^2}$ , where  $d$  is the distance from  $\alpha$  to  $(\alpha', y+1)$ . Since the original interval contributes  $\frac{d^2}{4}$  to the total average error, the contribution of all the new intervals is

$$\sum_{x=y+1}^{\infty} \frac{d^2}{4x^4} \approx \int_{y+1}^{\infty} \frac{d^2}{4x^4} = \frac{d^2}{12(y+1)^3}.$$

The contribution from each interval is multiplied by  $(3(y+1)^3)^{-1}$ ; also,  $y+1 > n+1$ . Thus, if  $f(n)$  is the average error for continued fractions of order  $n$ , we have the recurrence

$$f(n+1) < \frac{f(n)}{3(n+1)^3}.$$

Therefore,  $f(n) \in O(3^{-n}n!^{-3})$ .





## Chapter 6

### Future Work

The average-case error is only given for  $n = 1$  above, and the case for  $n = 2$  does not have a known closed form. Finding the exact average-case error for more small values of  $n$  may provide insight into a general formula for the average-case error. An upper bound on the asymptotic bound of the average-case error is given above; however, this is not known to be a strict upper bound, and no lower bound is given. Finding these bounds may also prove valuable.



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