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Dear Colleague,

This newsletter follows a three-day **Conference to Examine Mathematics as a Humanistic Discipline** in Claremont 1986 supported by The Exxon Education Foundation, and a special session at the AMS-MAA meeting in San Antonio January 1987. A common response of the thirty-six mathematicians at the conference was, "I was startled to see so many who shared my feelings".

Two related themes that emerged from the conference were 1) teaching mathematics humanistically, and 2) teaching humanistic mathematics. The first theme sought to place the student more centrally in the position of inquirer than is generally the case, while at the same time acknowledging the emotional climate of the activity of learning mathematics. What students could learn from each other, and how they might better come to understand mathematics as a meaningful rather than an arbitrary discipline were among the idea of the first theme.

The second theme was focused less upon the nature of the teaching and learning environment and more upon the need to reconstruct the curriculum and the discipline of mathematics itself. The reconstruction would relate mathematical discoveries to personal courage, relate discovery to verification, mathematics to science, truth to utility, and in general, to relate mathematics to the culture in which it is embedded.

Humanistic dimensions of mathematics discussed at the conference included:

a) An appreciation of the role of intuition, not only in understanding, but in creating concepts that appear in their finished versions to be "merely technical".

b) An appreciation for the human dimensions that motivate discovery – competition, cooperation, the urge for holistic pictures.

c) An understanding of the value judgements implied in the growth of any discipline. Logic alone never completely accounts for *what* is investigated, *how* it is investigated, and *why* it is investigated.

d) There is a need for new teaching, learning formats that will help wean our students from a view of knowledge as certain, to-be-received.

e) The opportunity for students to think like a mathematician, including a chance to work on tasks of low definition, to generate new problems and to participate in controversy over mathematical issues.

f) Opportunities for faculty to do research on issues relating to teaching, and to be respected for that area of research.

This newsletter, also supported by Exxon, is part of an effort to fulfill the hopes of the participants. Others who have heard about the conferences have enthusiastically joined the effort. The newsletter will help create a network of mathematicians and others who are interested in sharing their ideas and experiences related to the conference themes. The network will be a community of support extending over many campuses that will end the isolation that individuals may feel. There are lots of good ideas, lots of experimentation, and lots of frustration because of isolation and lack of support. In addition to informally sharing bibliographic references, syllabi, accounts of successes and failures, . . . , the network might formally support writing, team-teaching, exchanges, conferences,. . ..

Please send references, essays, half-baked ideas, proposals, suggestions, and whatever you think appropriate for this quarterly newsletter. Also send names of colleagues who should be added to the mailing list. All mail should be addressed to

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This issue contains some papers and excerpts of papers that were presented at the conferences.

APPLIED MATHEMATICS AS SOCIAL CONTRACT

Philip J. Davis

Abstract

The author takes the position that mathematical education must redefine its goals so as to create a citizenry with sufficient knowledge to provide social backpressure on future mathematizations. This can be accomplished by increasing the part of mathematical education that is devoted to the description and interpretation of the processes of mathematization and by allowing the technicalities of the formal operations within mathematics itself to be deemphasized or automated out by computer.

This Mathematized World

As compared to the medieval world or the world of antiquity, today's world is characterized as being scientific, technological, rational and mathematized. By "rational" I mean that by an application of reason or of the formalized versions of reason found in mathematics, one attempts to understand the world and control the world. By "mathematized", I shall mean the employment of mathematical ideas or constructs, either in their theoretical form or in computer manifestations, to organize, to describe, to regulate and to foster our human activities. By adding the suffix "ized", I want to emphasize that it is humans who, consciously or unconsciously, are putting the mathematizations into place and who are affected by them. It is of vital importance to give some

account of mathematics as a human institution, to arrive at an understanding of its operation and at a philosophy consonant with our experience with it, and on this basis to make recommendations for future mathematical education.

The pace of mathematization of the world has been accelerating. It makes an interesting exercise for young students to count how many numbers are found on the front page of the daily paper. The mere number of numbers is surprising, as well as the diversity and depth of the mathematics that underlies the numbers; and if one turns to the financial pages or the sports pages, one sees there natural language overwhelmed by digits and statistics. Computerization represents the effective means for the realization of current mathematizations, as well as an independent driving force toward the installation of an increasing number of mathematizations.

Philosophies of Mathematics

Take any statement of mathematics such as 'two plus two equals four', or any more advanced statement. The common view is that such a statement is perfect in its precision and in its truth, is absolute in its objectivity, is universally interpretable, is eternally valid and expresses something that must be true in this world and in all possible worlds. What is mathematical is certain. This view, as it relates, for example, to the history of art and the utilization of mathematical perspective has been expressed by Sir Kenneth Clark: ("Landscape into Art"): "The Florentines demanded more than an empirical or intuitive rendering of space. They demanded that art should be concerned with *certezza*, not with *opinioni*. Certezza can be established by mathematics".

The view that mathematics represents a *timeless* ideal of absolute truth and objectivity and is even of nearly divine origin is often called Platonist. It

conflicts with the obvious fact that we humans have invented or discovered mathematics, that we have installed mathematics in a variety of places both in the arrangements of our daily lives and in our attempts to understand the physical world. In most cases, we can point to the individuals who did the inventing or made the discovery or the installation, citing names and dates. Platonism conflicts with the fact that mathematical applications are often *conventional* in the sense that mathematizations other than the ones installed are quite feasible (e.g., the decimal system). The applications are often *gratuitous*, in the sense that humans can and have lived out their lives without them (e.g., insurance or gambling schemes). They are *provisional* in the sense that alternative schemes are often installed which are claimed to do a better job (Examples range all the way from tax legislation to Newtonian mechanics). Opposed to the Platonic view is the view that a mathematical experience combines the external world with our interpretation of it, via the particular structure of our brains and senses, and through our interaction with one another as communicating, reasoning beings organized into social groups.

The perception of mathematics as quasi-divine prevents us from seeing that we are surrounded by mathematics because we have extracted it out of unintellectualized space, quantity, pattern, arrangement, sequential order, change, and that as a consequence, mathematics has become a major modality by which we express our ideas about these matters. The conflicting views, as to whether mathematics exists independently of humans or whether it is a human phenomenon, and the emphasis that tradition has placed on the former view, leads us to shy away from studying the processes of mathematization, to shy away from asking embarrassing questions about this process: how do we install the mathematizations, why do we install them, what are they doing for

us or to us, do we need them, do we want them, on what basis do we justify them. But the discussion of such questions is becoming increasingly important as the mathematical vision transforms our world, often in unforeseen ways, as it both sustains and binds us in its steady and unconscious operation. Mathematics creates a reality that characterizes our age.

The traditional philosophies of mathematics: platonism, logicism, formalism, intuitionism, in any of their varieties, assert that mathematics expresses precise, eternal relationships between atemporal mental objects. These philosophies are what Thomas Tymoczko has called "private" theories. In a private theory, there is one ideal mathematician at work, isolated from the rest of humanity and from the world, who creates or discovers mathematics by his own logico-intuitive processes. As Tymoczko points out, private theories of the philosophy of mathematics provide no account either for mathematical research as it is actually carried out, for the applications of mathematics as they actually come about, or for the teaching process as it actually unfolds. When teaching goes on under the banner of conventional philosophies of mathematics, it often becomes to a formalist approach to mathematical education: "do this, do that, write this here and not there, punch this button, call in that program, apply this definition and that theorem." It stresses operations. It does not balance operations with an understanding of the nature or the consequences of the operations. It stresses syntactics at the expense of semantics, form at the expense of meaning. A fine place to read about this is in "L'age du Capitaine" by Stella Baruk, a mathematics supervisor in a French school system. Baruk writes

"From Pythagoras in antiquity to Bourbaki in our own day, there has been maintained a tradition of instruction - religion which sacrifices

full understanding to the recitation of formal and ritual chatechisms, which create docility and which simulate sense. All this has gone on while the High Priests of the subject laugh in their corners."

How many university lecturers, discoursing on numbers, say, allow themselves to discuss where they think numbers come from, what is one's intuition about them, how number concepts have changed, what applications they have elicited, what have been the pressures exerted by applications, how we are to interpret the consequences of these applications, what is the poetry of numbers is or their drama or their mysticism, why there can be no complete or final understanding of them. How many lecturers would take time to discuss the question put by Bertrand Russell in a relaxed moment: "What is the Pythagorean power by which number holds sway above the flux?"

Opposed to "private" theories, there are "public" theories of the philosophy of mathematics in which the teaching process is of central importance. Several writers in the past half century have been constructing public theories, and I should like to add a few bricks to this growing edifice and to point out its relevance for the future of mathematical education.

Applied Mathematics as Social Contract

I shall emphasize the applications of mathematics to the social or humanistic areas though one can make a case for applications to scientific areas and indeed to pure mathematics itself. (See, e.g., Spalt).

Today's world is full of mathematizations that were not here last year or ten years ago. There are other mathematizations which have been discarded (e.g., Ptolemaic astronomy, numerological interpretations of the cosmos, last years' tax laws) which have been discarded. How do these mathematizations come about? How are they implemented, why are they accepted? Some are so new, for example, credit cards, that we can actually document their installation. Some are so ancient, e.g., numbers themselves, that the historical scenarios that have been written are largely speculative. Are mathematizations put in place by divine fiat or revelation? By a convention of Elders? By the insights of a gifted few? By an evolutionary process? By the forces of the market place or of biology? And once they are in place what keeps them there? Law? Compulsion? Inertia? Darwinian advantage? The development of a bureaucracy where sole function it is to maintain the mathematization? The development of businesses whose function it is to create and sell the mathematization? Well, all of the above, at times, and more. But, for all the lavish attention that our historians of mathematics have paid to the evolution of ideas within mathematics itself, only token attention has been paid by scholars and teachers to the interrelationship between mathematics and society. A description of mathematics as a human institution would be complex indeed, and not be easily epitomized by a catch phrase or two.

The employment of mathematics in a social context is the imposition of a certain order, a certain type of organization. Government, as well, is a certain type of organization and order. Philosophers of the 17th and 18th-century (Hobbes, Locke, Rousseau, Thomas Paine, etc.) put forward an idea, known as social contract, to explain the origin of government. Social contract is an act by which an agreed upon form of social organization is

established. (Here I follow an article by Michael Levin.) Prior to the contract there was supposedly a "state of nature". This was far from ideal. The object of the contract, as Rousseau put it, was "to find a form of association which will defend the person and goods of each member with the collective force of all, and under which each individual, while counting himself with the others, obeys no one but himself, and remains as free as before". In this way, one may improve on a life which, as Hobbes put it in a famous sentence, was "solitary, poor, nasty, brutish, and short". The contract itself, whether oral or written, was almost thought of as having been entered into at a definite time and place. Old Testament history, with its covenants between God and Noah, Abraham, Moses, the Children of Israel, was clearly in the minds of contract theorists. In the United States, political thinking has often been in terms of contracts, as in the Mayflower Compact, the Constitutions of the United States and of the individual states, the Establishment of the United Nations in San Francisco in 1945, and periodic proposals for constitutional amendments and reform.

It was generally assumed by the contract theorists that "Human society and government are the work of man constructed according to human will even if sometimes operating under divine guidance". That "man is a free agent, rather than a being totally determined by external forces", and that society and government are based on mutual agreement rather than on force. (See Levin)

The acceptability of social contract as a historical explanation hardly lasted till the 19th century, even if political contracts continued to be entered into as instances of democratic polity. It is an instructive exercise, I believe, in order to get a grasp on the relationship between society and mathematics, to

take the outline of social contract just given and replace the words "government and society" by the word "mathematization". Though it is naive to think that most mathematizations came about by formal contracts, the "contract" metaphor is a useful phrase to designate the interplay between people and their mathematics and to make the point that mathematizations are the work of man, constructed according to human will, even if operating under a guidance which may be termed divine or logical or experimental according to one's philosophic predilection.

A number of authors, some writing about theology, and others about political or economic processes, have pointed out that contracts are continuously entered into, broken, and reestablished. I believe the same is the case for mathematizations. Consider, for example, insurance. This is one of the great mathematizations currently in place, and I personally, without adequate coverage, would consider myself naked to the world. Yet I am free to throw away my insurance policies. Consider the riders that insurance companies send me, unilaterally abridging their previous agreements. Consider also that in a litigious age, with a populace abetted by eager lawyers and unthinking juries, what appears as the 'natural' stability of the averages upon which possibility of insurance is based, emerges, on deeper analysis, to incorporate the willingness of the community to adjust its affairs in such a way that the averages are maintained. The possibilities of insurance can be destroyed by our own actions.

Another example that displays the relationship between mathematics, experience, and law is the highway speed limit in the United States. Before the gas shortage in 1974, the limit was 65 miles per hour. In 1974, the speed limits were reduced to 55 miles per hour in order to conserve gasoline.

As a side effect, it was found that the number of highway accidents was reduced significantly. Now (1987) the gas shortage is over, and there is pressure to raise the legal speed limit. Society must decide what price it is willing to pay for what some see as the convenience or the thrill of higher speeds. Here is mathematical contract at work.

The process of contract maintenance, renewal or reaffirmation, in all its complexities, is open to study and description. This is a proper part of applied mathematics and I shall argue that it should be a proper part of mathematical education.

Where is Knowledge Lodged?

There is an epistmological approach to the interplay between mathematics and society and that is to look at the way society answers the question that heads this section. According to how we answer this question, we will mathematize differently and we will teach differently.

Where, then, is knowledge lodged? (Here I follow an article by Kenneth A. Bruffee.) In the pre-Cartesian age, knowledge was often thought to be lodged in the mind of God. Those who imparted knowledge authoritatively derived their authority from their closeness to the mind of God, evidence of this closeness was often taken to be the personal godliness of the authority.

In the post-Cartesian age knowledge was thought to be lodged in some loci that are above and beyond ourselves, such as sound reasoning or creative genius or in the 'object objectively known'.

A more recent view, connected perhaps with the names of Kuhn and Lakatos, is that knowledge is socially justified belief. In this view, knowledge is not located in the written word or in symbols of whatever kind. It is

located in the community of practitioners. We do not create this knowledge as individuals but we do it as part of a belief community. Ordinary individuals gain knowledge by making contact with the community of experts. The teacher is a representative of the belief community.

In my view, knowledge as socially justified belief provides a fair description of how mathematical knowledge is ligitimized but we must keep clearly in mind that perceptions of what 'is', theory formation, validation, and utilization, are all part of a dynamic and iterative process. Knowledge once thought to be absolute, indubitable, is now seen as provisional or even probabilistic. Science is seen as a search for error as much as it is a search for truth. Eternally valid knowledge, may remain an ideal which we hold in our minds as a spur to inquiry. This view fits with the idea of applied mathematics as social contract, with the contractual arrangements being concluded, broken, and renegotiated in endless succession.

Another view of the locus of knowledge, not yet elevated to a philosophy, is that knowledge is located in the computer. One speaks of such things as 'artificial intelligence', 'expert systems', and more than one theoretical physicist has opined that all the essentials are now known (despite the fact that the same was asserted 100 years ago and 200 years ago) and that the computer can fill in the details and derive the consequences for the future.

Advocates of this view have asserted that while education is now teacher oriented, in the full bloom of the computer age, education will be knowledge oriented. These two contemporary views are not necessarily antithetical, provided we accommodate the computer into the community of experts, clarify whether 'belief' can reside in a computer, and decide whether mankind exists for the sake of the computers or vice versa.

Mathematical Education at a Higher Metalevel

A mathematized and computerized world brings with it many benefits and many dangers. It opens many avenues and closes many others. I do not want to elaborate this point as I and my co-author Reuben Hersh have done so in our book "Descartes' Dream", as have numerous other authors.

The benefits and dangers both derive from the fact that the mathematical/computational way of thinking is different from other ways. Philosopher and historian Sir Isaiah Berlin called attention to this divergence when he wrote "A person who lacks common intelligence can be a physicist of genius, but not even a mediocre historian." For the mathematical way to gain ascendancy over other modes is to create an imbalance in human life.

The benefits and dangers derive also from the fact that mathematics is a kind of language, and this language creates a milieu for thought that is hard to escape. It both sustains us and confines us. As George Steiner has written of natural language (1986): "The oppressive birthright is the language, the conventions of identification and perception. It is the established but customarily subconscious unargued constraints of awareness that enslave." One can assert as much for mathematics as a language. The subconscious modalities of mathematics and of its applications must be made clear, must be taught, watched, argued. Since we are all consumers of mathematics, and since we are both beneficiaries as well as victims, all mathematizations ought to be opened up in the public forums where ideas are debated. These debates ought to begin in the secondary school.

Discussions of changes in mathematics curricula generally center around (a) the specific mathematical topics to be taught, e.g., whether to

teach the square root algorithm, or continued fractions or projective geometry or Boolean algebra, and if so, in what grade, and (b) the instructional approaches to the specific topics, e.g., should they be taught with proofs or without; from the concrete to the abstract or vice versa, what emphasis should be placed on formal manipulations and what on intuitive understanding; with computers or without; with open ended problems or with "plug and chug" drilling.

Because of widespread, almost universal computerization, with handheld computers that carry out formal manipulations and computations of lower and higher mathematics rapidly and routinely, because also of the growing number of mathematizations, I should like to argue that mathematics instruction should, over the next generation, be *radically* changed. It should be moved up from subject oriented instruction to instruction in what the mathematical structures and processes mean in their own terms and what they mean when they form a basis on which civilization conducts its affairs. The emphasis in mathematics instruction ought to be moved from the syntactic-logico component to the semantic component. To use programming jargon, it ought to be "popped up" a metalevel. If, as some computer scientists believe, instruction is to move from being teacher oriented to knowledge-oriented - and I believe this would be disastrous - the way in which the role of the teacher can be preserved is for the teacher to become an interpreter and a critic of the mathematical processes and of the wce.ay these processes interact with knowledge as a database. Instruction in mathematics must enter an altogether new and revolutionary phase.

The Interpretive Component of Mathematical Education

Let me begin by asking the question: to what end do we teach mathematics? Over the millenia, answers have been given and they have differed. Some of them have been: we teach it for its own sake, because it is beautiful; we teach it because it reveals the divine; because it helps us think logically; because it is the language of science and helps us to understand and reveal the world; we teach mathematics because it helps our students to get a job either directly, in those areas of social or physical sciences that require mathematics; or indirectly, insofar as mathematics, through testing, acts as a social filter, admitting to certain professional possibilities those who can master the material. We teach it also to reproduce ourselves by producing future research mathematicians and mathematics teachers.

Ask the inverse question: what is it that we want students to learn? We may answer this by citing specific course contents. For example, we may say that we now want to emphasize discrete mathematics as opposed to continuous mathematics. Or that we want to develop a course in non-standard analysis on tape so that joggers may learn about hyperreal numbers even as they run. Or, we may decide for ourselves what the characteristic, constitutive ingredients of mathematical thought are: space, quantity, deductive structures, algorithms, abstraction, generalization, etc.., and simply assure that the student is fed these basic ingredients, like vitamins. All of these questions and answers have some validity, and tradeoffs must occur in laying out a curriculum.

Within an overcrowded mathematical arena with many new ideas competing for inclusion in a curriculum, I am asking for a substantial elevation in the awareness of the applications of mathematics that affect

society and of the consequences of these applications. If formal computations and manipulations can be learned rapidly and performed routinely be computer, what purpose would be served by tedious drilling either by hand or by computer? On-the-job training is certainly called for, whether at the supermarket checkout counter or on the blackboards of a hi-tech development company. If mathematics is a language, it is time to put an end to overconcentration on its grammar and to study the "literature" that mathematics has created and to interpret that literature. If mathematics is a logico-mechanism of a sort, then just as only a very few of us learn how to construct an automobile carburetor, but all of us take instruction in driving, so we must teach how to "drive" mathematically and *to interpret what it means when we have been driven mathematically in a certain manner.*

What does it mean when we are asked to create sex-free insurance pools? What are the consequences when people are admitted or excluded from a program on the basis of numerical criteria? How does one assess a statement that procedure A is usually effective in dealing with medical condition B? What does it mean when a mathematical criterion is employed to judge the quality of prose or the comprehensibility of a poem or to create music in a programmatic way? What are the consequences of a computer program whose output is automatic military retaliation? The list of questions that need discussion is endless. Each mathematization-computerization requires explanation and interpretation and assessment. None of these things are now discussed in mathematics courses in the concrete form that confronts the public. If a teacher were inclined to do so, the reaction from his colleagues would probably be "Well, that is not mathematics. That is applied mathematics or that is psychology or economics or social-anthropology or law or whatever." My

answer would be: I am trying, little by little, to bring in discussions of this sort into my teaching. It is difficult but important.

If the claim were made, with justice, that these matters cannot be discussed intelligently without deep knowledge both of mathematics and of the particular area of the real world, then I would agree, and point out that this claim forces into the open the conflict between democracy and "expertocracy". (See e.g., Prewitt). This conflict has received considerable attention in areas such as medicine, defense, and technological pollution, but has hardly been discussed at the level of an underlying mathematical language. The tension between the two claims, that of democracy and of expertocracy, could be made more socially productive by an education which enables a wide public to arrive at deeper assessments, moving from daily experience toward the details of the particular mathematizations. While we must keep in mind certain basic mathematical material, we must also learn to develop mathematical 'street smarts' which enable us to form judgements in the absence of technical expertise. (Cf. Prewitt).

A philosophy of mathematics which is "public" and not "private", lends support to introducting this kind of material into the curriculum. The discussion of such curriculum changes will be assisted by the perception of the mathematical enterprise as a human experience with contractual elements; and by the realization that every civilized person practices and utilizes mathematics at some level, and thereby enters a certain knowledge and belief community.

Again, following Kenneth Bruffees article (with additions and modifications), I would like to suggest several lines of inquiry.

(a) Identify and describe the mathematical beliefs, constructions, practices that are now in place. Where and how is mathematics employed in real life?

(b) Describe the mathematical beliefs, constructs, and practices that have been justified by the community. What are justifiable and unjustifiable? What are the modes of justification?

(c) Describe the social dimensions of mathematical practice. What constitutes a knowledge community? What does the community of mathematicians think are the best examples that the past has to offer?

As part both of (a) and (b) one should add: describe the nature of the various methods of *prediction* and the bases upon which prediction can become prescription (i.e., policy).

This type of inquiry is rarely carried out for mathematics. For example, the concrete question of where such and such a piece of university mathematics is used in practical life and how widespread is its use, is seldom answered. Many textbook claims are made in textbooks, but show me the real bottom line. It is important to know. How, in fact, would you define the bottom line?

The technical term for inquiries such as the above is 'hermeneutics'. This word is well established in theology, and in the last generation has been commonly employed in literary criticism. It means the principles or the lines along which explanation and interpretation are carried out. It is time that this word be given a mathematical context. Instruction in mathematics must enter a hermeneutic phase. This is the price that must be paid for the sudden, massive and revolutionary intrusion of mathematizations - computerizations into our daily lives.

Conclusion

Mathematics is a social practice. This practice must be made the object of description and interpretation. It is ill-advised to allow the practice to proceed blindly by "mindless market forces" or as the result of the private decisions of a cadre of experts. Mathematical education must find a proper vocabulary of description and interpretation so that we are enabled to live in a mathematized world and to contribute to this world with intelligence.

Acknowledgement

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Gresham's Law: Algorithm Drives Out Thought

by

S. K. Stein

(A talk delivered at the 1987 AMS meeting in San Antonio, Texas.)

Gresham's law in economics says, "Bad money drives good money out of circulation." Copper replaces silver; silver, gold. Gresham's law in mathematical pedagogy can be stated several ways. "Algorithm drives out thought." "The robotic displaces the humanistic." "Cultivation of algorithms replaces concern for thinking and writing."

We view colleges and universities ideally as places that develop the ability to think analytically, to probe independently, to resolve the open-ended problem, to write and speak clearly. Though the catalog may not mention them, these goals are in the back of our minds when we picture ourselves as teachers. In the catalog we find descriptions of courses couched in terms of their content, such as:

> Linear algebra. Matrices and linear transformations, determinants, complex numbers, quadratic forms

This list, with its focus on topics, illustrates the power of our version of Gresham's law. We can be sure that there will be definitions, theorems and proofs, and algorithms. Swept under the catalog is concern with the development of the ability to think and to communicate. So, without a battle, in

spite of our best intentions, the combination of curriculum, syllabus, and schedule seems to assure the triumph of Gresham's pedagogic law.

Algorithms, of course, are good and must be taught. After all, the world would be an unpleasant place if every time we added two fractions we had to discover the procedure from scratch.

But the temptation to emphasize drill over understanding is almost irresistable. It is much easier to teach the execution of an alogorithm than the ability to analyze. Furthermore, an algorithm can be described in just a few minutes and skill in its execution can be tested and scored easily.

Moreover, the incredible power of calculators and computers may entice us to shape our courses around them rather than around the students. As we incorporate these devices into our teaching, we must be sure that their role does not shift from servant to master and that skill in punching keys is not confounded with the ability to think and communicate.

The tendency for algorithm to displace reflection is not new. The student who shows up in our remedial or calculus class may already have experienced twelve years of robotics. Recently in my first-quarter freshman calculus class I assigned an exercise which asked the student to show that a polynomial of odd degree has a real root. The next day a student asked, "Could you work this problem?"

"What was the trouble?"

"Well, what's a polynomial of odd degree?"

"Didn't you take algebra in high school?"

Then a girl in the back raised her hand, "Professor Stein, you don't understand. In high school the teacher works one problem on the board and we then do twenty just like it. We don't have to know anything." A murmur of endorsement swept the room - from students who had graduated in the top eighth of their class from schools throughout California.

In one classroom in an above-average high school, logarithms were taught this way: "Logarithms are tough, but all you need to know is that when you press the log-key you get the logarithm." This is the complete triumph of algorithm over understanding.

Of course, educators have tried to resist the working of Gresham's law. The director of the California Curriculum Commission recently complained, "Youngsters need to know more than just computational skills. We want them to have a sense about what numbers mean." This announcement followed the Commission's rejection of all the textbooks submitted for adoption in grades K to 8 because they did not relate to the objectives that the Commission had published a year earlier, such as:

> "The focus of the program is on developing student understanding of concepts and skills rather than 'apparent understanding.'"
>
> "Students should be actively involved in problem-solving

in new situations."

"Nonroutine problems should occur regularly in the student pages."

These objectives, taken from the 1985 "framework", were not new. In 1980 an earlier Commission had urged,

"Problem solving has become the all-encompassing theme of mathematics instruction and is no longer a separate topic."

Twelve years earlier, in 1968, a still earlier Commission had said the same thing in different words:

"Textbooks shall facilitate active involvement of pupils in the discovery of mathematical ideas."

But even before that, in 1963, another Commission had insisted that:

"Pupils should make conjectures and guesses, experiment and formulate hypotheses, and seek meaning."

"Materials should elicit thoughtful responses and develop understanding."

So the texts submitted in 1986 not only failed to satisfy the demands of the current Commission, but they wouldn't even satisfy the demands set by any of the Commissions going back a quarter century.

However, concern with the displacement of thought by algorithm did not begin in 1963. In describing some of his experiments in the teaching of arithmetic, L. P. Benezet, a superintendent of schools, wrote in 1935 [1]:

"For some years I had noted that the effect of the early

introduction of arithmetic had been to dull and almost chloroform the child's reasoning faculties. [In my experiments] the teacher is careful not to let teaching of arithmetic degenerate into mechanical manipulation without thought... The objectives are first of all reasoning and estimating rather than mere ease in manipulation of numbers."

Incidentally, pupils in his program for one year caught up with pupils who had spent three years in the traditional arithmetic program.

This conflict between the thoughtful and the mechanical is as ubiquitous as the conflict between good and evil. Once you are sensitized to it, you see it everywhere. In one mail delivery recently I found an ad for a college algebra text and a sample of a new journal. The ad included this reassurance:

"Numerous algorithms for solving word problems are developed to help students learn and remember concepts."

So algorithm finally disposes of its arch enemy, the word problem.

There was an odd juxtaposition between this ad and the title of the journal that came in the same batch of mail: Teaching thinking and problem solving, with the peculiar implication that we need not think to problem-solve.

There seem to be two separate worlds. One is the world of Math Commissions with high aspirations, enrichment materials at publishers' booths, conferences on humanistic mathematics, articles that show how to teach thinking, books with the phrase "problem-solving" in their titles, and the exciting prefaces of texts. The other is the world of the typical classroom, whether

K to 12 or Freshman to Senior at college. Vast storms of reform rage in the first world, but they stir scarcely a faint breeze in the second world. The first corresponds roughly to the world of "thinking;" the second to the world of "plugging in."

The fashionable terms are now "problem-solving" and "algorithms." Whatever the terminology, students know the difference. In anonymous course evaluations they write, "This course made me think." They do not write, "This course made me problem-solve." The word "think," loose though it may be, is good enough.

But there are many obstacles to teaching "thinking." Some are external to any particular course. As individuals, we can't do much about them: that for twelve years most of our students have learned robotics, with even word problems resolved by mnemonic devices; that society rewards the seemingly practical more than the fundamental; that many students go to college only to get a good job at a time when the economy no longer even promises everyone a job.

The internal obstacles are quite different. The prescribed syllabus may move so fast that there isn't time to address such fine points as "thinking." The midterms and final are squeezed into such narrow time slots that we dare not pose problems that demand fresh thought. The text may offer almost exclusively exercises that cultivate algorithms. Indeed, if you thumb through many a high school or college text, you can come upon section after section where every single exercise is

routine.

Everything seems to conspire to favor algorithm over thought. The syllabus is worked out and expressed in terms of topics, not in terms of processes. Texts, by their very structure, offer answers before the students have absorbed the questions. Homework assignments draw the students' attention to individual exercises rather than to underlying concepts. To cap it off, we're so busy or the classes are so large that we read neither the daily homework (read by undergraduates), nor the midterms (read by graduate students). So, captivated by the clarity of our own lectures, we assume that all is well.

For some twelve years most students have been strapped to a table. No wonder they cannot walk on their own two feet. We must remember that thinking in a mathematics classroom may be a novel or at least unusual experience.

In spite of these obstacles, external and internal, there are actions we can take as individuals to subvert Gresham's pedagogic law.

As we propose a day-by-day syllabus we can <u>delete topics</u> to provide more time to give attention to "thinking."

We may even propose a new<u>course whose main purpose is the cultivation of the student's ability to analyze and write</u>. It can be smuggled into the catalog under the guise of, say, "discrete mathematics."

And we can make a conscious choice as we begin teaching a course. Are we going to emphasize facts and algorithmic skills,

hoping that incidentally the students will mature? Or are we going to emphasize independence, analysis, and communication, hoping that along the way the students will pick up the facts and algorithmic skills?

In the first case we plan more in terms of our lectures, in terms of what we will do. In the second case we plan more in terms of the homework, in terms of what the students will do.

In the second case we would examine the exercises and ask "What is the purpose of this exercise?" Is it to check a definition or a theorem or the execution of an algorithm? Such exercises have their place, but they should not be the last word. They represent one coin of Gresham's law; they are designed to have a closed field. Blinders are placed on the student to focus attention on particular facts or skills. For instance, we may ask the student to factor $x^4 - 1$.

An open-field exercise puts no blinders on the student. We might ask, "For which positive integers n does $x^2 - 1$ divide $x^n - 1$?" An open-field exercise may not connect with the section covered that day; it may not even be related to the course. Such an exercise may require a student to devise experiments, make a conjecture, and prove it. If it has all three parts, it is a "triex," which is short for "explore, extract, explain" or for "try the unknown." But it may have only the first two parts, amounting to "find the pattern." Or it may

have only the last two parts. For instance: "If a continuous function defined on the x-axis is one-to-one must it be a decreasing function or else an increasing function?" This could be reworded to become just the third part of a triex: "Prove that a one-to-one continuous function defined on the x-axis is either an increasing function or a decreasing function." Since experiments with such functions are not feasible, this exercise does not lend itself to the full triex form. However, the following exercise does.

"Does every convex closed curve in the plane have a circumscribing square?"

The way we word a problem may determine how closely it approximates a full triex and where it stands on the "closed-open" scale. Here is an illustration in which each variation enlarges the field from closed to open. At each stage the student is offered more responsibility, more chance to develop self reliance.

> <u>First formulation</u>: Prove that if 3 divides the sum of the digits of an integer, then 3 divides the integer.
> (This is the narrowest form, just the last part of a triex.)
>
> <u>Second formulation</u>: If 3 divides the sum of the digits of an integer, must it divide the integer?
> (This opens up a bit of the second part of a triex, but the student can guess, "Of course, why else would the instructor ask?")
>
> <u>Third formulation</u>: Let d be one of the integers 2 through

9. If d divides the sum of the digits of an integer, must it divide the integer?

(This is a full triex. There are no clues to the answer. The student must experiment and conjecture.)

The following exercise has a closed field: Prove that when a segment AB is cut into segments by dots labelled either A or B, then the number of segments having both labels is odd. It can be recast to have an open field: (a) Draw a segment AB and and cut it into segments by dots that you label A or B. Count the number of segments AA, the number AB, and the number BB. (b) Do this several times and on the basis of your experiments make at least one conjecture. (c) Prove your conjecture. See [2, 3, 4] for more examples.

So the simplest way to resist the assault of Gresham's law is to include exercises that are not simply routine. To do this, it helps to go beyond the usual ways we contrast exercises as "easy" versus "hard," "short" versus "long," "new" versus "review," but to think in such dichotomies as "computation only" versus "exposition required," or "closed field" versus "open field."

But choice of exercises comes late in the game. Other steps can be taken earlier.

1. <u>Curriculum reform</u> As we propose a new course or curriculum, we should think in terms of the student, not just in terms of the topics. The temptation is to make a neat outline of chapters and sections, leaving skills in analysis and

communication to develop magically on their own.

2. Planning a course As we work out the day-by-day schedule of a course we should put concern for the student's growth at least on a par with concern for particular topics. This means that we may sacrifice some traditional topics to make time for other matters.

3. Texts When writing or adopting texts, we should pay attention to the exercises that provide an opportunity to explore, conjecture, and write. This means checking that there are enough open-field exercises.

4. Feedback The student's work on open-ended exercises requires more careful reading and criticism than do routine computations. An instructor who does not have the assistance of prematurely wise undergraduates or graduate students will have to read papers carefully. This requires time.

These are a few ways to resist Gresham's law of mathematical pedagogy. Perhaps there is another law that reads,"If each of us tries, we can repeal Gresham's pedagogic law."

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Teaching with a Humanist

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This is a report about a team-taught course at San Francisco State University, and what I have learned teaching it for eleven years with, at different times, three different Professors of English. [For the record, their names were Edwin Nierenberg, Judy Breen, and David Renaker. They have not seen this paper and bear no responsibility for it.] We taught a course, titled "The Newtonian Revolution," dealing with the impact of Newtonian science on the literature and philosophy of the eighteenth century British enlightenment. A rough outline of the course follows.

Weeks	Subject	Activity
2	Intro to 17'th century science and culture	read Burton
3	Newton's accomplishment	read Newton "Optics" etc. Do experiment and report
10	Newton's influence	Read Locke, Pope, Swift, Franklin, Sterne, Shelley. Write term paper

Class periods are balanced between informative lectures and discussions of the readings. Usually the only exam is a final with essays and short-answer questions. About twenty students from a large number of different majors enroll in the course each time it is offered.

This course is part of program of team-taught interdisciplinary courses at SFSU called NEXA, or the Science-Humanities Convergence Program. Begun in 1975 with a five-year grant of $850,000 from the National Endowment for the Humanities, NEXA promised a curriculum of ten team-taught courses exploring the boundaries between the sciences and the humanities, a series of public programs about contemporary issues of science and values, and a weekly staff seminar for the NEXA faculty. Public programs have included national colloquia on sociobiology and the heritage of Jacob Bronowski.

NEXA has survived 12 years, or seven years past the end of its major external funding. The campus currently supports a curriculum that has grown to sixteen team-taught courses, occassional public events are still presented, the

staff seminar is taking a sabbatical this spring after meeting weekly for twelve years, and NEXA has assisted the founding of similar programs on other California State University campuses.

The focus of NEXA has always been and remains its program of team-taught courses. But what happens when a humanist and a scientist come together in a classroom? Superficially what results is a humanities-style course about science and culture and their mutual interrelationships. The scientist explains scientific facts and practices; the humanist disucsses how these are reflected in art and history. In my course, for example, the students receive a few weeks lecture (and an exam question) about Newton's scientific accomplishment, and they perform an experiment which is as open-ended and realistic as possible. However, they don't learn to solve any science or math problems. Instead they listen, read, discuss, and write about the effect of Newtonian science on British literature much as they would in any humanities class.

I'll digress to describe the experiment that the students perform, because it is my attempt to have them do a little bit of realistic scientific research.

Students are asked to construct a pendulum and to report on the relationships between the weight of the bob, the length of the string, the period of oscillation, and the angle of oscillation. They do this experiment individually or in teams, at their choice, and they do the experiment in their own homes with equipment they construct themselves. The students are put in the position of a scientist who has only a vague question, and who must develop experiments that simultaneously sharpen the question and answer it.

Sometimes students who did well in a prior physics course get the wrong answer to the pendulum question, because they have learned that the period is independent of the angle, which it demonstrably is not. A few students are overwhelmed by the huge quantity of data they produce by measuring all possible combinations of length, weight, and angle. They never get their data organized into a demonstration of a conclusion. But most students do find the expected pattern and explain coherently how their data justifies their conclusion. Some manage to draw some graphs to accompany their tables, and a few even find the square-root law relating period to length.

To return to my main issue, what is so special about a NEXA course? What is it that distinguishes a NEXA course from a course in the history of science or intellectual history? The answer is team-teaching. I want to describe how team-teaching works and what I've learned from it.

My partner and I are both present for all class sessions, and we spend about one hour each week outside of class planning the course. We listen to each other's lectures, and we do not hesitate to interrupt with questions or even respectful disagreements. Our discussions teach

students that there may be more than one answer to a question, more than one interpretation of a poem or novel or even a scientific theory. We sometimes plan our disagreements ahead of time to demonstrate the different points of view characteristic of the sciences and the humanities. Having both of us present, seeing both of us discuss common issues, brings home to the students the different values of our disciplines and the different ways we view the world. No single instructor could model the two views we represent or make their contract so evident to the students. At least that is the claim and the justification for team teaching.

There are real differences between my partners and myself. The first difference, which appears trivial but which I think is crucial, is that humanists expect good writing on tests and homework. For them, the quality of student expression is as important as the correct answer in determining a student's mark. The sort of answers mathematicians are accustomed to receiving to homework and exam questions in calculus and other computational courses would not be accepted in a humanities course, even if the answers contained all the correct elements. Humanists expect students to demonstrate their grasp of a subject by explaining it clearly and completely.

The second difference is that the humanists seem to value and encourage opinion. Humanities faculty express opinions about books and ideas, and students are encouraged to do the same in class discussion. These discussions help students learn to rely on themselves to make informed judgements instead of looking to their instructors for an immediate verdict about their ideas.

Thirdly, humanists give more complex assignments than mathematicians do, but they engage in less direct methodological instruction. They assign several original works for students to read and interpret; mathematicians assign a single textbook specially designed for student comprehension. Humanists assign term projects; mathematicians assign a weekly list of short exercises. Even the humanists' exam questions are less direct and more open to interpretation than our own. To generalize, humanities students, compared to math students, are asked to engage in intellectually more sophisticated activities but are given less direct instruction in how to accomplish them.

Good writing, class discussion, and term projects can be adopted to mathematics classes. If mathematics is to become more humanistic, it will have to borrow from the humanities. In my classes, I use lots of writing assignments. Student homework must be written using complete sentences organized into paragraphs. This homework sometimes requires students to explain key concepts in their own words. For example, calculus students have beenasked to explain "rate of change". When students complain, "It's not fair to take off for grammar. This isn't an English

class.", I can mollify most of them by pointing out that they will have to combine mathematics with writing in their professional lives.

I also attempt class disscussion about issues where opinion is appropriate. We discuss issues in aesthetics when we have two answers to the same problem. We also discuss ethical matters when these arise, sometimes in the context of a statistics course, and sometimes when a particularly relevant article is published in the local paper. Last term my statistics class discussed the controversy about conflicting statistical studies of women marrying after age 30.

Some of my classes are assigned term projects based on open-ended problems that allow students to develop their own ideas. The problems require not cleverness but straightforward elaboration, an element often missing in math assignments. For example, linear algebra students have described an application of linear algebra, and abstract algebra students will find subgroups of GL(2,R).

Sometimes students are discomfited when faced with humanities-style assignments in a math class. They expect a sharp dichotomy between mathematics and the humanities. In a math class, they expect to learn theorems and algorithms, and to be tested on how well they have assimilated them. A humanities class is a place for discussing books and expressing their own ideas.

This expectation was brought home to me last year after I had assigned a short paper in a calculus course. I asked my students whether this paper was easier or harder to write than a paper in English composition. "Much harder," they responded. "In English, you can say anything. In math, you have to be right."

If students think that, in humanities, anything goes, they make a complementary mistake about mathematics when they believe that all methods are known and just have to be learned. Why else would we see students asking for more and more examples, hoping as they do to find every possible problem exemplified and therefore solved?

Certainly English instructors are not trying to teach their students to say just "anything", but it is not easy for them to teach the standards of rhetoric. Students resist the distinction between ideas and prejudice, between justification and mere repetition, between thinking and feeling. They sometimes seem to believe that discourse involving values or judgment has no rules at all. When studying Defoe's Moll Flanders in NEXA, students are quick to judge Moll, but they are less successful in defending their judgments by analyzing her actions or social background.

Math students are notoriously no more successful than humanities students in making logical arguments. Once in calculus, when I pointed out to a student that she defined a term one way in her first sentence and used it in the

opposite sense in the next, she responded: "Well, you can define anything any way you want!" So you can, but how can we stop students from changing their definitions in mid-argument?

As I try to overcome the limited student view of mathematics as the application of rules, I find myself in league with my humanistic colleagues who are attempting to instill standards in students. I believe we share the following four goals:

1. Students should have a clear conception in mind when they write, if not when they start at least when they finish. How often do math students successfully finish a calculation with little idea of what they have accomplished?

2. Student writing should be organized and focussed on the issue at hand, and it must be grammatically correct. Math students should write complete explanations, not incomplete notes incomprehensible to anyone who doesn't already know the answer.

3. Questionable assertions should be recognized and justified. This perhaps is more difficult for humanists to achieve than mathematicians, since students so often assume their personal prejudices are obvious universal truths, but math students too have to learn to distinguish between statements that require justification from those that are obviously correct.

4. Students must have command of all the relevant information about a question. In mathematics, that means using not just basic definitions but also new theorems and prior coursework when solving a problem.

These four common goals for mathematics and humanities education don't mean that the sciences are the same as the humanities. Manifold differences exist, and the NEXA staff seminar, after twelve years, has not yet exhausted its discussion of them. However, the differences as perceived by students are not the right ones, but rather are symptomatic of the degree to which mathematicians and humanists have failed to communicate to students that the same intellectual rigor and flexibility is required in all academic subjects. Team teaching with humanists has shown me how many goals I share with them, and it has taught me some of the ways I can try to achieve these goals in my math classes.

PATTERNS OF EMOTION WITHIN MATHEMATICS PROBLEM-SOLVING

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I like the clever twists of logic that turn a two page proof into a one-half page proof. There are lots of clever little insights. There's something very satisfying about a nice tight argument that no one can doubt is correct...I've worked on a research problem for over six months with no results...now I'm starting to dream about it and that's too much..the mathematics is taking too much control over me. (Angrily.) (Rosamond, 1982)

Mathematics often is viewed as the ideal discipline-pure rational thought dealing with ideal objects to produce irrefutable arguments, not coloured by any emotion. Training in mathematics is seen as producing students capable of such clear thinking in all disciplines. So why don't all mathematics teachers present mathematics in the ultimate, Bourbaki style? To mathematize is supposedly part of the human condition, so how can there be such a thing as math anxiety, when feelings should clearly not be a part of learning in mathematics?

Or does mathematic arouse emotion because it was conceived out of emotion in the first place?...What is the link between the affective and the cognitive?
(CMESG Announcement, 1985)

PATTERNS OF EMOTION IN MATHEMATICS PROBLEM-SOLVING

In an effort to understand and explicate the feelings of satisfaction and anger expresssed by the mathematics graduate student in the first quotation, a Workshop on the Role of Feelings in Learning Mathematics was held during the Canadian Mathematics Education Study Group annual meetings of 1985 and 1986. We engaged in a problem-solving exercise that also was given to six mathematics education graduate students at a State college and on two occasions to six people who met in a private home.

We are all (with the exception of two people) involved in mathematics as professional mathematicians, as teachers, as graduate students or as people who use mathematics in our work. We believe that thinking, feeling and acting work together, that true understanding implies feeling the significance of an idea, and that our experiences are not far from that of our students. We decided to examine our own feelings in depth in hopes of finding outstanding commonalities that could be used to improve classroom teaching.

Studies on cognitive science (Davis, 1984. Papert, 1980), problem-solving (Silver, 1985), metacognition (Schoenfeld, 1983) and belief systems (Perry, 1970) offer some insight into the role of emotions in problem-solving but only indirectly. We are not sure we have even a vocabulary with which to describe feelings at a specific moment as a function of many variables.

To begin with, we made a list of relevant positive and negative emotion descriptors (see appendix). This list was adjusted by the results of the exercise. The exercise is a simple one. We went in pairs to different parts of the room where one person agreed to be the problem-solver and the other the observer. The rules were 1) The solver do his or her best to provide a running commentary on feelings. 2) The observer keep quiet, pay attention, take notes.

After a fixed amount of time (15 minutes, in later sessions changed to 30 minutes) all gathered and each observer reported on what the solver had done, focussing on the feelings. The solver also reported.

The roles were then switched, observer became solver. Solver became observer. Another problem was presented and the observation and reporting process repeated.

We feel many positive emotions (challenge, hope, zest, satisfaction, etc.) when doing mathematics and wish to promote these in our students. Lazarus is a noted psychologist at University of California at Berkeley who has done extensive analysis of the theory of emotions. In his paper, "Emotions: a Cognitive - Phenomenological Analysis", he describes some of the contributions positive emotions make to coping. Before describing our exercise and the implications that we found for teaching, I will briefly outline some of Lazarus' position and make some connections to mathematics.

LAZARUS ON POSITIVE EMOTIONS

Lazarus points out that negative emotions have been studied almost exclusively. Some reasons for this are that emotions have been studied as evolutionary and that negative emotions such as fear or stress influence our capacity to survive. Another reason is that emotion is studied by therapists who may view emotion as pathological. In this case happiness may be seen as hysteria, concern as paranoia and hugs as evidence of nymphomania. A third reason is that it is more difficult to measure arousal for joy, delight, and feelings of peace than it is for rage, disgust or anxiety.

Because we are trying to promote good problem-solving, we feel it is appropriate to focus on the positive feelings associated with our goal: on hope rather than hopelessness, challenge rather than threat, zest rather than dispair although negative emotions do need to be recognized.

Positive emotions tend to be frowned upon or viewed as "childish." Not many people exhort optimism like Ray Bradbury does: "We are matter and force turning into imagination and will! I am the center of a miracle! Out of the things I am crazy about I've made a life!...Be proud of what you're in love with. Be proud of what you're passionate about! (Bradbury, 1986) It is even hard to hear people shout gladly onto the Lord; but we were just trying to hear people shout gladly about mathematics. People who exhibit positive emotions often are accused of playing, of not being serious.

Yet playing with ideas is inherent in mathematics problem-solving. What emotions should we expect to feel when engaging in problem-solving? Lazarus answers this by saying that the essence of play is that it is highly stimulating. It is accompanied by pleasurable emotions such as joy, a sense of thrill, curiosity, surprise, wonder, emotions exploratory in nature. We recognize that we do experience these positive toned emotions when doing mathematics.

As educators we wish to know the optimum conditions that encourage problem-solving. Lazarus says, "...exploratory activity ocurs more readily in a biologically sated, comfortble and secure animal than in one greatly aroused by a homeostatic crisis. The human infant will not venture far from a parent unless it is feeling secure, at which point it will play and explore, venturing farther and farther away but returning speedily if threatened or called by the mother." As shall be discussed in more detail in the next section, mathematics problem-solving requires playing in an almost "other-world" of intense concentration. Insecurities in terms of math ability or other issues (world peace) inhibits problem-solving by interferring with the level of concentration.

USES OF POSITIVE EMOTIONS

Lazarus sees at least three ways in which a person uses positive emotions: as "breathers" from stress, to sustain coping, and to act as restorers to facilitate recovery from harm or loss. Lazarus' discussion may be interpreted with mathematics in mind.

BREATHERS OR TIMES OF INCUBATION

"Breathers" are times when positive emotion occurs as during vacations, coffee breaks or school recess. They can also be thought of as times of incubation.

Lazarus quotes the noted mathematician Poincare to suggest that it may be the good feelings themselves that allow a solution to emerge from the subconscious to the conscious.

Poincare made the surprising comment that unconscious creative mathematical ideas "are those which, directly or indirectly, affect most profoundly our emotional sensibility." By this he meant that, since creative thoughts are aesthetically pleasing, the strong, positive emotional reaction to such ideas provides an opening through which they are ushered into consciousness.

Lazarus reminds us of another relevant description of a "breather" made by the great German physicist Helmholtz:

> He (Helmholtz) said that after previous investigations of the problem "in all directions...happy ideas come unexpectedly without effort, like an inspiration. So far as I am concerned, they have never come to me when my mind was fatigued, or when I was at my working table...They came particularly readily during the slow ascent of wooded hills on a sunny day."

The acceptance of the role of a breather is reflected in the usual advice given by teachers to their students: "Concentrate long enough to get the problem firmly in your mind and to try several approaches. But then take a walk or do some pleasant activity and let your mind work on the problem for you."

SUSTAINERS OR MOTIVATORS

Positive emotions act to sustain problem-solving in the sense that good feelings build on good feelings. Mathematics and the word "challenge" often are linked together as in "The problem is a challenge." A challenge can be viewed as a threat and in our exercise, problem-solvers were momentarily worried about failure in front of an observer. However, in challenge, a person's thoughts can center on the potential for mastery or gain. This challenge is accompanied by excitement, hope, eagerness, and the "joy of battle." All these positive emotions were mentioned by problem-solvers. One solver summarized the feeling as "the joy of mental engagement and the bringing of all mental force to bear in a cohesive way." Solvers who perceived their problem as too easy felt disappointment even before they began to work on the problem. Those who felt the problem worth working felt an immediate joy even before proceeding. This joy was a signal to bring all mental force to bear on the problem, which in itself produced pleasure and therefore motivation to continue.

Lazarus describes "flow" ,an extremely pleasant, sustaining emotion, as in the case of the basketball player who is "hot" or the inspired performance of a musician.Lazarus claims flow arises when one is totally immersed in an activity and is utilizing one's resources at peak efficiency. Mathematical problem-solving requires total immersion and we found that a comfort with notation was important in maintaining this flow. Comfort with notation will be discussed later in this paper.

The positive emotion of hope also provides motivation to keep going. Occasionally during a problem-solving episode the solver lost control of the problem. Solvers said, "I've lost control of the problem." or "This is too complicated, too many angles to label." or "I feel this is getting a little out of hand. This one and that one cancel out and I haven't used fact that it's a prime." Hope, the belief that there is even a slim chance things will work out, helps one continue. Ambibuity nurishes hope. One cannot be hopeful when the outcome is certain. We would like to know how ambiguity can serve classroom mathematics. The emotions of challenge and hope are powerful motivations in problem-solving and deserve further research.

A more obvious way in which emotions sustain actions is in terms of longer range goals. The student who has a positive feeling solving one math problem is more likely to try another. The confidence that comes from understanding mathematics empowers the student to attempt new ventures also, as in the case of a geometry student who attributes his decision to help in crime prevention directly to his success in his geometry class.

RESTORERS

Lazarus offers a third. function of positively toned emotions, that of restorer. Lazarus' descriptions of recovery from depression or restorations of self-esteem might be useful to the teacher dealing with math-anxious students. Lazarus quotes Klinger:

> At some time during clinical depression patients become unusually responsive to small successes. For instance, depressed patients working on small laboratory tasks try harder after successfully completing a task than after failing one, which is a pattern opposite to that of nondepressed individuals, who try harder after failure.

It would be worthwhile for the classroom teacher to know when small successes are more likely to evoke positive emotions. Offering a small task to a math anxious student may foster optimism and incentive while the same problem may seem trivial to a non-anxious student and provoke anger or disappointment. This is an area for more research.

Much of the information on emotion in problem-solving is obtained by having students fill out questionnaires. While the information is useful, a rating on a scale from one to five of confidence in doing math, liking for math, or usefulness of math is very general. Questionnaries also are remote from the actual process of problem-solving. Recollections of feelings might not be quite the same as the feelings at the time. Also, mathematical problem-solving requires intense attention to the problem. It is likely that without some help a solver will not even be aware of his or her emotions. The above reasons together with the belief that our own feelings when doing mathematics are

the same as those of our students prompted us to do an exercise utilizing a close observer and introspection.

OBSERVATIONS FROM THE EXERCISE

Altogether the exercise of observing, reporting, solving, reporting was done by 19 pairs. Problems initially were of the puzzle variety (Gardiner,1967, 1979. Mott-Smith, 1954) but in later sessions more substantial problems were chosen from Honsberger. One person kept track of time for the whole group. A group of six people (three pairs) seems the best size. We posture...laughter...intent stillness" but that description is not used in this paper.

EMBARKING ON THE PROBLEM-SOLVING

Solvers accepted their problems with curiosity and positive anticipation. These were people who did formal mathematics frequently. Two people who had not done formal math recently reported terror.

The initial reading of the problem provoked a reaction to its type followed by a sense of its difficulty. "I anticipate I will enjoy this problem but may not make much progress." or "I loathe this type of problem . It is do-able but will require a big effort. I think I will have to go through many tedius decompositions."

The word "do-able" was used often and meant either that the problem was solvable or that progress could be made in understanding the question. For one of the people who reported terror, a person who rarely uses formal mathematics rarely and who was talked into coming to the workshop, considerable time was spent blocking the reading of the problem. Emotion can be regulated by avoidance or denial. This person acknowledged feeling bad but then felt bad about feeling bad so that "Even if I could do it I couldn't." Considerable time was spent recalling past history of problem solving failures all the while avoiding (somewhat consciously) making the decision to try to do the problem. Another solver also reported "I felt unhappy and then felt unhappy about feeling unhappy." Emotions tend to feed on and reinforce each other. The math oriented solvers were predisposed to extend effort on the problem. They had much more commitment to do math.

After reading the problem, all began to develop a notation, to draw a diagram or to write some hypothesis. This was the beginning of a cycle of attention on problem - attention on self or distraction by environment - attack on problem - attention to self or environment - problem - self - problem - self, etc.

When preparing to choose a method of attack, there was considerable emotion tied in with "not cheating." Each person placed the problem in a certain context and at a certain level of difficulty and felt it would be cheating, bad sport, to use a technique that was too powerful. One solver says, "Can I use fancy stuff?... Then I'll use Jordan Curve Theorem....laughs". Backtracks. "Maybe an easier way." Another solver resisted but finally made a grudging commitment to using calculus for a problem entitled, "An Obvious Maximization."

Using brute force was considered almost as bad as using a too powerful method. "I'm annoyed because I can't see any other way than brute force and that would not yield for me any understanding of the problem...there must be an easier way." Solvers wanted to find solutions that were generalizable. Using a too powerful method, brute force, or an "obvious method" brought forth comments of feeling embarrassed or annoyed.

A less conscious resistance to cheating was the seen in the imposition of ridiculous restrictions on oneself. For example, one solver had Honsberger's book in hand and was to "Use the 'Method of Reflection' to...". (Honsberger, p.70). The solver's reaction was, "I understand the problem but don't know this method...I wish I could read the chapter..." . Instead of simply reading the chapter, the solver tries to invent a plausible 'Method of Reflection'.

Another solver spent long moments seemingly aimless. "I'm feeling a little out of control of the problem...lots of parameters...seems to be a lot of ways to define this problem...I'd like to clarify the problem by asking whoever wrote it." Finally with a forced air, "I could break it up into cases myself and come to grips on my own terms and get partial solutions...got control back."

Self imposed restrictions would slow a solver down until there were reports of, "I'm squandering time. I really haven't done anything." Then there would be a squaring of the shoulders and a businesslike assertion to "...take a stand and try to prove it ..." even though this might mean grinding out a meaningless, albeit correct, solution.

INVOLVEMENT WITH THE PROBLEM

Once commitment was made to attempt the problem, there was a lorelai seductiveness about it, a delicious slipping off into another world. Solver became oblivious to self, observer, or environment. This total immersion was a wonderful release from daily life. Poland (CMESG, 1985) used mathematics to help him ignore the pain of an illness. Some people use the other-world quality of doing mathematics to avoid interaction with peers. Mathematics can help with depression as the famous mathematician Kovalevskaya said in a letter: "I am too depressed...in such moments, mathematics comes in handy, and one enjoys the existence of a world completely outside of oneself." (Knopp, 1985).

Mingled with the charm of seduction there was a dangerous quality, a frightening isolation if one stayed immersed too long. Rosamond (1982) gives examples in which the solver feels consumed by a too dominating mathematics. As one mathematics graduate student said with tears in his eyes, "What do you do if you are 80 - 90% mathematics? If you've let yourself become consumed by mathematics so that that is what you are. And then you want to let someone get to know you. What do you do when you can't explain that much of yourself to them?" The presence of the observer comforted the solver and lessened the dangerous quality in the isolation.

There was a letdown feeling of disapointment if the solution came so easily that little emotion needed to be invested in in the problem. Typical is the remark, "The problem must have been too easy, I got it. So what's the big deal? I feel let down."

or "It was fun but not intense because not a challenge. I feel let down because I didn't spend a lot of emotion." The complexity of the problem came like a revelation to one solver who then responded with a BIG smile. Overall, the amount of satisfaction with the problem correlated directly with the intensity of concentration. The perceived level of difficulty of the problem also influenced satisfaction and this will be discussed later.

However, one cannot maintain a constant level of intensity throughout the solving of a problem. The use of notation in a ritualistic manner provided a "breather" or moments of relaxation while allowing the solver to remain in the "other-world". When no progress was being made on a problem, the solver remained in the intense state by writing out some formal routine. Some solvers would rewrite the definition of the variable. One solver began, "There are two cases: a) the problem is solveable and b) the problem is not solveable." Almost everyone used x's and y's at one time and then decided to switch to a's and b's (or vice versa). Some would say, "I'm going to try induction." and then write out the induction hypotheses. The rote writing out of hypotheses or the rote switching of variables afforded a lull within the other-world state and continued the flow. The importance of these rituals was to help focus on the problem. To sit too long without progress or a ritual meant the solver would think about self again.

Other pauses also bump one out of concentration. When the solver paused overlong in appreciation of some success, then attention tended to turn to self or environment. The jolt of finding a counterexample to a hoped-for truth caused one to notice the ticking of the clock or the coldness of the room. Extended frustration of method caused recall of poor geometric visualization in the past and then embarrassment. Attention was diverted from the problem to the self. This usually was for a brief amount of time, less than a minute. Solvers would look around, sigh, stroke the pen, scratch, talk a little and then go back into the problem.

Most solvers were engrossed in the problem when time was called and these people were irritated at being interrupted. They almost all mumbled "I'll continue later." Solvers who were in an attention-outward part of the problem-solving cycle just prior to time being called generally sat back and waited out the time. They did not work on the problem further while waiting but mentioned that they would return to it later. There was reluctance to allow oneself to get lost in a train of thought and then yanked out of it.

IMPLICATIONS FOR THE CLASSROOM:
VARIABLES THAT INFLUENCE ENGAGEMENT

The primary goal of our exercise is to improve classroom teaching. It would be useful for a teacher to know what a particular emotion looks like. For example, a teacher who knows that yawning is a release of nervous tension and not an indication of boredom have an immediate and obvious clue that a student needs help. (And the teacher knows not to get personally insulted by the yawn.) In the opposite direction, the teacher who wants to indicate positive emotions to the students would know how to do it because he or she would know what they look like.

To this end we took notice of some physiological indications of emotive arousal (flushed face, sweaty palms, muscle tension, etc.) and of body movement (twitching, sighing, laughing, etc.) but more work should be done here and these indications are not elaborated on in this paper.

We found that overall satisfaction in problem solving is directly related to the intensity of engagement with the problem. The engagement is influenced by several variables: the nature of the problem, the perceived usefulness of mathematics, the role of the observer, the use of mathematics rituals, and the testing situation. Each of these variables will be discussed along with their implications for the classroom.

NATURE OF THE PROBLEM

All solvers were more encouraged by harder problems than by ones marked "obvious" or ones perceived as easy. There had to be a sense of value of the problem, not that it must be directly applicable to daily life, but rather that one needed to think in order to understand the problem. If one could get the answer just by asking someone else or by looking it up then that made the problem artificial and was almost an affront to the solver.

Surprisingly, solvers felt threatened whenever they saw the words, "Clearly", "It is easy.", or "Obviously". Most felt that teachers should not say, "This is easy." and that textbooks should not indicate the easy exercises. Solvers sometimes worried that the problem looked so simple. They felt they were missing the point and that their solution was not elegant enough. One solver found three solutions by varying the constraints and then felt less humiliated.

One solver exhibited obvious arousal with eyes wide open, clear face and a slight laugh. "Hey, there's an infinite process..." Exploration didn't bear out infinite process and then there was "That was neat. What was the problem?" together with a clear drop of interest and rather emotionless settling again into the problem. The challange of the infinite process stimulated playing around in the "math-world."

The math-world is a mental out-of-body arena of intense concentration in which a person can play with ideas. Trivial problems do not make good play-mates. One solver's most satisfactory experience of problem-solving came after having spent a week on a problem only to have the professor tell the class that the problem was not solveable.

Solvers felt initial relief at seeing an easy problem but were quickly bored, disappointed or insulted. The classroom teacher must pay careful attention to the quality of problems offered and should not label them easy or difficult.

USEFULNESS OF MATHEMATICS

Doing mathematics is seductive but one must allow oneself to be seduced. Three different participants at three different sessions (all women) felt that going off and doing mathematics was a luxury. A teacher of older women said she had to convince her students that they were not squandering time while problem solving. Women are always productive. They even knit while watching TV. She got around her students' hesitancy by saying, "I'm going to show you some games to teach your kids and improve their math."

The notion of usefulness was mentioned by only three women but it is a construct that has been singled out as the most important attitudinal factor in decisions to take math classes (Sherman and Fennema, 1977.)

Usefulness was elaborated on at length by one solver waho was able to solve the assigned problem in a short time and with no intense engagement. The solver was disappointed and felt letdown. It was not clear if the following remarks would have been made had the solver been given a more engaging problem. I asked at the time but the solver was very agitated and insisted that another problem would have made no difference.

> "What would have beeen a meaningful problem? How come I'm not satisfied? I had an expectation about solving that problem that did not get fulfilled. It didn't make me happy. There were some moments of tension and some of excitement but not intense. It was entertaining like a grade C movie.

"Math has no social relevance to me...I am willing to solve math problems, even ready but it feels completely disjoint from what interests me. I still love it (This solver has a Masters degree in math and is an active MD.) but its importance seems miniscule compared to world problems...beautiful but frivolous to use my mind in this way." (It would be useful for other people to do math but there were more pressing issues for this particular solver.)

Usefulness of mathematics in terms of careers or its sometimes theraputic value as a means of escape is an affective variable that may be easy for teachers to influence. Teachers can present information about the mathematics required by various careers as well as the mathematics courses that should be taken to keep options open in the future.

THE USE OF RITUALS

The use of formal routines that keep one's attention on task while providing a sort of restful interlude speaks directly to the classroom teacher. Students must have a comfort with notation not only becausd the notation itself sometimes points to the solution but because that comfort sustains concentration.

ROLE OF OBSERVER

Contrary to almost everyone's expectation, having someone observe while working the mathematics was positive. At first, some solvers felt less inclined to free associate with ideas in front of an observer who might have the problem already all figured out or the solver sometimes felt that the observer must be bored. Some solvers wanted to talk things over with their observer or would look up at the observer hoping for confirmation.

It turned out that the presence of the observer was an impetus to persistence in doing the problem. This is a very important point. Liking the problem was directly and positively related to the amount of time spent working on it. Almost everyone liked their problem more the longer they worked. Those that did not like their problem initially began to like it after all and to get interested in it. Without an observer, those solvers might have quit.

Being observed evoked other feelings. As noted earlier, the presence of an observer reduced the feeling of danger in isolation that lengthy immersion in the problem sometimes brought. There was a feeling of honor. "I felt honored that another peson was taking the time to observe me." Another feeling was intimacy. "It felt intimate to have someone commited to watch the workings of my mind."

While more emotion seemed to come from being watched, it was

also important to be the watcher. Watching seemed to take away some of the secret charge of the observer's own problem-solving anxieties. The observer could recognize his or her own feelings in the other person and see how the feelings influenced their actions. Watching another person struggle with anxieties made the solver think, "Why don't they just get on with it."

One participant reported, "The most poignant part of the exercise was hearing the observer say what I'd done. I did not feel intimidated. I didn't get any of the bad response I expected. The observer demystified my emotional and intellectual engagement by simply listing what I did: 1, 2, 3, 4. This cut it down to size, gave it true proportion."

This exercise of being observer then reporter, then switching to being solver then recipient of report should be explored as a means of eliminating math anxieties in our students. The real key is the switching. This exposes and throws out the power of negative feelings while encouraging positive ones.

It should be noted that no one argued with their observer. A few points of clarification were made but there were no misinterpretations. It is possible that finer gradations or other categories of feelings can be made, but there was good correspondence within our vocabulary.

THE TESTING SITUATION

Concern about the nature of the problem carries over into the testing situation. One solver commented on the problems found on math tests. "A test is an almost random set of narrow problems where one thing must trigger another. It is not about figuring things out. Test questions do not show that math is a process." This solver had as a partner a professional research mathematician. The solver was not intimidated by being observed even though the problem was not solved because "The observer could hear that I have math training. He could see how my math mind works, how I assimilate information, manipulate, and use an asrsenal of strategies. This is so much different from taking a math test where I am not tested on how my mind works. On a math test, I could expect not to be able to show what I know. I would feel shame."

Part of almost any testing situation is a time constraint. Having only 15 or 30 minutes annoyed and inhibited these solvers. Some reported feeling "hemmed in...I do best by playing around...ordinarily would draw pictures and really understand...build up a pattern." Another felt pressure to categorize a solution method quickly. "Without a time constraint I probably would have been more impulsive...would have guessed and then worked backward. I felt forced to be more systematic, meticulous, more step-by-step and mechanical. I think I could have solved this in a shorter amount of time if there had been no

time limit."

When the timing in itself counts, it is as though what the problem means in itself is not enough. Perhaps the discomfort of a time constraint forces one's attention to be divided between the math-world and present time. Not only are different methods of solution chosen at the onset of the process, but also the total immersion into the problem-world is not as possible or as deep.

CONCLUSION

It is important to state that a basic assumption of this experiment is that we professional teachers and mathematicians have at least the same feelings that students have. We may experience a difference in intensity (less anxiety, more confidence) or have other feelings in addition (sense of commitment) but overall how we respond gives an indication of how our students respond. A mathematics educator refused to participate in our exercise saying that it might be worthwhile for "personal growth" but that it would give no insight into how students feel. He believes that teacher feelings are completely different from student feelings.

But imagine your feelings if the Chair of your Math Department suddenly announced that you must take a test. If you have not taught a particular course in the past two years you must pass a test before you can teach it. What course are you scheduled to teach that you have not taught recently? What is your reaction to your Chair's announcement? You are not being tested on how well you review the materal during the semester or on how carefully you prepare your lessons. You are not being asked to share ideas with a colleague. You are being evaluated on questions someone else has chosen and already knows the answers to. I think your reaction to this thought-experiment may show that seasoned teachers can feel anxiety in a test situation similar to what their students feel in their test situations.

The act of knowing is not antiseptic; rather it is wrapped in feelings. It is the engagement of feelings. The primary goal of our work is to improve classroom teaching. This paper indicated only a few of the emotions inseparately connected within mathematical activity and specifically calls the classroom teacher's attention to the nature of the problems, the perceived usefulness of mathematics, the role of observer, the use of mathematics rituals and the testing situation.

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APPENDIX

SATISFACTION SIGNS

relaxed

complacent

proficient

sense of familiarity

supported

encouraged

in-charge, in-control

comfortable

sense of enjoyment

sense of doing something useful

confident

elated

surprise

sense of persistence

sense of adequate energy

sense of elation

sense of significance

pride in accomplishment

feeling of having fun

sense of appropriateness

zest

vigor

sense of curiosity

lo'
se.

ANXIETY DESCRIPTORS

nervous

dread

fearful

hostile

stuck

rigid, uncreative

frustrated

all alone, isolated

sense of urgency

sense of impending failure

desire for approval

feel in the minority

sense of holding back

don't know enough

desire to finish first

desire to finish last

desire not to be noticed

desire to be noticed

feel wasting time

feel stupid

vague depression

worry about making mistake

impulsiveness

feeling of boredom

muscle tension in
neck
jaw
eyes
shoulders
back
stomach

adrenolin rush

blur of vision

sweaty palms, face

increased blood pressure

fight or flight syndrome

stomach butterflies

sense of euphoria

sense of paranoia

taste

tactile

auditory

smell

sensual

sigh

laughter

twitching

foot, finger tapping

flushed face

wiggles

ball (clutched, rolled)

other

MENTAL IMAGES

Shapes
dots
curves
numbers
symbols
colors
sizes
scenes
other

Movement
expanding, exploding
shrinking, collapsing
rotating, revolving
moving
other