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Group Frames and Partially Ranked Data

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May, 2010

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Abstract

We give an overview of finite group frames and their applications to calculating summary statistics from partially ranked data, drawing upon the work of Rachel Cranfill (2009). We also provide a summary of the representation theory of compact Lie groups. We introduce both of these concepts as possible avenues beyond finite group representations, and also to suggest exploration into calculating summary statistics on Hilbert spaces using representations of Lie groups acting upon those spaces.

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Chapter 1

Introduction and Background

Consider an election where voters rank n candidates $A, B, C, \ldots$ in order of preference. Each voter selects a permutation of the n candidates, and we can represent the election results as a single $n!$-dimensional vector, called a *voting profile*, where each dimension corresponds to a permutation of the n candidates, and each entry counts the number of voters who chose the corresponding permutation. Such a data vector is an example of *rank data*.

Given a particular data vector, we can describe attributes of the data using *summary statistics*. Summary statistics are typically related to the lengths of projections of the data vector onto meaningful subspaces of the sample space; depending on the subspace, these lengths can measure the uniformity of a data vector, the effects of single candidates or pairs of candidates on a voting profile, and so on. For example, the means summary statistic for rank data describes the average rank of each candidate compared to the overall average rank; if a candidate was ranked very high or very low on average, the means statistic will be larger than if all candidates had roughly equal average rank. Therefore, a large means statistic indicates that one or more candidates was ranked differently than the others on average, and thus it was unlikely the voters voted uniformly across all possible permutations of candidates.

As data vectors grow in size and dimension, computing the lengths of the projections to obtain these summary statistics can quickly become unwieldy. In the case of rank data, certain "shortcuts" to calculating the most common summary statistics are well-known (Marden, 1995). Recent work by Cranfill (2009) has shown that two of these shortcuts can be interpreted as statements about Parseval group frames.

We wish to explore extensions of these ideas to partially ranked data, with the ultimate goal of discovering new shortcuts or new summary statistics. As the name implies, *partially ranked data* reveal a subset of the information provided by rank data. In particular, partially ranked data describe an election where voters are asked to partition candidates into ordered subsets of fixed size. We will focus on partially ranked data that arise when voters choose their top k candidates out of n choices; the corresponding sample space has $\binom{n}{k}$ dimensions, each describing a particular k-subset of candidates. However, before we describe this exploration, we must first provide some background information on these concepts.

1.1 Finite Representation Theory

For a finite group G, a *representation* of G is a homomorphism of groups,

$$\rho : G \to GL_n(\mathbb{F}),$$

from G into the group of $n \times n$ invertible matrices over a field $\mathbb{F}$. This mapping induces an action of the group on the vector space $\mathbb{F}^n$ by the product

$$v \mapsto \rho(g)v.$$

for an element $g \in G$. We can naturally define a multiplication between group elements $g \in G$ and vectors in $v \in \mathbb{F}^n$ by

$$gv \mapsto \rho(g)v,$$

producing another vector in $\mathbb{F}^n$. Scaling v before applying g simply scales the result, and the action of g distributes over a sum of vectors in $\mathbb{F}^n$; additionally, for $g, h \in G$, we see that

$$(hg)v = \rho(hg)v = \rho(h)\rho(g)v = \rho(h)(\rho(g)v) = h(gv),$$

because ρ is a homomorphism. The group identity 1 must correspond to the identity matrix, or else $\rho(g) = \rho(1 \cdot g) = \rho(1)\rho(g)$ would not hold. These properties give $\mathbb{F}^n$ the structure of an $\mathbb{F}G$-module. This module is also referred to as a representation of G. For the purposes of this thesis, I will choose $\mathbb{C}$ as the field $\mathbb{F}$ unless otherwise specified.

For example, consider the action of the cyclic group $Z_3 = \{1, a, a^2\}$ on $\mathbb{C}^3$ by permutation of the three dimensions. We define a map $\rho : Z_3 \to GL_3(\mathbb{C})$ by

$$1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad a \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad a^2 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We have

$$\rho(1 \cdot a) = \rho(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \rho(1)\rho(a)$$

(and the similar result for $\rho(1 \cdot a^2) = \rho(1)\rho(a^2)$), and

$$\rho(a \cdot a^2) = \rho(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \rho(a)\rho(a^2).$$

Because $\rho(gh) = \rho(g)\rho(h)$ for any $g, h \in Z_3$, ρ is a representation of Z_3.

The primary method of understanding group representations, and the groups they represent, is through *irreducible* representations, which are representations whose associated $\mathbb{F}G$-modules have only trivial submodules (i.e., the only subspaces of V invariant under the action of G are the trivial subspace $\{0\}$ and the entire module). The following theorem (known as *Schur's Lemma*) constrains the relations between two irreducible representations.

Theorem 1.1 (Schur's Lemma)**.** *If U and V are complex finite-dimensional irreducible representations of a group G, then any $\mathbb{C}G$-module homomorphism $f : U \to V$ (a linear map such that $f(gu) = gf(u)$) is either an isomorphism or the trivial map $u \mapsto 0$. Moreover, if f is an isomorphism, it is a scalar multiple λ times the identity endomorphism 1_U.*

Proof. (Given in Carter and colleagues (1995: Book 2, Lemma 6.6)) The first statement holds because $\ker(f)$ and $f(U)$ are $\mathbb{C}G$-submodules of U and V, respectively; because both U and V are irreducible, then $\ker(f)$ must be either U or 0 and $f(U)$ must be either 0 or V, respectively. Therefore, f is injective if and only if it is surjective, and f is either surjective or the trivial map.

The second statement holds because eigenspaces of isomorphisms are $\mathbb{C}G$-submodules; therefore, all of U is an eigenspace for some eigenvalue λ of f, and thus applying f is equivalent to scaling the argument by λ. □

Furthermore, we have that every representation of a finite group G is decomposable into a direct sum of irreducible submodules.

Theorem 1.2 (Maschke's Theorem)**.** *If V is a representation of a finite group G, and U is a subrepresentation of V, then there is a complementary G-invariant subspace W of V such that $V = U \oplus W$.*

Proof. (Given in Fulton and Harris (2004: Proposition 1.5)) Let W' be any subspace complement to U in V, so $V = U \oplus W'$. Consider the projection $\pi_0 : V \to U$ from the direct sum decomposition and average π_0 over the group G to obtain

$$\pi(v) = \frac{1}{|G|} \sum_{g \in G} g(\pi_0(g^{-1}v)),$$

which is a G-linear map from V onto U equivalent to the identity on U. Then $\ker(\pi)$ is a G-invariant subspace of V complementary to U, as desired. □

It follows from Schur's Lemma and Maschke's Theorem that every representation of a finite group G has a unique decomposition into irreducible subrepresentations; repeated application of Maschke's Theorem decomposes the representation into irreducible subrepresentations, and Schur's Lemma shows that the decomposition is unique (Fulton and Harris, 2004).

1.1.1 Representation Theory of Partially Ranked Data

For rank data and partially ranked data of n candidates, we have a natural unitary action of the symmetric group S_n acting on the sample space by permuting candidates. In fact, this action defines a representation of S_n, making the sample space into a $\mathbb{C}S_n$-module. Fortunately, this representation is well-understood. Each irreducible $\mathbb{C}S_n$-module corresponds to a certain partition of n elements, and for the module M corresponding to the action of S_n on partially ranked data consisting of k-subsets of n candidates, we have

$$M = S_n^{(n)} \oplus S_n^{(n-1,1)} \oplus \cdots \oplus S_n^{(n-k,k)} \tag{1.1.1}$$

where the S_n^{λ} are the irreducible representations of S_n corresponding to the partitions λ of n elements (James and Kerber, 1981; Diaconis, 1988). In particular, every irreducible module in this representation occurs only once in the direct sum, so it is extremely easy to isolate each irreducible (due to Schur's Lemma).

1.2 Group Frames

With group representations refreshed, we can turn our attention to frames. Frames are similar in many ways to bases and spanning sets; they are sets of elements within a space that can be used to construct the entire space in which they reside. However, frames are more flexible than bases, because they do not require linear independence, while still providing several basis-like relations that are not guaranteed by arbitrary spanning sets. To begin, we will examine the spaces from which we can obtain a frame.

Recall that an *inner product space* is a vector space V over the real or complex field $F = \mathbb{C}$ or $\mathbb{R}$, paired with a binary operation $\langle \, , \, \rangle : V \times V \to F$ such that

1. For all vectors $v \in V$, $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$;
2. For all vectors $u, v \in V$, $\langle u, v \rangle = \overline{\langle v, u \rangle}$; and
3. For all vectors $u, v, w \in V$ and scalars $a, b \in F$,

$$\langle au + bv, w \rangle = a \langle u, w \rangle + b \langle v, w \rangle .$$

The inner product also induces a *norm* on the vector space V, defined by

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

Note that when V is a vector space over $\mathbb{R}$, $\overline{\langle v, u \rangle} = \langle v, u \rangle$, and the inner product is symmetric.

Many familiar vector spaces are inner product spaces; the Euclidean spaces $\mathbb{R}^n$ are inner product spaces, for example, under the standard Euclidean dot product

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = x_1 y_1 + \cdots + x_n y_n.$$

Also, recall that a *metric space* is a set M with a function $d : M \times M \to \mathbb{R}$ (called a *metric*) that satisfies, for all $x, y, z \in M$,

1. $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$; and
3. $d(x, y) \leq d(x, z) + d(z, y)$.

Note that any inner product space is a metric space using the metric

$$d(u, v) = \|u - v\| = \sqrt{\langle u - v, u - v \rangle}.$$

Finally, recall that a sequence (x_n) in M is called a *Cauchy sequence* if, for any $\varepsilon > 0$, there exists some $N > 0$ such that for all $m, n \geq N$, $d(x_m, x_n) < \varepsilon$; a metric space M in which each Cauchy sequence converges to some point in M is a *complete* metric space.

A *Hilbert space* $\mathcal{H}$ is an inner product space that is complete with respect to the metric induced by the inner product. Frames arise in these spaces.

Again, many familiar spaces are Hilbert spaces; $\mathbb{R}^n$ is a Hilbert space, as the metric induced by the standard Euclidean inner product is the same as that given by the distance formula, and $\mathbb{R}^n$ is complete under that metric (the proof of the completeness of $\mathbb{R}^n$ is not relevant to this thesis, and has been omitted).

While we normally discuss group representations as maps into the general linear group, we can also define the *unitary representation of a group G* as a group homomorphism ρ from the group G into the group of unitary operators on a Hilbert space $\mathcal{H}$ (an *operator* is a Hilbert space transformation from $\mathcal{H}$ to itself). When $\mathcal{H}$ is $\mathbb{R}^n$ or $\mathbb{C}^n$, ρ is simply a homomorphism from G into the group of real or complex unitary $n \times n$-matrices, as expected.

A *frame* for a Hilbert space $\mathcal{H}$ is a collection $\{x_1, x_2, \ldots\} \subset \mathcal{H}$ with constants $0 < A \leq B < \infty$ such that for any $x \in \mathcal{H}$,

$$A \|x\|^2 \leq \sum_i |\langle x, x_i \rangle|^2 \leq B \|x\|^2 .$$

If $A = B$, we call this a *tight frame*; if $A = B = 1$, we call it a *Parseval frame*.

When $\mathcal{H}$ is finite-dimensional, a frame is equivalent to a spanning set of $\mathcal{H}$. While this is not true for an infinite-dimensional space, it greatly simplifies dealing with frames in finite-dimensional spaces.

Theorem 1.3. *If $\mathcal{H}$ is a finite-dimensional Hilbert space, then $\{x_1, \ldots, x_k\} \subset \mathcal{H}$ is a frame for $\mathcal{H}$ if and only if it is a spanning set of $\mathcal{H}$.*

Proof. (Given in Han and colleagues (2007: Proposition 3.18)) ($\Rightarrow$) Suppose to the contrary that the frame $\{x_1, \ldots, x_k\}$ does not span $\mathcal{H}$; then there is some nonzero vector $v \in M^\perp$, the perpendicular complement of the subspace $M = \text{span}\{x_1, \ldots, x_k\}$. But then v is orthogonal to each x_i, and thus

$$\sum_{i=1}^{k} |\langle v, x_i \rangle|^2 = 0,$$

violating the lower bound condition of a frame.

($\Leftarrow$) To prove the converse, suppose to the contrary that the spanning set

$\{x_1, \ldots, x_k\}$ is not a frame. Because this set is finite, there is no way to violate the upper bound condition of a frame, so this set must violate the lower bound condition; therefore, for every positive integer m, there is some $y_m \in \mathcal{H}$ such that $\|y\|_m = 1$ and

$$\sum_{i=1}^{k} | \langle y_m, x_i \rangle |^2 < \frac{1}{m}.$$

The sequence $\{y_m\}$ is bounded, and therefore it must have a convergent subsequence $\{y_{m_j}\} \to y$ as $j \to \infty$. Then because $\lim_{j\to\infty} \frac{1}{m_j} = 0$,

$$0 = \lim_{j\to\infty} \sum_{i=1}^{k} \left| \left\langle y_{m_j}, x_i \right\rangle \right|^2 = \sum_{i=1}^{k} |\langle y, x_i \rangle|^2$$

and thus y is orthogonal to each x_i. But $\|y_m\| = 1$ and $y_m \to y$, so $\|y\| = 1$ and thus there is a nonzero $y \in \mathcal{H}$ that is orthogonal to the spanning set $\{x_1, \ldots, x_k\}$, a contradiction. □

Theorem 1.3 gives us a great number of accessible examples of frames. In $\mathbb{R}^n$, any spanning set is a frame, so every basis is a frame. In addition, the collection $(1, 0), (0, 1), (-1, 0), (0, -1)$ forms a frame for $\mathbb{R}^2$, because the collection spans the entire space.

Parseval frames have the convenient property that they can be used to reconstruct vectors like orthonormal bases (a proof of Theorem 1.4 is given in Han and colleauges (2007: Proposition 3.11)).

Theorem 1.4. *$\{x_1, \ldots, x_k\} \subset \mathcal{H}$ is a Parseval frame for a Hilbert space $\mathcal{H}$ if and only if*

$$x = \sum_{i=1}^{k} \langle x, x_i \rangle \, x_i$$

for all $x \in \mathcal{H}$.

A unitary representation ρ is called a *frame representation* when there is some vector $\phi \in \mathcal{H}$ such that the action of G on ϕ sweeps out a frame for $\mathcal{H}$ (that is, when the set $\{\rho(g)\phi\}$ is a frame). Such a frame is called a *group frame*. Group frames have extremely strict conditions for being Parseval frames.

Theorem 1.5. *Let G be a finite group with a unitary representation ρ on $\mathcal{H}$, and let $\mathcal{H} = V_1 \oplus \cdots \oplus V_k$ be an orthogonal direct sum of $\mathbb{F}G$-modules. Then for*

$v = v_1 + \cdots + v_k$, where $v_i \in V_i$, $Gv = \{\rho(g)v\}_{g\in G}$ is a Parseval frame for $\mathcal{H}$ if and only if

$$\frac{\|v_i\|^2}{\|v_j\|^2} = \frac{\dim(V_i)}{\dim(V_j)}$$

for all $1 \leq i, j \leq k$ and

$$\sum_{g\in G} \langle v_i, gv_i \rangle \, gv_j = 0$$

for all $i \neq j$.

A proof is given in Vale and Waldron (2004: Lemma 6.7).

Chapter 2

Group Frames for Partially Ranked Data

We would like to find a Parseval group frame for the sample space of partially ranked data of k-subsets of n candidates. Why do we seek such an object? As shown in Cranfill (2009), shortcuts to computing two summary statistics can be explained as statements about Parseval frames; we want to use a similar construction for a summary statistic on partially ranked data, and thus we must begin with an appropriate Parseval frame on the sample space. A Parseval frame gives us a way to easily compute the projections and corresponding norms to obtain certain summary statistics, as shown in Cranfill (2009). Moreover, we want our statistics to be invariant under permutations of the candidates; if we conclude our voting profile is nonuniform, we should give the same conclusion for a voting profile with the same vote totals, but where the candidates have been permuted. Therefore, we want to find subspaces and frames that are invariant under permutations of candidates, which are precisely the $\mathbb{C}S_n$-modules corresponding to permutations of the candidates and the group frames for that module, respectively. Because these representations decompose into direct sums of irreducible $\mathbb{C}S_n$-modules, we need only consider the irreducible $\mathbb{C}S_n$-modules.

We have from Theorem 1.5 a condition for determining when a vector v will generate a Parseval frame when acted upon by S_n. To use this condition, we need to know the norm of the projection of our data vector v onto the irreducible submodules of the representation, and the dimensions of those irreducible submodules. Because the representation theory of the action of S_n on partially ranked data is well-known, we have the dimension of each irreducible submodule of our representation. If we define M as in Equation 1.1.1 and N in a similar way for the $\binom{n}{k-1}$ case, we see that $M = N \oplus S_n^{(n-k,k)}$. Furthermore, M has dimension $\binom{n}{k}$ and N has dimension $\binom{n}{k-1}$, because each module has dimension equal to the number of subsets available to voters to choose. It follows that $S_n^{(n-k,k)}$ has dimension $\binom{n}{k} - \binom{n}{k-1}$.

We can also easily obtain the projections of our data vector v onto the relevant irreducible submodules of our representation by using the *centrally primitive idempotents* of those irreducible submodules; these centrally primitive idempotents are the summands in the decomposition of the group identity e_G into elements of the irreducible submodules of the representation. Moreover, the action of a centrally primitive idempotent e_U for an irreducible module U on a vector v is to isolate only the part of v within U (the proof of this statement is omitted here, and can be found in James and Liebeck (2001: Chapter 14)). Furthermore, if U is an irreducible $\mathbb{C}G$-module under a representation ρ, the corresponding centrally primitive idempotent e_U is given by

$$e_U = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) g,$$

where $\chi(g)$ is the trace of the matrix $\rho(g)$ (James and Liebeck, 2001: Proposition 14.10).

For partially ranked data of the $\binom{n}{k}$ variety, we see that the indicator vectors for the k-subsets of candidates form an orthonormal basis for the corresponding $\mathbb{C}S_n$-module. This basis is a Parseval frame, as all orthonormal bases satisfy Theorem 1.4, and (by definition) must be spanning sets of their spaces. Because the representations of smaller subsets of n candidates nest within the $\binom{n}{k}$ representation, it would be convenient if the indicator vector for j-subsets of candidates formed a Parseval frame for the submodule $S_n^{(n)} \oplus \cdots \oplus S_n^{(n-j,j)}$ of $S_n^{(n)} \oplus \cdots \oplus S_n^{(n-k,k)}$, even when $j \neq k$.

Unfortunately, this is not the case in general. Consider an election between four candidates where each voter chooses a pair; the sample space S of voting profiles is six-dimensional, and the action of S_4 on S gives us the $\mathbb{C}S_4$-module

$$S \cong S_n^{(4)} \oplus S_n^{(3,1)} \oplus S_n^{(2,2)}.$$

We have the four indicator vectors

$$\begin{pmatrix}1\\1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\1\\1\end{pmatrix}$$

for the four respective candidates (assuming subsets of candidates are assigned to dimensions in lexicographical order); these form an orbit under the action of S_4 by permutation of candidates. We examine the projections of the first of these indicator vectors onto each of the irreducible submodules of S; these are

$$\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix}, \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}$$

for $S_4^{(4)}$, $S_4^{(3,1)}$, and $S_4^{(2,2)}$, respectively. Note that the projection of this indicator vector onto $S_4^{(2,2)}$ is zero, because the indicator vectors for single candidates reside fully within the part of S isomorphic to the $\binom{4}{1}$ case (i.e., $S_4^{(4)} \oplus S_4^{(3,1)}$).

$S_4^{(4)}$ has dimension 1 and $S_4^{(3,1)}$ has dimension 4, so Theorem 1.5 tells us that the norm squared of the projection into $S_4^{(4)}$ should be one fourth that of the norm squared of the projection into $S_4^{(3,1)}$. However, both projections have a norm squared of $\frac{3}{2}$, so these indicator vectors do not form a Parseval frame for $S_4^{(4)} \oplus S_4^{(3,1)}$.

Given the decomposition of a vector into its projections onto irreducible submodules of S, we can reassemble a vector that will generate a Parseval frame by scaling the projections so that the squares of their norms are in the correct ratios, and taking the sum of the scaled projections. For the single-candidate indicator vectors, we need to scale the projection into $S_4^{(3,1)}$ by 2, thus scaling the square of its norm by a factor of 4, and satisfying Theorem 1.5.

In this case, we can scale by an integer to obtain the desired ratios of projections, but this is not true in other cases. If we consider the effect of pairs of candidates in an election where voters choose a set of 3 candidates from a pool of 6, the indicator vector decomposes into projections whose norms squared are in a $1 : \frac{5}{2} : \frac{3}{2}$ ratio (as the sample space has dimension 20, I will refrain from reproducing the vectors themselves). To be a Parseval frame, the ratio of norms squared must be $1 : 5 : 9$; therefore, we need to scale the projection onto $S_n^{(5,1)}$ by $\sqrt{2}$ and the projection onto $S_n^{(4,2)}$ by $\sqrt{6}$. When considering the effect of pairs of candidates where voters choose a set of 3 candidates from a pool of 8, the resulting ratios of norms squared is $1 : \frac{35}{9} : \frac{40}{9}$, while the ratios required to generate a Parseval frame are $1 : 7 : 20$.

While it is straightforward to compute Parseval frames for these spaces, the resulting frames lack a straightforward statistical interpretation. It is possible that using some generalized indicator vector, where partial rankings are assigned different weights based on how many elements from a desired subset appear, might generate a Parseval group frame. It is unclear at best if Cranfill's work can be extended to generate summary statistics for partially ranked data with group frames. However, we have completed only introductory work, and the question remains mostly unexplored.

Chapter 3

Lie Theory

Beyond applications to partially ranked data, it may be possible to generalize Cranfill's work further to gain insight into "data" vectors in some arbitrary Hilbert space that is homogeneous under the action of a Lie group. While we merely introduce the possibility of such a generalization, we provide background on Lie groups and Lie algebras in the hope that a more focused investigation into Lie groups, group frames, and representation theory might begin.

Lie groups differ from more familiar finite groups (such as the symmetric group, the cyclic group, etc.) in two primary ways. Lie groups may contain infinitely many elements, and they are endowed with a topological structure as well as an algebraic structure. Although these attributes might seem unintuitive, many Lie groups are familiar objects; the real numbers, $\mathbb{R}$, form a Lie group, as does the sphere S^2 living in $\mathbb{R}^3$. Like finite groups, Lie groups can also arise as the symmetries of an object; for example, the set of rotations of the sphere S^2 in $\mathbb{R}^3$ is the Lie group SO_3. Because Lie groups have a topological structure, before giving a formal definition of a Lie group it is useful to briefly review some basic topological ideas, with attention to similarities to group structures.

3.1 Topological Spaces

Just as a group is defined as a set paired with an operation that relates elements in that set, a topological space is defined as a set paired with a collection that describes which subsets define the "closeness" of elements of that set.

Specifically, a set X is a *topological space* when it is paired with a collection T of subsets of X where

1. $\emptyset$ and X are contained in T;
2. The union of any arbitrary subcollection of elements in T is also in T; and
3. The intersection of finitely many elements of T is contained in T;

then T is a *topology* on X, and the elements of T are the *open sets* in X under the topology T.

There are many familiar topological spaces. $\mathbb{R}$ is one good example that uses distance to define a topology; a set is open in $\mathbb{R}$ if every element of the set has some nonzero radius around it that contains only points in the set.

Like we define a basis for a group that generates the rest of the group through application of the group operation, we can define a *basis* for a topology as a collection B of subsets of X (*basis elements*) that satisfy the following properties:

1. For each $x \in X$, there is some basis element B_x containing x, and
2. If for basis elements B_1 and B_2, $x \in B_1 \cap B_2$, there is some basis element B_3 containing x with $B_3 \subseteq B_1 \cap B_2$.

Again using $\mathbb{R}$ as an example, the collection of all open intervals forms a basis for $\mathbb{R}$, because every point $x \in \mathbb{R}$ is contained in some open interval, and the intersection of any two open intervals is another open interval.

Given a basis B for a topology on X, we can recover the full topology T by taking a subset U of X to be open if and only if for each $x \in U$ there is some basis element $B_x \subseteq U$ with $x \in B_x$. In this case, T is called the *topology generated by* B.

Theorem 3.1. *If B is a basis for a topology T on a set X, then T is the collection of all unions of elements of B.*

Proof. (Given in Munkres (1975), Lemma 2.1) Because T is closed under arbitrary unions of its elements, and B is a subcollection of T, any union of elements of B is an element of T. Conversely, for some $U \subseteq X$ in T, any $x \in U$ is contained in some basis element $B_x \subseteq U$, and thus

$$U = \bigcup_{x \in U} B_x,$$

proving the theorem. □

A subgroup of a group is a subset of the group that inherits the group's structure; likewise, if Y is a subset of a topological space X under topology T, then

$$S = \{U \cap Y | U \in T\}$$

is the *subspace topology* on Y, and Y is a *subspace* of the topological space X.

Note that the identities $\emptyset \cap Y = \emptyset \in S$, $X \cap Y = Y \in S$,

$$\bigcap_{i=1}^{n}(U_i \cap Y) = \left(\bigcap_{i=1}^{n} U_i\right) \cap Y,$$

and

$$\bigcup_{\alpha}(U_\alpha \cap Y) = \left(\bigcup_{\alpha} U_\alpha\right) \cap Y$$

show that S satisfies the requirements of a topology on Y.

As taking the Cartesian product of two groups yields a group, taking the Cartesian product of two topological spaces X and Y generates a topology on the Cartesian product $X \times Y$ called the *product topology*, making it a topological space as well. The product topology itself is defined from its basis B, which consists of all sets $U \times V$ where U and V are open subsets of X and Y, respectively. This collection satisfies the requirements of a basis because $X \times Y$ is a basis element, and for two basis elements (U_1, V_1) and (U_2, V_2) their intersection is also a basis element,

$$(U_1, V_1) \cap (U_2, V_2) = ((U_1 \cap U_2), (V_1 \cap V_2)),$$

the product of open sets $U_1 \cap U_2$ and $V_1 \cap V_2$.

3.1.1 Continuity

A homomorphism is a map between two groups that in a sense preserves the group structure; roughly speaking, two points in the image interact the in same way that their preimages interact. The topological counterpart is a map between two topological spaces such that points in the preimage form an open set if the images of those points form an open set.

Formally, if $f : X \to Y$ is a map between topological spaces, it is *continuous* if for any open set $V \subseteq Y$, its inverse image $f^{-1}(V)$ is open in X.

A bijection between topological spaces $f : X \to Y$ is a *homeomorphism* if it and its inverse $f^{-1} : Y \to X$ are continuous; as group isomorphisms relate equivalent groups, homeomorphisms relate equivalent topological spaces.

The topological analog of a quotient group requires more preparation than the previous concepts. It first depends on the idea of a *quotient map*

$$p : X \to Y,$$

which is a surjective map where a subset V of Y is open in Y if and only if $p^{-1}(V)$ is open in X; it follows immediately that p is continuous.

Note that a quotient map does not guarantee that every open subset of X has an open image in Y; if $U \subset X$ is open, it might have a nonopen image $p(U)$, but $p^{-1}(p(U))$ is a nonopen strict superset of U. For example, consider the topological space $Z = \{0,1\}$ with open subsets $\emptyset, \{1\}, Z$. If $p : \mathbb{R} \to Z$ is defined by

$$x \leq 1 \mapsto 0$$

$$x > 1 \mapsto 1,$$

then p is a quotient map, because the open sets in Z all have open preimages in $\mathbb{R}$; however, the image of the open set $(0,1) \subset \mathbb{R}$ is $\{0\}$, which is not open in Z.

However, if p is continuous and $p(U)$ is open for every open $U \subseteq X$, it follows that p is a quotient map; any open set V in Y has an open preimage $p^{-1}(V)$ by continuity, and any subset W of Y with an open preimage $P^{-1}(W)$ must be open, because p maps open sets to open sets.

If $p : X \to A$ surjectively maps a topological space X onto an arbitrary set A, then the unique topology on A relative to which p is a quotient map is called the *quotient topology*. Uniqueness follows from the definition of a quotient map, because open sets in A must be precisely those sets $W \subseteq A$ for which $p^{-1}(W)$ is open in X. The properties of a topology are fulfilled because $p^{-1}(A) = X$, $p^{-1}(\emptyset) = \emptyset$,

$$p^{-1}\left(\bigcup_{\alpha} W_\alpha\right) = \bigcup_{\alpha} p^{-1}(W_\alpha),$$

and

$$p^{-1}\left(\bigcap_{i=1}^{n} W_i\right) = \bigcap_{i=1}^{n} p^{-1}(W_i).$$

If a topological space X is partitioned into equivalence classes, X^* is the set of those equivalence classes, and $p : X \to X^*$ is the natural projection mapping each point $x \in X$ to its containing element in the equivalence set X^*, then X^* is a *quotient space* of a topological space X under the quotient topology induced by the projection p.

Specifically, suppose G is a group and a topological space, and $N \trianglelefteq G$ is a normal subgroup of G. Then the cosets of N partition G into the quotient group G/N, and under the topology induced by the natural projection $g \mapsto gN$, the quotient G/N becomes a topological quotient space of G.

3.1.2 Manifolds and Smoothness

An *n-manifold* is a topological space M such that for every $m \in M$ there is an open set U_m containing m that is homeomorphic to an open set $V_m \subseteq \mathbb{R}^n$, given by a homeomorphism ψ_m (called a *chart*).

If there is some collection of charts on a manifold M that covers all of M and has the property that for any two charts ψ_l and ψ_m, the composition map

$$\psi_{lm} = \psi_m \circ \psi_l^{-1}$$

mapping $\mathbb{R}^n \to \mathbb{R}^n$ on $\psi_l(U_l \cap U_m)$ is infinitely differentiable (*smooth*), then M is a *smooth manifold*, and the collection of charts is called an *atlas*.

Supposing $f : X \to Y$ is a map from a smooth m-manifold to a smooth n-manifold, f is a *smooth map* if, for any charts ψ_x about some $x \in X$ and ψ_y about $f(x) = y \in Y$, the composition

$$\psi_y \circ f \circ \psi_x^{-1} : \mathbb{R}^m \to \mathbb{R}^n$$

defined on $\psi_x(U_x \cap f^{-1}(U_y))$ is smooth.

Every point x of a smooth n-manifold M has a *tangent space* T_xM that is a real n-dimensional vector space. If M is a subset of some Euclidean space $\mathbb{R}^k$, then the tangent space at x is the space of velocity vectors of all curves on M that pass through x. Alternatively, the tangent space can be defined as a space of triples,

$$(x, \psi, \xi),$$

where ψ is a chart for a set containing x, and ξ is a vector in $\mathbb{R}^n$ we choose to represent a vector in T_xM. If ϕ is another chart whose set contains x, then the triple (x, ϕ, η) is equivalent to (x, ψ, ξ) if and only if in some neighborhood of $y = \psi(x)$,

$$\eta = (D\theta(y))\,(\xi)$$

where $\theta = \phi \circ \psi^{-1}$.

3.2 Definitions

With the necessary background described, the definition of a *Lie group* is straightforward enough; it is a smooth manifold G where

1. The group operation $G \times G \to G$ mapping $(g, h) \mapsto gh$, and
2. The taking of inverses $G \to G$ mapping $g \mapsto g^{-1}$

are smooth.

Examples of Lie groups include

1. $\mathbb{R}^n$, with the standard Euclidean topology and pointwise addition;
2. $\mathbb{T} = S^1$, the circle group viewed as a 1-manifold with the group operation given by the multiplication of associated complex exponentials;
3. $GL_n\mathbb{R}$ (or $\mathbb{C}$), the set of invertible $n \times n$ matrices, by viewing the matrices as vectors in $\mathbb{R}^{n^2}$ (or $\mathbb{C}^{n^2}$) and the usual matrix multiplication;
4. U_n, the set of $n \times n$ unitary matrices, and its subgroup SU_n of unitary matrices of determinant 1, as a subgroup and subspace of $GL_n\mathbb{R}$; and
5. O_n, the set of $n \times n$ orthogonal matrices, and its subgroup SO_n of orthogonal matrices of determinant 1, also a subgroup of $GL_n\mathbb{R}$.

A *Lie algebra* is a vector space $\mathfrak{g}$ and a map (called the *Lie bracket*),

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

that is skew-symmetric, bilinear, and satisfies the Jacobi identity,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

The tangent space at the identity T_1G of a Lie group G naturally has the structure of a Lie algebra through a construction given in Fulton and Harris (2004: 8.1), with the Lie bracket $[X, Y]$ given by the function

$$\text{ad} = d(d(\Psi_g))_1(X)(Y),$$

the second differential of Ψ_g (or the derivative of $\text{Ad}(g) = d\Phi_g$, the adjoint representation of G), where $\Psi_g(h) : h \mapsto ghg^{-1}$ is an automorphism of G. Then ad is a map

$$T_1G \to \text{End}(T_1G)$$

from the tangent space to the group of linear transformations of the tangent space. By applying ad to a vector $X \in T_1G$ and then applying the resulting transformation to another vector $Y \in T_1G$, we construct a function of two variables

$$T_1G \times T_1G \to T_1G.$$

While the proof that this construction satisfies the requirements of the Lie bracket is not obvious in the general case, it is enlightening to examine the construction for the Lie group $GL_n\mathbb{R}$ of invertible matrices (also given in (Fulton and Harris, 2004: 8.1)). For $G = GL_n\mathbb{R}$, $T_1G = \mathrm{End}(\mathbb{R}^n) = M_n\mathbb{R}$ (with the ordinary differentiation of matrices) and the action by conjugation given by Ψ_g extends to the tangent space as

$$\mathrm{Ad}(g)(M) = gMg^{-1}$$

for $M \in M_n\mathbb{R}$. Then for any tangent vectors X and Y to $GL_n\mathbb{R}$ at the identity, define a function $\gamma : \mathbb{R} \to G$ to be a curve with $\gamma(0) = 1_G$ and $\gamma'(0) = X$. It follows from the construction of the bracket that

$$[X, Y] = \mathrm{ad}(X)(Y) = \left.\frac{d}{dt}\right|_{t=0} (\mathrm{Ad}(\gamma(t))(Y))$$

and application of the product rule to $(\mathrm{Ad}(\gamma(t))(Y)) = \gamma(t)Y\gamma(t)^{-1}$ yields

$$\begin{aligned}[X, Y] &= \gamma'(0)Y\gamma(0) + \gamma(0)Y(-\gamma(0)^{-1}\gamma'(0)\gamma(0)^{-1}) \\ &= X \cdot Y \cdot 1 + 1 \cdot Y(-1 \cdot X \cdot 1) \\ &= X \cdot Y - Y \cdot X. \end{aligned} \tag{3.2.1}$$

From its construction, the bracket is bilinear (since it is defined in terms of linear operations). From Equation 3.2.1, transposing X and Y shows skew-symmetry as follows:

$$[Y, X] = Y \cdot X - X \cdot Y = -(X \cdot Y - Y \cdot X) = -[X, Y]. \tag{3.2.2}$$

Using Equation 3.2.2, it is also straightforward to show the Jacobi identity holds, as

$$\begin{aligned}&[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] \\ &\qquad = [X, YZ - ZY] + [Y, ZX - XZ] + [Z, XY - YX]\end{aligned}$$

which reduces to

$$\begin{aligned}&XYZ - XZY - YZX + ZYX + YZX - YXZ \\ &\qquad - ZXY + XZY + ZXY - ZYX - XYZ + YXZ = 0.\end{aligned}$$

3.3 Lie Group Actions

Consider the groups SU_2 and SO_3. We wish to show that there is a double covering of SO_3 by SU_2 by considering their actions on the sphere S^2.

In Carter and colleagues (1995: Book 2, Chapter 2), Segal gives the following proof. First, note that there is a diffeomorphism between the sphere S^2 and $\mathbb{C} \cup \{\infty\}$ given by the stereographic projection

$$(x, y, z) \in S^2 \leftrightarrow \frac{x + iy}{1 - z} \in \mathbb{C} \cup \{\infty\}.$$

Let each element

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU_2$$

act on $\mathbb{C} \cup \{\infty\}$ by

$$c \mapsto \frac{ac + b}{-\bar{b}c + \bar{a}},$$

which, when viewed as an action on S^2, is a rotation of the sphere. To complete the double covering, we project again from the point $(0, 0, -1)$ to $\frac{x+iy}{1+z}$ and continue as before.

This action is more easily understood as an action on the complex projective space $\mathbb{CP}^1$ as follows: Define three maps,

$$\pi : w \in \mathbb{C} \cup \{\infty\} \mapsto \begin{cases} \begin{pmatrix} w \\ 1 \end{pmatrix} \in \mathbb{CP}^1, & w < \infty \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{CP}^1, & w = \infty \end{cases},$$

$$\psi_g : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{CP}^1 \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ -\bar{b}x + \bar{a}y \end{pmatrix} \in \mathbb{CP}^1,$$

and

$$\varphi : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{CP}^1 \mapsto \frac{x}{y} \in \mathbb{C} \cup \{\infty\}$$

for some $g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU_2$. ψ_g is an invertible homomorphism, so its image remains in $\mathbb{CP}^1$.

It is worth examining the edge cases where $w_\infty = \infty$ and $w_g = \frac{\bar{a}}{\bar{b}}$, in order to check that the maps behave nicely when dealing with infinity. In the former case, where $w = \infty$,

$$\pi(w_\infty) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

the limit of $\begin{pmatrix} k \\ 1 \end{pmatrix}$ as $k \to \infty$; thus

$$(\psi_g \circ \pi)(w_\infty) = \begin{pmatrix} a \\ -\overline{b} \end{pmatrix} = \begin{pmatrix} -a \\ \overline{b} \end{pmatrix}$$

and

$$(\varphi \circ \psi_g \circ \pi)(w_\infty) = \frac{-a}{\overline{b}}.$$

In the latter case, $w_g = \frac{\overline{a}}{\overline{b}}$ and

$$\pi(w_g) = \begin{pmatrix} \overline{a} \\ \overline{b} \end{pmatrix},$$

$$(\psi_g \circ \pi)(w_g) = \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

giving

$$(\varphi \circ \psi_g \circ \pi)(w_g) = \frac{1}{0} = \infty.$$

While these two cases involve infinities, note that all three maps are well-defined. Also note that the composition of the three maps is the transformation above, denoting the action of SU_2 on S^2.

We see that $SL_2\mathbb{R}$ acts transitively on the complex upper half-plane by a similar action given in Carter and colleagues (1995: Book 2, Chapter 3),

$$z \mapsto \frac{az + b}{cz + d}$$

for $ad - bc = 1$. For some arbitrary $z = x + iy$ in the upper half-plane we can reach i by the action of

$$A = \begin{pmatrix} \frac{1}{\sqrt{y}} & -\frac{x}{\sqrt{y}} \\ 0 & \sqrt{y} \end{pmatrix} \in SL_2\mathbb{R},$$

and, from i, applying the matrix

$$B = \begin{pmatrix} \sqrt{t} & s \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix} \in SL_2\mathbb{R}$$

to obtain some other point $s + it$ in the upper half-plane. Composing the two transformations gives the desired action from one point to another, and remains in $SL_2\mathbb{R}$. Note that these transformations are real if and only if y and t are nonnegative, but for any x and s, or precisely on the upper half-plane.

Another example from the same chapter in Carter and colleagues (1995) asks us to construct a bijection between S^{n-1} and O_n/O_{n-1}. Consider the set of orthonormal transformations in O_n that fix a specific vector x_0. Without loss of generality, we can choose to fix the first basis vector e_0, because we can change the basis of $\mathbb{R}^n$ to make x_0 the first basis vector. Then the other $n-1$ vectors are transformed orthonormally, giving a matrix of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix}$$

for some $A \in O_{n-1}$, and thus this space is isomorphic to O_{n-1}. Then the quotient O_n/O_{n-1} leaves all those matrices that shift the vector x_0, and matrices that map x_0 to the same vector y_0 are equivalent. Because the matrix is orthonormal, then $\|y_0\| = 1$ but has no other restrictions; therefore, the quotient is the set of all vectors in $\mathbb{R}^n$ with length 1, or exactly S^{n-1}.

3.4 Representation Theory of Lie Groups

Recall that for a finite group G, a *representation* of G is a homomorphism of groups

$$\rho : G \to GL_n(\mathbb{F})$$

from G into the group of $n \times n$ invertible matrices over a field $\mathbb{F}$. Similarly, a representation of a Lie group G is a homomorphism from G into the group of invertible linear transformations of a topological vector space (simply a vector space with an appropriate topology such that vector addition and scalar multiplication are continuous) V over $\mathbb{F}$ with the added condition that the map

$$(g, v) \mapsto \rho(g)v : G \times V \to V$$

is continuous. As in the finite case, a representation makes V into an $\mathbb{F}G$-module.

For finite groups, we can completely decompose any representation of G over a complex vector space (a homomorphism from G to the group of complex invertible matrices) into a direct sum of its irreducible representations. A counterpart to this statement for representations of the circle group $\mathbb{T}$ decomposes functions on the circle into a sum of complex exponentials.

Theorem 3.2 (Fourier)**.** *A smooth function $f : \mathbb{T} \to \mathbb{C}$ can be expanded into a Fourier series*

$$f(\theta) = \sum_{n \in \mathbb{Z}} a_n e^{in\theta},$$

where

$$a_n = \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta) e^{-in\theta} d\theta.$$

If V is a representation of $\mathbb{T}$, then any $v \in V$ can be expanded as

$$v = \sum_{n \in \mathbb{Z}} v_n,$$

where

$$v_n = \frac{1}{2\pi} \int_{\mathbb{T}} (R_\theta(v)) e^{in\theta} d\theta$$

and

$$R_\theta(v_n) = e^{-in\theta} v_n.$$

Proof. (Given in Carter and collegaues (1995: Book 2, Equation 6.2)) The result follows from the fact that as $N \to \infty$, the function

$$s_N(\theta) = \frac{1}{2\pi} \sum_{|n| \leq N} e^{in\theta}$$

tends to the delta function $\delta(\theta)$. □

Though the Peter-Weyl Theorem allows us to decompose a representation V of a compact Lie group G into isotypic submodules (an isotypic submodule is the direct sum of isomorphic irreducible submodules), in general, it is not possible to decompose an arbitrary Lie group (Carter et al., 1995).

Chapter 4

Future Work

Despite our discouraging preliminary investigation, it may be possible to find a simple and statistically meaningful interpretation of a generator for a Parseval group frame that spans the space of j-subset effects within $\binom{n}{k}$ partially ranked data. It may also be possible to construct summary statistics for partitions of n candidates into multiple subsets.

In addition, we hope to generalize some of our methods for dealing with finite groups acting on finite-dimensional Hilbert spaces to arbitrary Lie groups acting on potentially arbitrary Hilbert spaces. Imagine, for example, a function that describes the temperature at every point in some Hilbert space. Suppose we wanted to calculate some summary statistic regarding the temperature within this space (average temperature, say). It would be convenient if, as in the finite case, there were some group frame we could leverage in the computation of that statistic, but it is possible that we would need to use a non-finite Lie group to act on the space instead of a finite group, in order generate a large enough frame to span the entire space.

Perhaps the most important step in this generalization is generalizing Theorem 1.5 to handle infinite groups. If the Lie group has finite dimension, this equation might remain unchanged, but it will not necessarily hold in the case of an infinite Lie group.

In addition, it could be interesting to consider finite group frames as samples of an infinite group's action on a space, and explore possible benefits of that approach to generalizing finite groups into infinite groups. If a Lie group acting on a Hilbert space had some finite subgroup, it might be computationally or analytically easier to examine a frame for the space generated by that smaller subgroup; we would like to know what information would be lost in that restriction, and what information would be preserved.

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