2010

Combinatorial Proofs Using Complex Weights

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May, 2010

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Abstract

In 1961, Kasteleyn, Fisher, and Temperley gave a result for the number of possible tilings of a $2m \times 2n$ checkerboard with dominoes. Their proof involves the evaluation of a complicated Pfaffian. In this thesis we investigate combinatorial strategies to evaluate the sum of evenly spaced binomial coefficients, and present steps towards a purely combinatorial proof of the 1961 result.

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Acknowledgments

My appreciation goes to Arthur Benjamin and Kimberly Tucker for their guidance through the year, Claire Connelly for her help with the template, bibliography, and other formatting issues, and Robin Dobashi for her help with OmniGraffle.

Chapter 1

Introduction

The study of regular lattices is important in solid-state physics for calculating properties of semiconductors and other materials. Of particular interest is the number of different configurations of *dimers* (molecules that occupy two adjacent slots in a crystal lattice) in a lattice. Of course, this quantity can be studied via the model of domino tilings of a checkerboard. Kasteleyn (1961) and Temperley and Fisher (1961) independently found the closed form for the number configurations of full tilings of a finite $2m \times 2n$ board to be

$$\prod_{j=1}^{m}\prod_{k=1}^{n}\left[4\cos^2\frac{j\pi}{2m+1}+4\cos^2\frac{k\pi}{2n+1}\right]. \tag{1.1}$$

Their proof, examined in Section 3.1, involves the computation of the Pfaffian of a certain skew-symmetric matrix derived in a complicated manner. In this thesis, we will attempt to find a purely combinatorial proof of Equation 1.1.

1.1 Structure of This Document

In this thesis, we attempt to find a combinatorial interpretation of the Expression 1.1. Chapter 2 explores the combinatorial strategy of weighted enumeration. In Section 2.3 we find and prove combinatorially the closed form of a sum of binomial coefficients, illustrating the power of our approach. Chapter 3 outlines previous proofs of the domino tiling result, then details our work so far towards a combinatorial proof. Chapter 4 discusses future work to be done on this problem.

Chapter 2

Sums of Binomial Coefficients

The binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

counts the number of distinct ways to choose a subset of size k out of a population of size n. It is one of the most basic combinatorial functions (second perhaps only to the factorial) and yet gives rise to a myriad of complicated identities (for example, see Gould (1972)).

We begin in Section 2.1 with two well-known identities involving sums of binomial coefficients, illustrating two classic strategies of combinatorial proof. We will introduce a new combinatorial technique in Section 2.2, and then apply this technique to evaluate a generalization of the sum in Section 2.3.

2.1 Basics

The most basic strategy for producing combinatorial identities is to count a set in two ways, producing a different quantity each time.

Example 1. We recall that, for $n \geq 0$,

$$\sum_{k \geq 0} \binom{n}{k} = 2^n.$$

While this identity is simply the $x = 1$ case of the Binomial Theorem, we will prove this identity instead by a counting argument. First, note that the left-hand side is actually a finite sum, as $\binom{n}{k} = 0$ when $k > n$. This sum

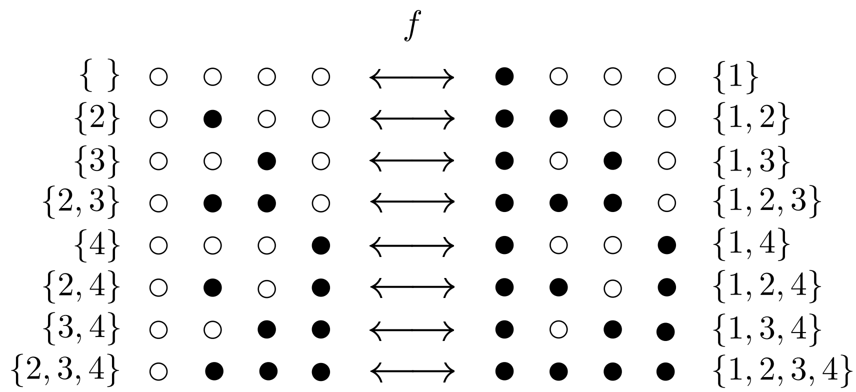

Figure 2.1: The action of f on the power set of $[4]$ is an involution.

counts the number of subsets of the set $[n] = \{1, 2, 3, \ldots, n\}$ conditioning on the size of the subset. On the other hand, we know that the total number of subsets (of any size) of $[n]$ is simply 2^n, because every element may either be included or excluded. Since the two answers enumerate the same set (namely, the power set of $[n]$), they must be equal.

Another combinatorial strategy involves reasoning about *involutions*, or functions for which $f(f(x)) = x$. Involutions are used to match elements in one set to another, thereby showing that two sets have the same size, or that the size of the set in question is half of some known quantity.

Example 2. Let us again consider the sum of binomial coefficients, this time restricted to subsets of even (or odd) size. Then we have

$$\sum_{k\geq 0} \binom{n}{2k} = \sum_{k\geq 0} \binom{n}{2k+1} = 2^{n-1}.$$

We will prove this identity with an involution argument.

For any subset $S \subseteq [n]$, define $f(S)$ to be the symmetric difference of S with the set $\{1\}$. In other words, if $1 \in S$, then f removes it; if $1 \notin S$ then f adds it. Figure 2.1 shows f acting on the subsets of $[4]$. Note that f is an involution on the power set of $[n]$, because taking the symmetric difference twice with any fixed set leaves our original subset unchanged. But $|f(S)|$ always has a different parity than $|S|$, so the involution f pairs even subsets of $[n]$ with odd subsets. Hence, the number of even (or odd) subsets of $[n]$ is just half the total number, or $\frac{1}{2}2^n = 2^{n-1}$, as desired.

Having proved these two identities, we may expect the sum

$$\sum_{k\geq 0}\binom{n}{3k}$$

to be similarly easy to evaluate. But such is not the case. Gould lists

$$\sum_{k\geq 0}\binom{n}{rk}=\frac{1}{r}\sum_{j=1}^{r}(1+\omega^{j})^{n} \tag{2.1}$$

as identity 1.55 in Gould (1972), where $\omega = e^{\frac{2\pi i}{r}}$ is a primitive rth root of unity. Various proofs of this identity are known, but they involve either eigenvalues or other algebraic manipulations (see for example Guichard (1995). An equivalent form is proved in Dsouza and Krebs (2009)).

2.2 Weighted Enumeration

We turn our attention to a third combinatorial strategy, less commonly encountered than the first two. This will be the key to finding a combinatorial proof of Equation 2.1.

Let A be a collection of objects, and suppose we want to count the size of a certain subset $B \subseteq A$. Instead of finding the size by counting

$$|B| = \sum_{b\in B} 1,$$

we define a *weight function* $f : A \mapsto \mathbb{R}$ or $\mathbb{C}$ and compute instead

$$f(A) = \sum_{a\in A} f(a).$$

We call $f(a)$ the *weight* of an element $a \in A$ and $f(A)$ the weight of the set A.

How does this idea help us count the set B? If $f = I_B$, the *indicator function* on B, defined by

$$I_B(a) = \begin{cases} 1 & a \in B \\ 0 & a \notin B \end{cases},$$

Then of course $f(A) = |B|$. But it seems that if we could describe explicitly the indicator I_B we may as well have enumerated B to begin with.

However, we do not *need* for f to be the indicator function on B. For $f(A) = |B|$ to hold, it is enough, for example, if $f(b) = 1$ for $b \in B$ and

$$\sum_{a \notin B} f(a) = 0.$$

As we will see, it is often possible to pick an f such that f vanishes on B^C, the complement of B, assigns unit weight to elements of B, and is easy to sum over A.

2.3 Evenly Spaced Binomial Coefficients

We return to the problem of evaluating

$$\sum_{k \geq 0} \binom{n}{rk}$$

for $r \geq 3$. First, we will review from West (2001) some basic terminology regarding directed graphs.

Definition 3. A *directed graph* (hereafter graph) $G = (V, E)$ is an ordered pair of a set V of *vertices* and a set E of *edges*. The elements of E are ordered pairs of elements of V. If $x, y \in V$ are vertices of G such that $(x, y) \in E$, we say x *points* to y.

We now define a family of graphs utilized in our proof.

Definition 4. For $r \geq 2$, define $G_r = ([r], E_r)$, where

$$E_r = \{(1,2), (2,3), \ldots, (r,1)\} \cup \{(1,1), (2,2), \ldots, (r,r)\}.$$

This is the *looped cycle* on r vertices.

For the reader familiar with graphs, G_r is simply the directed cycle on r vertices, with an additional r edges pointing from each vertex back to itself.

Definition 5. A *walk* on a graph G is a sequence

$$X = x_0 x_1 \cdots x_n$$

of vertices such that for $1 \leq i \leq n$, x_{i-1} points to x_i. Such a walk has length n.

A walk is *closed* if $x_0 = x_n$ and is *open* otherwise. If G is a graph, we write $W_n(G)$ for the set of walks on G of length n and $\mathrm{Cl}_n(G)$ for the corresponding subset of closed walks.

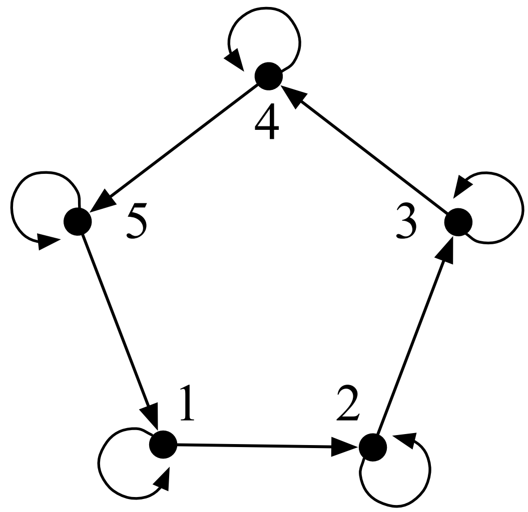

Figure 2.2: G_5, the looped graph on 5 vertices.

Walks are aptly named: we may think of an imaginary pedestrian traveling from vertex to vertex along the paths denoted by the edges.

Example 6. Consider the graph G_5, seen in Figure 2.3. The sequence

$$X_1 = 12234445$$

is an open walk of length 8. If we append the steps 11 to X_1, the resulting walk

$$X_2 = X_1 11 = 1223444511$$

is a closed walk of length 10.

We shall introduce an alternate way of notating walks on G_r, called *step notation*, which will prove convenient in the following applications.

Definition 7. Let $X = x_0 x_1 \cdots x_n$ be a walk of length n on the graph G_r. At each vertex, we have two possible choices: to move to the next vertex (modulo r), or to travel along the loop and remain at the same vertex. That is, for every i, either $x_i = x_{i-1} + 1$ or $x_i = x_{i-1}$ (here $+$ denotes addition modulo r). Hence, we may specify a particular walk simply by indicating the value of x_0 as well as the subset $D(X) \subseteq [n]$ of indices for which $x_i = x_{i-1} + 1$. So we may write

$$X = (x_0, D(X)).$$

We call x_0 the *initial vertex* and $D(X)$ the *instructions* for X.

Example 8. Let $r = 4$, and let X be the closed walk

$$X = 3341223.$$

Then, using step notation we may write

$$X = (3, \{2, 3, 4, 6\}).$$

A walk is completely determined by its initial vertex (r choices) and instructions (2^n choices). Thus, the total number of walks of length n on G_r is

$$|W_n(G_r)| = r2^n.$$

It is also not difficult to count the number of closed walks on G_r; we just need to ensure that we count only the correct instructions.

Lemma 9. *For $r \geq 1$, the number of closed walks of length n on G_r is*

$$|\operatorname{Cl}_n(G_r)| = r \sum_{k \geq 0} \binom{n}{rk}.$$

Note that when we take $r = 1$, every walk is a closed walk, and we recover

$$|W_n(G_r)| = |\operatorname{Cl}_n(G_r)| = r2^n$$

after applying Example 1.

Proof. We may choose the initial vertex freely, and this gives r choices.

For a walk $X \in W(G_r)$ to be closed, the number $|D(X)|$ of forward steps must be a multiple of r. Hence there are

$$\binom{n}{0} + \binom{n}{r} + \binom{n}{2r} + \cdots = \sum_{k \geq 0} \binom{n}{rk}$$

subsets of $[n]$ which are the instructions of a closed walk.

The total number of closed walks is just the product,

$$r \sum_{k \geq 0} \binom{n}{rk},$$

as desired. □

Now, to evaluate this sum, we shall take a weighted enumeration approach. We use the following weight function.

Definition 10. Let $\omega = e^{\frac{2\pi i}{r}}$ be a primitive rth root of unity. Note that $\omega^j = 1$, but that $\omega^j \neq 1$ for $1 \leq j < r$. We define $f : W_r(G_r) \to \mathbb{C}$ by

$$f(X) = f(x_0, I) = \omega^{x_0|I|}.$$

We can interpret f as assigning weight ω^{x_0} to every forward step $i \in I$, (implicitly) assigning weight 1 to every move $i \notin I$, and finally assigning to a walk the product of the weights of its steps. That is, f keeps track of the number $|I|$ of forward moves for a particular walk ('scaled' by the initial vertex x_0).

Example 11. Let $r = 4$, and again consider the closed walk

$$X = 3341223 = (3, \{2, 3, 4, 6\}).$$

Then $\omega = i$, a primitive 4th root of unity, and

$$\begin{aligned} f(X) &= \omega^{x_0|I|} \\ &= \omega^{3\cdot 4} \\ &= i^{12} \\ &= 1. \end{aligned}$$

We now compute the sum $f(W_n(G_r)) = \sum_{X \in W_n(G_r)} f(X)$ of the weight of all walks of length n on G_r.

Lemma 12. *Let f, ω be defined as in Definition 10. Then*

$$f(W_n(G_r)) = \sum_{j=1}^{r} (1 + \omega^j)^n.$$

Proof. By summing over $W_n(G_r)$ in two steps, we see that

$$\begin{aligned} f(W_n(G_r)) &= \sum_{j=1}^{r} \sum_{I \subseteq [n]} f(j, I) \\ &= \sum_{j=1}^{r} (1 + \omega^j)^n. \end{aligned}$$

The evaluation of the inner sum arises from observing that we may choose a forward or stationary move for each of our n steps in the walk. □

Recall that our goal in choosing a weight function is to find one that is

(a) easy to sum,

(b) assigns unit weight to elements we are interested in counting (in this case $\mathrm{Cl}_n(G_r)$, the subset of closed walks on G_n of length n), and

(c) assigns total weight 0 to everything we don't care to count (the open walks of length n on G_r).

Condition (a) has already been met (recall that a task is *easy* if we have already accomplished it). We now verify Conditions (b) and (c).

Proposition 13. *Closed walks have unit weight.*

Proof. Recall that a walk X on G_r is closed if and only if its instructions $D(X)$ have cardinality kr for some $k \geq 0$. As ω is an rth root of unity,

$$f(X) = \omega^{x_0 kr} = 1,$$

and the result follows. □

Proposition 14. *Let $I \subseteq [n]$ be the instructions for an open walk on G_r of length n. Then*

$$\sum_{j=1}^{r} f(j, I) = 0.$$

Thus, for any open walk $X = (x_0, I)$, we call $\{(j, I)\}_{j \in [r]}$ the orbit *of X.*

Proof. Since I does not create a closed walk, $|I|$ is not a multiple of r. Then $\omega^{|I|} \neq 1$, and we sum a finite geometric series to get

$$\begin{aligned} \sum_{j=1}^{r} f(j, I) &= \sum_{j=1}^{r} \omega^{j|I|} \\ &= \sum_{j=1}^{r} \left(\omega^{|I|}\right)^j \\ &= \frac{1 - \omega^{r|I|}}{1 - \omega^{|I|}} \\ &= 0 \end{aligned}$$

because the denominator is nonzero and the numerator is. □

Corollary 15. *The sum of weights over all open walks is zero.*

Proof. Every open walk can be placed into an orbit as described in Proposition 14. □

We now have the necessary tools to prove Equation 2.1.

Theorem 16. *Let $r \geq 1$. Then*

$$\sum_{k\geq 0} \binom{n}{rk} = \frac{1}{r} \sum_{j=1}^{r} (1+\omega^j)^n,$$

where ω is a primitive rth root of unity.

Proof. We have shown combinatorially that

$$r \sum_{k\geq 0} \binom{n}{rk} = |\operatorname{Cl}_n(G_r)| = f(W_n(G_r)) = \sum_{j=1}^{r} (1+\omega^j)^n,$$

and now dividing by r yields the desired theorem. (For those who prefer a direct proof without rearranging terms, simply count the number of closed walks with initial vertex, say, 1.) □

2.4 Shifted Coefficients

One very nice feature of proofs via weighted enumeration is that oftentimes we can generalize our result by changing the weight function in a small way. These generalizations may be hard to notice, let alone prove, using other methods. We will now generalize the result from Section 2.3 and evaluate

$$\sum_{k\geq 0} \binom{n}{a+rk}$$

for $0 \leq a < r$.

Theorem 17. *Let $0 \leq a < r$, and let $r \geq 1$. Then*

$$\sum_{k\geq 0} \binom{n}{a+rk} = \frac{1}{r} \sum_{j=1}^{r} \omega^{-aj} (1+\omega^j)^n.$$

Note that we do recover Theorem 16 by taking $a = 0$.

Proof. Consider the length n walks on G_r that move forward a vertices; that is, the walks $X \in W_n(G_r)$ for which $x_n - x_0 = a$ (modulo r). How many are there? We must choose an initial vertex x_0 (in r ways) and then a subset $I \subseteq [n]$ with cardinality congruent to a modulo r. The total number of ways to accomplish this selection is

$$r \sum_{k \geq 0} \binom{n}{a + rk}.$$

Now, define the weight function $f_a : W_n(G_r) \to \mathbb{C}$ by

$$f_a(X) = f(x_0, I) = \omega^{x_0(|I|-a)} = \omega^{-ax_0} f(X).$$

We take the same weight function from our previous proof, and add in an 'offset' of a steps. If we go through the exact same arguments as in Section 2.3 (just using f_a in place of f), we see that walks which advance a vertices have unit weight, while other walks can be placed into orbits with vanishing weight. So $f_a(W_n(G_r))$ counts the desired subset of walks.

Meanwhile, we can compute the total weight of all walks with initial vertex j simply by shifting the old sum by ω^{-aj}, so that we have

$$f_a(W_n(G_r)) = \sum_{j=1}^{r} \omega^{-aj}(1 + \omega^j)^n.$$

The rest of the proof proceeds as before. □

2.5 Alternative Forms

When r divides n, we can make simplifications to our results, expressing the sum of binomial coefficients using cosines instead of complex numbers. Recall that we may write

$$\cos\theta = \frac{1}{2}\left(e^{-i\theta} + e^{i\theta}\right).$$

Corollary 18. *Let r divide n. If $\frac{n}{r}$ is even, then*

$$\sum_{k \geq 0} \binom{n}{rk} = \frac{1}{r} \sum_{j=1}^{r} \left(2\cos\frac{\pi j}{r}\right)^n,$$

and if $\frac{n}{r}$ is odd then

$$\sum_{k \geq 0} \binom{n}{rk} = \frac{1}{r} \sum_{j=1}^{r} (-1)^j \left(2\cos\frac{\pi j}{r}\right)^n.$$

Proof. Let

$$\nu = \sqrt{\omega} = e^{\frac{\pi i}{r}}$$

be a primitive $2r$th root of unity. Write $n = rs$. From Theorem 16, we have

$$\begin{aligned}
\sum_{k\geq 0} \binom{n}{rk} &= \frac{1}{r}\sum_{j=1}^{r} (1+\omega^j)^n \\
&= \frac{1}{r}\sum_{j=1}^{r} (1+\nu^{2j})^n \\
&= \frac{1}{r}\sum_{j=1}^{r} \nu^{jn}(\nu^{-j}+\nu^j)^n \\
&= \frac{1}{r}\sum_{j=1}^{r} (\nu^r)^{sj}\left(e^{\frac{-i\pi j}{r}}+e^{\frac{i\pi j}{r}}\right)^n \\
&= \frac{1}{r}\sum_{j=1}^{r} (-1)^{sj}\left(2\cos\frac{\pi j}{r}\right)^n.
\end{aligned}$$

If $\frac{n}{r} = s$ is even, then $(-1)^{sj} = 1$ for every j and hence vanishes as a factor. Otherwise, $(-1)^{sj} = (-1)^j$, and the result follows. □

If we run this argument in reverse, we see that cosine terms can be rewritten into sums of complex numbers, which may permit a combinatorial argument by weighted enumeration. This is the motivation for our approach in Chapter 3.

Chapter 3

Checkerboards and Dominoes

We recall that Kasteleyn, Fisher, and Temperley had shown that the number of ways to cover a checkerboard with dominoes (or equivalently, to cover a *square lattice* with *dimers*, molecules that occupy two adjacent spaces in a lattice) is given by

$$N(2m,2n) = \prod_{j=1}^{m} \prod_{k=1}^{n} \left[4\cos^2 \frac{j\pi}{2m+1} + 4\cos^2 \frac{k\pi}{2n+1} \right], \tag{3.1}$$

where the checkerboard measures $2m \times 2n$. In this chapter we will examine a slightly simplified version by Propp (1997). We then present an incomplete combinatorial argument for this result.

3.1 A Simplified Kasteleyn Proof

We (following Propp (1997)) shall consider the version of the proof that asks for the number of perfect matchings of a graph.

Definition 19. A graph is an ordered pair (V, E), where V is a set and E is a collection of 2-subsets of V. The elements of V are called *vertices* and those of E are called *edges*. If the set $\{u, v\} \in E$, then we say u is *adjacent* to v and write $u \sim v$ (and of course v is adjacent to u as well!). The *neighbors* of a vertex v are simply the vertices adjacent to v.

The adjacency matrix of a graph G is

$$A = \left(a_{ij}\right),$$

where

$$a_{ij} = \begin{cases} 1 & u \sim v \\ 0 & u \nsim v. \end{cases}.$$

Let G be the grid graph of size $2m \times 2n$, so that

$$V = \{(j,k) \mid 1 \leq j \leq 2m, 1 \leq k \leq 2n\},$$

and that

$$E = \{\{(a,b),(c,d)\} \mid |a-c| + |b-d| = 1\}.$$

We color the vertices black and white according to the parity of the sum of the coordinates, so that vertices are adjacent only to those of the opposite color. Now, we label the vertices so that $v_1, \ldots, v_{2mn}$ are black and $v_{2mn+1}, \ldots, v_{4mn}$ are white.

The adjacency matrix A of G is

$$A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

for some matrix B, because no two vertices of the same color are adjacent.

Propp then goes on to show that if we replace the 1s in A that denote a vertical edge with is, then the determinant of the resulting matrix counts exactly the number of possibly perfect matchings. Evaluating the determinant yields Equation 3.1.

3.2 The Combinatorial Idea

We shall now attempt to prove Kasteleyn's result combinatorially. To do so, we rewrite Equation 3.1 into the following form:

$$N(2m,2n) = \prod_{j=1}^{m}\prod_{k=1}^{n}\left[4 + \omega_{2m+1}^{j} + \omega_{2m+1}^{-j} + \omega_{2n+1}^{k} + \omega_{2n+1}^{-k}\right], \qquad (3.2)$$

where ω_r is a primitive rth root of unity. As in Section 2.3, we made use of the fact that

$$2\cos\theta = e^{i\theta} + e^{-i\theta}.$$

Now, private correspondence has provided the following theorem.

Theorem 20. *(Benjamin, Tucker, Klawe.) Choose a (not necessarily proper!) cover of the* doubly even cells *(those with coordinates of the form* $(2j,2k)$*). That is, for every doubly even cell, cover it with a domino that extends in one of the four directions. Then two dominoes overlap, some domino hangs over the edge, a cycle of dominoes enclose an odd region, or there is a unique way to extend the cover to a tiling of the entire board.*

This theorem, combined with the fact that the result takes a product over mn objects, suggests that our weight function should be a product of what 'happens' at each doubly even cell, because their cover determines already whether or not we can cover the entire board. This line of reasoning is precisely the one we will adopt.

First, we explain our notation. We use both black and white dominoes, for reasons that will soon become evident. We write a specific cover C of the doubly even cells as

$$C = \left[c_{jk}\right],$$

where $1 \leq j \leq m, 1 \leq k \leq n$, and

$$c_{jk} \in \{U, D, L, R, U', D', L', R'\},$$

which are the possible *orientations* of a domino. The value of c_{jk} indicates how the domino covering the cell $(2j, 2k)$ is oriented. U indicates a white domino covering $(2j, 2k)$ and the cell above, R' indicates a black domino covering $(2j, 2k)$ and the cell to the right, and so on.

It will be useful to define some notation with which we can write orientations in terms of the other orientations.

Definition 21. We may write, for shorthand,

$$U^{-1} = U', \qquad D^{-1} = D', \qquad L^{-1} = L', \qquad R^{-1} = R',$$

and vice versa. We also write

$$-U = D, \qquad -U' = D', \qquad -L = R, \qquad -L' = R',$$

and vice versa.

Definition 22. A doubly even tiling C is *good* if it can be extended to a complete tiling in the manner specified by Theorem 20 and every domino in the tiling is white; otherwise it is *bad*.

If a tiling is bad, then the dominoes that prevent Theorem 20 from applying are called the *impurities* of C. Impurities can be due to color, overlap, hanging (Ds in the last row or Rs in the last column), or some combination thereof.

If, say, $c_{jk} = R, c_{j(k+1)} = L$, then we say both c_{jk} and $c_{j(k+1)}$ are overlap impurities, for a total of two impurities.

To count the total number of tilings of the checkerboard, it suffices to count just the number of good tilings, because each such tiling extends to a tiling of the complete board. Due to the uniqueness of the extension, no two complete tilings arise from the same doubly even tiling, and so the number of good tilings precisely equals the number of tilings of the board. We use the following weight function to count the number of good tilings.

Definition 23. For a cover C of the doubly even cells, define the weight function

$$f(C) = \prod_{j=1}^{m} \prod_{k=1}^{n} f_{jk}(c_{jk}),$$

where

$$f_{jk}(c) = \begin{cases} 1, & c \in \{U, D, L, R\} \\ \omega_{2m+1}^{j} & c = U' \\ \omega_{2m+1}^{-j} & c = D' \\ \omega_{2n+1}^{k} & c = L' \\ \omega_{2n+1}^{-k} & c = R'. \end{cases}$$

Remark 24. Note that only colored orientations have nonunit weight; in particular, if a tiling has no impurities, then it has no color impurities and hence has unit weight. Moreover, the weight of a vertical color impurity depends only on its row, and not its horizontal position; likewise, the weight of a horizontal color impurity depends only on its column, and not on its position within that column.

It is clear, then, that the double product in Equation 3.2 simply enumerates the total weight of all 8^{mn} (not necessarily proper) doubly even tilings of the checkerboard. Moreover, good tilings, by definition, have weight 1. It remains to be shown that every bad tiling can be put into an orbit with zero weight. What follows is partial work towards this result.

3.3 $m = 1$

We first consider the simpler case of $m = 1$. When a doubly even tiling has a vertical impurity, we can cycle between the three possible vertical impurities at that location to create an orbit with vanishing weight.

Proposition 25. *Let $m = 1$ and let n be arbitrary. The total weight of all doubly even tilings with vertical impurities is 0.*

Proof. By definition, a covering C has a vertical impurity if and only if

$$c_{1k} \in \{D, U', D'\}$$

for some k. Let k_0 be the first horizontal position with a vertical impurity. Write C as

$$C = \begin{bmatrix} A & c_{1k_0} & B \end{bmatrix},$$

where A has no vertical impurities, c_{1k_0} is a vertical impurity, and B is any doubly even tiling.

Now, consider the orbit

$$\mathcal{T} = \{ \begin{bmatrix} A & c & B \end{bmatrix} \mid c \in \{D, U', D'\}\}.$$

Note that $\mathcal{T}$ is in fact an orbit because it preserves the location of the first vertical impurity. Then

$$\begin{aligned}
\sum_{T \in \mathcal{T}} f(C) &= f(\begin{bmatrix} A & D & B \end{bmatrix}) + (\begin{bmatrix} A & U' & B \end{bmatrix}) + (\begin{bmatrix} A & D' & B \end{bmatrix}) \\
&= f(\begin{bmatrix} A \end{bmatrix}) f(\begin{bmatrix} B \end{bmatrix}) \left[f(D) + f(U') + f(D') \right] \\
&= f(\begin{bmatrix} A \end{bmatrix}) f(\begin{bmatrix} B \end{bmatrix}) \left[1 + \omega_3 + \omega_3^{-1} \right] \\
&= 0,
\end{aligned}$$

as desired. Note the slight abuse of notation here: by $f([A])$ we mean the total weight of the subtiling A, and not the weight if A were treated as a complete tiling by itself, and similarly for $f([B])$.

The result follows because *every* tiling with a vertical impurity can be put into an orbit of this form, which has 0 weight. □

Now consider a covering with only one horizontal impurity and no vertical impurities. We may cycle the horizontal impurity through all $2n + 1$ possible choices of impurity: n positions with R', n positions with L', and finally R in the last position on the strip.

Proposition 26. *Let $m = 1$ and n be arbitrary. The total weight of all doubly even tilings with exactly one horizontal impurity, and no vertical impurities, is 0.*

Proof. For a doubly even tiling

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \end{bmatrix}$$

when $m = 1$, define

$$C + a = \begin{bmatrix} c_{1(1+a)} & c_{1(2+a)} & \cdots & c_{1n} & c_{11} & \cdots & c_{1a} \end{bmatrix}.$$

That is, $C + a$ is simply C shifted cyclically by a steps.

Let C be a tiling with exactly one horizontal impurity and no others, and let k_0 be the horizontal location of the horizontal impurity. Then we may write

$$C = \begin{bmatrix} A & c_{1k_0} & B \end{bmatrix},$$

where A, B do not have impurities and $c_{1k_0} \in \{L', R'\}$. Note that either A or B may be empty. (If C has one hanging horizontal impurity and no others, instead write

$$C = \begin{bmatrix} A & R' \end{bmatrix}$$

so that C has the form above. We will put these two configurations into the same orbit in the next step, so this rewriting is allowed.)

Define

$$D = \begin{bmatrix} A & -c_{1k_0} & B \end{bmatrix}.$$

Let $\mathcal{T}$ be the orbit

$$\{C + a \mid 0 \leq a \leq n-1\} \cup \{D + a \mid 0 \leq a \leq n-1\} \cup \{\begin{bmatrix} B & A & R \end{bmatrix}\}.$$

Note that $\mathcal{T}$ is, in fact, an orbit, as it preserves the relative positions of every domino except the horizontal impurity, and that we keep exactly one horizontal impurity. Of course, changing the horizontal positions of the vertical impurities does not affect their contribution to the weight of the tilings, as per Remark 24.

It follows that

$$\begin{aligned} \sum_{T \in \mathcal{T}} f(T) &= f(\begin{bmatrix} B & A & R \end{bmatrix}) + \sum_{a=0}^{n-1} f(C+a) + f(D+a) \\ &= 1 + \sum_{k=1}^{n} \omega_{2n+1}^{2k} + \omega_{2n+1}^{-2k} \\ &= 0. \end{aligned}$$

The second equality follows from the fact that the horizontal impurity is the only contributor of nonunit weight, and it finds itself in each position $(1, k)$ for $1 \leq k \leq n$, in both L' and R' orientations. The third equality follows because we are just adding up all $2n$th roots of unity.

Because every tiling with exactly one horizontal impurity will be put into an orbit of the above form, the proposition follows. □

However, we have been unable to complete the combinatorial argument for the $m = 1$ case, which involves putting those doubly even tilings with more than one horizontal impurity (and no vertical impurities) into orbits with vanishing weight.

3.4 $m > 1$

Theorems 25 and 26 are just special cases of the following theorem, which applies to checkerboards of arbitrary size.

Theorem 27. *Let m, n be arbitrary. If a tiling C of the doubly even cells contains some column with exactly one vertical impurity, or contains a row with exactly one horizontal impurity, then C can be put into an orbit with total weight zero.*

Proof. We shall prove only the case where C has a row with precisely one horizontal impurity (the other case is analogous).

If j is the row that contains one horizontal impurity, then we may write C as

$$C = \begin{bmatrix} A \\ c_j \\ B \end{bmatrix},$$

where c_j is a $1 \times n$ tiling. Now, take the orbit $\mathcal{T}$ of c_j given by the proof of Proposition 26. Then the orbit

$$\mathcal{S} = \left\{ \begin{bmatrix} A \\ T \\ B \end{bmatrix} \middle| \; T \in \mathcal{T} \right\}$$

preserves the row on which C has exactly one horizontal impurity, and moreover has total weight

$$\begin{aligned} \sum_{S \in \mathcal{S}} f(S) &= f(A) f(B) \sum_{T \in \mathcal{T}} f(T) \\ &= f(A) f(B) \cdot 0 \\ &= 0, \end{aligned}$$

because the total weight of the orbit $\mathcal{T}$ is zero by Proposition 26. □

Chapter 4

Conclusion

We discovered a combinatorial proof of the identity

$$\sum_{k \geq 0} \binom{n}{rk} = \frac{1}{r} \sum_{j=1}^{r} (1 + \omega^j)^n,$$

where ω is a primitive rth root of unity. We noticed that we could easily convert from sums of roots of unity to cosine expressions and back, which led to our idea for a combinatorial proof of

$$N(2m, 2n) = \prod_{j=1}^{m} \prod_{k=1}^{n} \left[4 \cos^2 \frac{j\pi}{2m+1} + 4 \cos^2 \frac{k\pi}{2n+1} \right].$$

We detailed the combinatorial argument that should prove this equation, though there are gaps which remain to be filled in.

4.1 Future Work

This thesis related two distinct areas of research: binomial coefficient identities and the Kasteleyn Fisher Temperley result on domino tilings. Each area has its own directions for future research.

4.1.1 Identities Relating Binomial Coefficients and Roots of Unity

Gould (1972) contains a large number of binomial coefficient identities. Many of them involve roots of unity. However, not all such identities can be proved combinatorially using a weighted enumeration approach.

For example,

$$\sum_{k=1}^{\infty} \frac{1}{\binom{kr}{r}} = \sum_{k=1}^{r-1} -\omega^k (1-\omega^k)^{r-1} \log \frac{1-\omega^k}{-\omega^k}$$

is Identity 2.24 (again, ω is a primitive rth root of unity). The reciprocals in the left sum, along with the $\omega_k(1-\omega_k)^{n-1}$ factor in the right sum, suggest a probabilistic approach.

4.1.2 Domino Tilings of a Checkerboard

Section 3.2, of course, remains to be completed.

To complete the combinatorial proof of Equation 3.1, it remains only to consider the doubly even tilings which have at least two horizontal impurities in any row that contains a horizontal impurity, and which have at least two vertical impurities in any column that contains any vertical impurity. This suggests that we may be able match up these impurities to outline some sort of polygonal cycle of dominoes. These cycles play an important role in the proof of Theorem 20, and deeper analysis of the proof may yield an insight.

Another direction in which we might obtain an insight is to consider wraparound boards; specifically, cylindrical and toroidal boards. We noted that an issue which presented difficulties was the way in which black dominoes wrapped around off the edge of the board, in that their weight would suddenly change by a great deal (from $e^{\frac{2m\pi i}{2m+1}}$ to $e^{\frac{2\pi i}{2m+1}}$). If we could define some new weight function that smooths out this discontinuity as the domino wraps around the edge, orbits for the bad tilings may be easier to find.

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