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Abstract: We make use of linear operators to derive the formulae for the general solution of elementary linear scalar ordinary differential equations of order n. The key lies in the factorization of the linear operators in terms of first-order operators. These first-order operators are then integrated by applying their corresponding integral operators. This leads to the solution formulae for both homogeneous- and nonhomogeneous linear differential equations in a natural way without the need for any ansatz (or "educated guess"). For second-order linear equations with nonconstant coefficients, the condition of the factorization is given in terms of Riccati equations.

1 Introduction

In this paper we introduce linear differential operators and show that a linear differential equation can be solved by factoring these operators in terms of linear first-order operators. The factorization and integration of these operators then leads to a direct method and solution formulae to integrate any linear differential equation with constant coefficients without the need of an ansatz for its solutions. This method provides an alternative to the *method of variation of parameters* or the *method of judicious guessing* which one finds in most standard textbooks on elementary differential equations, e.g [1]. A more detailed discussion of the method of linear operators for differential equations is given in [2].

2 Definitions

In this section we introduce linear operators and introduce a integral operator that corresponds to a general first-order linear differential operator. This integral operator is the key to the integration of the linear equations.

Let $C(I)$ denote the vector space of all continuous functions on some domain $I \subseteq \mathbb{R}$ and let $C^n(I)$ denote that vector subspace of $C(I)$ that consist of all functions with continuous nth order derivatives on I.

Definition 2.1. *We define the linear mapping*

$$L : C^n(I) \to C(I)$$

for all $f(x) \in C^n(I)$ on the interval $I \subseteq \mathbb{R}$ as

$$L : f(x) \mapsto Lf(x), \tag{2.1}$$

where L is the linear differential operator of order n

$$L := p_n(x)D_x^n + p_{n-1}(x)D_x^{(n-1)} + \cdots + p_1(x)D_x + p_0(x). \tag{2.2}$$

Here $n \in \mathbb{N}$ and $p_j(x)$ $(j = 0, 1, \ldots, n)$ are differentiable functions with

$$D_x^{(k)} := \frac{d^k}{dx^k}, \qquad D_x \equiv D_x^{(1)}$$

so that

$$Lf(x) = p_n(x)f^{(n)}(x) + p_{n-1}(x)f^{(n-1)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x).$$

Example 2.1. Consider the second-order linear operator

$$L = \cos x D_x^2 + e^x D_x + x^2.$$

As an example, let L act on both e^{2x} and on $u(x)\,e^{-x}$:

$$\begin{aligned} L\,e^{2x} &= 4e^{2x}\cos x + 2e^{3x} + x^2e^{2x} \qquad \text{and} \\ L\big(u(x)\,e^{-x}\big) &= \big(e^{-x}\cos x\big)\,u'' + \big(1 - 2e^{-x}\cos x\big)\,u' + \big(e^{-x}\cos x + x^2e^{-x} - 1\big)\,u \\ &\equiv \tilde{L}\,u(x), \end{aligned}$$

where $\tilde{L}$ is another linear operator given by

$$\tilde{L} := e^{-x}\cos x D_x^2 + \big(1 - 2e^{-x}\cos x\big)\,D_x + e^{-x}\cos x + x^2e^{-x} - 1.$$

Next we define the composite linear operator and the integral operator D_x^{-1}.

Definition 2.2.

a) *Consider two linear differential operators of the form (2.2), namely* L_1 *of order* m *and* L_2 *of order* n*, and consider a function* $f(x) \in C^{m+n}(I)$ *with* $I \subseteq \mathbb{R}$*. The* **composite operator** $L_1 \circ L_2$ *is defined by*

$$L_1 \circ L_2\, f(x) := L_1\left(L_2 f(x)\right) \tag{2.3}$$

where $L_1 \circ L_2$ *is a linear differential operator of order* $m + n$*.*

b) *The* **integral operator**, D_x^{-1}*, is defined by the linear mapping*

$$D_x^{-1}:\ C(I) \to C^1(I)$$

for all $f(x) \in C(O)$ *on the interval* $I \subseteq \mathbb{R}$ *as*

$$D_x^{-1} : f(x) \mapsto D_x^{-1} f(x) \tag{2.4}$$

where

$$D_x^{-1} f(x) := \int f(x) dx \tag{2.5}$$

Note that, in general, $L_1 \circ L_2\ f(x) \neq L_2 \circ L_1 f(x)$.

Example 2.2. We consider the two linear differential operators

$$L_1 = D_x + x^2, \qquad L_2 = x D_x^2 + 1.$$

Then

$$\begin{aligned} L_2 \circ L_1\, f(x) &= \left(x D_x^2 + 1\right)\left(f' + x^2 f\right) \\ &= x f^{(3)} + x^3 f'' + (4x^2 + 1) f' + (x^2 + 2x) f \equiv L_3\, f(x), \end{aligned}$$

where $L_3 := x D_x^3 + x^3 D_x^2 + (4x^2 + 1) D_x + x^2 + 2x$. Furthermore

$$L_1 \circ L_2\, f(x) = x f^{(3)} + (x^3 + 1) f'' + f' + x^2 f \equiv L_4\, f(x)$$

where $L_4 := x D_x^3 + (x^3 + 1) D_x^2 + D_x + x^2$. Clearly $L_2 \circ L_1\, f(x) \neq L_1 \circ L_2\, f(x)$.

Let f be a differentiable function. Then, following Definition 2.2 b), we have

$$D_x^{-1} \circ D_x f(x) = D_x^{-1} f'(x) = f(x) + c, \tag{2.6}$$

where c is an arbitrary constant of integration and, furthermore,

$$D_x \circ D_x^{-1} f(x) = \frac{d}{dx}\left(\int f(x)\,dx + c\right) = f(x). \tag{2.7}$$

In the next section we shall introduce a first-order linear operator and a corresponding integral operator which is essential for our application.

In terms of the linear operator (2.2), the nth-order linear nonhomogeneous differential equation,

$$p_n(x)u^{(n)} + p_{n-1}(x)u^{(n-1)} + \cdots + p_1(x)u' + p_0(x)u = f(x) \tag{2.8}$$

takes the form $L\,u(x) = f(x)$, where L is the linear operator (2.2).

In the next section we introduce a method to solve $L\,u = f$ by factoring L into first-order linear operators and then use the corresponding integral operators to eliminate all derivatives. For this purpose the following integral operator plays a central role.

Definition 2.3. *Let a and b be continuous real-valued functions on some interval $I \subseteq \mathbb{R}$, such that $a(x) \neq 0$ for all $x \in I$ and consider the first-order linear operator*

$$L = a(x)D_x + b(x). \tag{2.9}$$

We define $\hat{L}$ as the **integral operator corresponding to** *L, as follows:*

$$\hat{L} := e^{-\xi(x)} D_x^{-1} \circ \frac{1}{a(x)}\, e^{\xi(x)}, \tag{2.10}$$

where

$$\frac{d\xi(x)}{dx} = \frac{a(x)}{b(x)}. \tag{2.11}$$

By Definition 2.3, this proposition follows.

Proposition 2.1. *For any continuous function, $f(x)$, we have*

$$\hat{L} f(x) = e^{-\xi(x)}\left[\int \frac{f(x)}{a(x)}\, e^{\xi(x)}\,dx + c\right] \tag{2.12}$$

and

$$\hat{L} \circ L\, f(x) = f(x) + e^{-\xi(x)}\,c, \tag{2.13}$$

where c is an arbitrary constant of integration.

We show that relation (2.13) holds:

$$\begin{aligned}\hat{L}\circ L\,f(x) &= \hat{L}\,(a(x)f'(x)+b(x)f(x))\\ &= e^{-\xi(x)}\left(\int \frac{a(x)f'(x)+b(x)f(x)}{a(x)}\,e^{\xi(x)}\,dx + c\right)\\ &= e^{-\xi(x)}\left(\int f'(x)e^{\xi(x)}\,dx + \int \frac{b(x)}{a(x)}\,f(x)\,e^{\xi(x)}\,dx + c\right).\end{aligned}$$

Note that $f(x)'e^{\xi(x)} + f(x)\left(e^{\xi(x)}\right)' = \left(f(x)e^{\xi(x)}\right)'$, so that (2.13) follows.

Example 2.3. We consider $L = xD_x + x^2$ and $f(x) = e^{-x^2/2}$. Then the corresponding integral operator is $\hat{L} = e^{-x^2/2}D_x^{-1}\circ e^{x^2/2}/x$ so that

$$\hat{L}\,e^{x^2/2} = e^{-x^2/2}\,(\ln|x|+c) \quad \text{and} \quad \hat{L}\circ L\,e^{-x^2/2} = e^{-x^2/2}(1+c).$$

3 Application to linear differential equations

3.1 First-order linear equations

Consider the first-order linear differential equation in the form

$$u' + g(x)u = h(x). \tag{3.1}$$

In terms of a linear differential operator (2.2) we can write (3.1) in the form

$$Lu(x) = h(x), \tag{3.2}$$

where L is the first-order linear operator

$$L = D_x + g(x). \tag{3.3}$$

Following Definition 2.3 we now apply the corresponding integral operator $\hat{L}$ to both sides of (3.2) to gain the general solution of (3.1). We illustrate this explicitly in the next example.

Example 3.1. We find the general solution of

$$u' + g(x)u = h(x).$$

Following Definition 2.3 we apply $\hat{L}$, given by

$$\hat{L} = e^{-\xi(x)}D_x^{-1}\circ e^{\xi(x)} \quad \text{with} \quad \xi(x) = \int g(x)dx, \tag{3.4}$$

to the left-hand side and the right-hand side of (3.2). For the left-hand side we obtain

$$\hat{L}\circ Lu(x) = u(x) + c_1 e^{-\xi(x)}, \quad (c_1 \text{ is an arbitrary constant})$$

and for the right-hand side

$$\hat{L}h(x) = e^{-\xi(x)} \int \left[h(x)e^{\xi(x)}dx + c_2\right] \qquad (c_2 \text{ is an arbitrary constant}).$$

Thus

$$u(x) + c_1 e^{-\xi(x)} = e^{-\xi(x)} \int \left[h(x)e^{\xi(x)}dx + c_2\right],$$

or, equivalently

$$u(x) = e^{-\xi(x)} \left[\int h(x)e^{\xi(x)}dx + c\right],$$

where $c = c_2 - c_1$ is an arbitrary constant and $\xi(x) = \int g(x)dx$.

3.2 Linear differential equations with constant coefficients

In order to solve higher-order linear differential equations, we need to factorize the differential equatons into first-order factors. This is in principle always possible, but in practise there are some obstacles.

We first look at the constant-coefficient case and then the more general case which allows nonconstant coefficient functions in the linear operator.

It should be clear that the linear constant-coefficient homogeneous equation factors in terms of first-order differential operators in the same manner as the corresponding characteristic equation does. Take, for example, the second-order equation

$$u'' + pu' + qu = f(x), \qquad p \in \mathbb{R},\ q \in \mathbb{R}, \tag{3.5}$$

with characteristic equation (following the substitution $u = e^{zx}$)

$$z^2 + pz + q = 0 \tag{3.6}$$

and roots

$$z_1 = \frac{1}{2}\left(-p + \sqrt{p^2 - 4q}\right), \qquad z_2 = \frac{1}{2}\left(-p - \sqrt{p^2 - 4q}\right). \tag{3.7}$$

Equation (3.5) then takes the form $L\,u(x) = f(x)$, where

$$L = D_x^2 + pD_x + q. \tag{3.8}$$

It is now easy to show that the second-order operator (3.8) factors as

$$L = (D_x - z_1) \circ (D_x - z_2) \equiv L_1 \circ L_2, \tag{3.9}$$

where $L_1 = D_x - z_1$ and $L_2 = D_x - z_2$. Since

$$\begin{aligned} L_1 \circ L_2\, u(x) &= (D_x - z_1)(u' - z_2 u) \\ &= u'' - (z_1 + z_2)u' + z_1 z_2 u \end{aligned}$$

and, by (3.7), $z_1 + z_2 = -p$ and $z_1 z_2 = q$. Thus

$$\begin{aligned} L\,u(x) &= L_1 \circ L_2\, u(x) \\ &= u'' + pu' + qu = f(x). \end{aligned}$$

This easily extends to linear constant-coefficient equations of order n.

Proposition 3.1. *Consider a constant coefficient nth-order nonhomogeneous differential equation of the form*

$$L\,u(x) = f(x), \tag{3.10}$$

where

$$\begin{aligned} L = a_n D_x^n + a_{n-1} D_x^{n-1} + \cdots + a_1 D_x + a_0 \\ a_j \in \mathbb{R}, \quad j = 0, 1, \ldots, n. \end{aligned} \tag{3.11}$$

The characteristic equation of (3.10), following the substitution $u(x) = e^{zx}$, is the nth degree polynomial

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0 \tag{3.12}$$

with n roots, $\{z_1, z_2, \ldots, z_n\}$, so that (3.12) can be factorized as

$$(z - z_1)(z - z_2) \cdots (z - z_n) = 0.$$

Then equation (3.10) factors in the form

$$L_1 \circ L_2 \circ \cdots \circ L_n\, u(x) = f(x), \tag{3.13}$$

where

$$L_j = D_x - z_j, \qquad j = 1, 2, \ldots, n. \tag{3.14}$$

The general solution of (3.10) then follows by applying, successively, the corresponding integral operators $\hat{L}_1, \hat{L}_2, \ldots, \hat{L}_n$ to (3.13).

Applying Proposition 3.1 to second-order equations leads to Proposition 3.2.

Proposition 3.2. *Consider the 2nd-order equation*

$$u'' + pu' + qu = f(x), \qquad p,\ q \in \mathbb{R} \tag{3.15}$$

with characteristic equation

$$z^2 + pz + q = 0. \tag{3.16}$$

a) *If the two roots, z_1 and z_2, of (3.16) are real and distinct, then the general solution of (3.15) is*

$$u(x) = c_1 e^{z_1 x} + c_2 e^{z_2 x} + u_p(x), \tag{3.17}$$

where $u_p(x)$ is a particular solution of (3.15) given by

$$u_p(x) = \left(\frac{1}{z_1 - z_2}\right)\left(e^{z_1 x}\int f(x)e^{-z_1 x}dx - e^{z_2 x}\int f(x)e^{-z_2 x}dx\right) \tag{3.18}$$

and c_1 and c_2 are arbitrary constants.

b) *If the two roots, z_1 and z_2, of (3.16) are complex numbers, then $z_2 = \bar{z}_1$ (z_2 is the complex conjugate of z_1) and the general solution of (3.15) is*

$$u(x) = c_1\, Re\{e^{z_1 x}\} + c_2\, Im\{e^{z_1 x}\} + u_p(x), \tag{3.19}$$

where $u_p(x)$ is a particular solution of (3.15) given by

$$u_p(x) = \left(\frac{1}{z_1 - \bar{z}_1}\right)\left(e^{z_1 x}\int f(x)e^{-z_1 x}dx - e^{\bar{z}_1 x}\int f(x)e^{-\bar{z}_1 x}dx\right) \tag{3.20}$$

and c_1 and c_2 are arbitrary constants. Note that although z_1 and $\bar{z}_1$ are complex numbers, the solution $u(x)$ is always a real-valued function.

c) *If the two roots for (3.16) are equal, i.e. $z_1 = z_2 \in \mathbb{R}$, then the general solution of (3.15) is*

$$u(x) = c_1 e^{z_1 x} + c_2\, x\, e^{z_1 x} + u_p(x) \tag{3.21}$$

where c_1 and c_2 are arbitrary constants and a particular solution, $u_p(x)$, of (3.15) is

$$u_p(x) = e^{z_1 x}\int\left(\int f(x)e^{-z_1 x}dx\right)dx. \tag{3.22}$$

Proof: Equation (3.15) can be written in the form

$$L_1 \circ L_2\, u(x) = f(x), \quad L_1 = D_x - z_1, \quad L_2 = D_x - z_2, \tag{3.23}$$

where z_1 and z_2 are the roots of the characteristic equation (3.16). The corresponding integral operators for L_1 and L_2 are

$$\hat{L}_1 = e^{z_1 x} D_x^{-1} \circ e^{-z_1 x}, \quad \hat{L}_2 = e^{z_2 x} D_x^{-1} \circ e^{-z_2 x}, \tag{3.24}$$

respectively. From (3.23) we obtain

$$\begin{aligned} L_2 u(x) + e^{z_1 x} k_1 &= e^{z_1 x} \int f(x) e^{-z_1 x}\, dx + k_2 e^{z_1 x} \\ \text{or} \quad L_2 u(x) &= e^{z_1 x} \int f(x) e^{-z_1 x}\, dx + k_3 e^{z_1 x}, \end{aligned} \tag{3.25}$$

where $k_3 \equiv k_2 - k_1$ is another constant of integration. Applying now $\hat{L}_2$ to each side of (3.25) we have

$$\begin{aligned} u(x) + e^{z_2 x} k_4 &= e^{z_2 x} \int \left(k_3 e^{(z_1 - z_2)x} + e^{(z_1 - z_2)x} F(x) \right) dx + k_5 e^{z_2 x} \\ \text{or} \quad u(x) &= e^{z_2 x} \int \left(k_3 e^{(z_1 - z_2)x} + e^{(z_1 - z_2)x} F(x) \right) dx + k_6 e^{z_2 x}, \end{aligned} \tag{3.26}$$

where $k_6 \equiv k_5 - k_4$ is a constant of integration and

$$F(x) := \int f(x) e^{-z_1 x}\, dx. \tag{3.27}$$

If $z_1 \neq z_2$, then (3.26) reduces (after integration by parts) to (3.17) for $z_1 \in \mathbb{R}$ and $z_2 \in \mathbb{R}$ and to (3.19) for complex roots z_1 and $z_2 = \bar{z}_1$, or to (3.21) for equal roots $z_1 = z_2 \in \mathbb{R}$. Note that $c_1 \equiv k_3$ and $c_2 \equiv k_6$ are the two arbitrary constants for the general solution. □

Example 3.2. We find the general solution of

$$u'' + 4u = 8x^2. \tag{3.28}$$

The characteristic and its roots are $z^2 + 4 = 0,\ z_1 = 2i,\ z_2 = -2i$ so the equation can be presented in factorized form

$$L_1 \circ L_2\, u(x) = 8x^2, \quad \text{where} \quad L_1 = D_x - 2i, \quad L_2 = D_x + 2i.$$

Following Proposition (3.2), the general solution, $\phi_h(x)$, of the homogeneous part of (3.28) is

$$\phi_h(x; c_1, c_2) = c_1 \cos(2x) + c_2 \sin(2x). \tag{3.29}$$

For a particular solution, $u_p(x)$, we use formula (3.20) and calculate the integrals:

$$u_p(x) = \frac{1}{4i} \left(e^{2ix} \int 8x^2 e^{-2ix}\, dx - e^{-2ix} \int 8x^2 e^{2ix}\, dx \right) = 2x^2 - 1.$$

The general solution of (3.28) is thus $u(x) = c_1 \cos(2x) + c_2 \sin(2x) + 2x^2 - 1$, where c_1 and c_2 are arbitrary constants.

For an nth-order linear constant-coefficient equations we can obtain this result.

Proposition 3.3. *Consider a constant coefficient nth-order nonhomogeneous differential equation with $n > 2$ of the form*

$$L\,u(x) = f(x), \tag{3.30}$$

where

$$L = a_n D_x^n + a_{n-1} D_x^{n-1} + \cdots + a_1 D_x + a_0 \tag{3.31}$$
$$a_j \in \mathbb{R}, \quad j = 0, 1, \ldots, n.$$

Equation (3.30) factors in the form

$$L_1 \circ L_2 \circ \cdots \circ L_n\, u(x) = f(x), \tag{3.32}$$

where $L_j = D_x - z_j$ ($j = 1, 2, \ldots, n$) and $\{z_1,\ z_2,\ \ldots, z_n\}$ are the roots of its characteristic equation. The general solution of (3.30) is then

$$u(x) = \phi_h(x; c_1, c_2, \ldots, c_n) + u_p(x), \tag{3.33}$$

where $\phi_h(x)$ is the general solution of the homogeneous part of (3.30), namely

$$\phi_h(x; c_1, c_2, \ldots, c_n) = c_1 e^{z_n x}\left(\int e^{(z_{n-1}-z_n))x} G_{12}^{\{n-2\}}(x)\,dx\right) + c_2 e^{z_n x} G_{23}^{\{n-2\}}(x)$$
$$+ c_3 e^{z_n x} G_{34}^{\{n-3\}}(x) + \cdots + c_{n-1} e^{z_n x} G_{(n-1)n}^{\{1\}}(x) + c_n e^{z_n x} \tag{3.34}$$

($c_1,\ c_2,\ \ldots,\ c_n$ are arbitrary constants) and $u_p(x)$ is a particular solution,

$$u_p(x) = e^{z_n x} \int \left(e^{(z_{n-1}-z_n)x} F_{n-1}(x)\,dx\right). \tag{3.35}$$

Here

$$G_{k\ell}^{\{1\}}(x) := \int e^{(z_k - z_\ell)x}\,dx, \qquad G_{k\ell}^{\{j\}}(x) := \int e^{(z_{\ell+j-2} - z_{\ell+j-1})x} G_{k\ell}^{\{j-1\}}(x)\,dx,$$
$$j = 2, 3, \ldots; \quad k = 1, 2, \ldots; \quad \ell = 2, 3, \ldots\ .$$
$$F_1(x) := \int f(x) e^{-z_1 x}\,dx, \qquad F_i(x) := \int e^{(z_{i-1} - z_i)x} F_{i-1}(x)\,dx$$
$$i = 2, 3, \ldots\ .$$

To prove Proposition 3.3 we apply the corresponding integral operators on (3.32) and identify the patterns. The details are ommitted.

Remark: For complex numbers, $\{z_1,\ z_2,\ \ldots,\ z_n\}$, the expression for $\phi_h(x; c_1, c_2, \ldots, c_n)$, (3.34), will be a complex-valued solution for (3.30) for which the real- and the imaginary parts are real-valued solutions of (3.30). We therefore need to combine the real- and imaginary parts of ϕ_h such that we have n linear independent real-valued solutions. We remark further that (3.35) is a real particular solution for (3.30), even for the case where $\{z_1,\ z_2,\ \ldots,\ z_n\}$ are complex roots of the characteristic equation.

Example 3.3. We find the general solution for

$$u^{(3)} - u'' + u' - u = e^{2x}\cos x. \tag{3.36}$$

The characteristic equation is $z^3 - z^2 + z - 1 = 0$, with roots $z_1 = 1,\ z_2 = i,\ z_3 = -i$. Thus (3.36) can be presented in the factorized form

$$L_1 \circ L_2 \circ L_3\, u(x) = e^{2x}\cos x, \quad \text{where} \quad L_1 = D_x - 1,\ \ L_2 = D_x - i,\ \ L_3 = D_x + i.$$

Following Proposition 3.3 the general solution of the homogeneous part of (3.36) is

$$\phi_h(x; c_1, c_2, c_3) = c_1 e^{z_3 x}\left(\int e^{(z_2 - z_3)x} G_{12}^{\{1\}}\, dx\right) + c_2 e^{z_3 x} G_{23}^{\{1\}} + c_3 e^{z_3 x},$$

where

$$G_{12}^{\{1\}} = \int e^{(z_1 - z_2)x}\, dx = \int e^{(1-i)x}\, dx = \left(\frac{1}{1-i}\right) e^{(1-i)x}$$
$$G_{23}^{\{1\}} = \int e^{(z_2 - z_3)x}\, dx = \int e^{2ix}\, dx = \left(\frac{1}{2i}\right) e^{2ix}.$$

We find

$$\phi_h(x; c_1, c_2, c_3) = \left(\frac{1}{2}\right) c_1 e^{x} + \left(\frac{1}{2i}\right) c_2 e^{ix} + c_3 e^{-ix},$$

where

$$\operatorname{Re}\{\phi_h\} = \frac{1}{2} c_1 e^{x} + \frac{1}{2} c_2 \sin x + c_3 \cos x$$
$$\operatorname{Im}\{\phi_h\} = -\frac{1}{2} c_2 \cos x - c_3 \sin x$$

are real-valued solutions of the homogeneous part of (3.36). Since $\{e^x,\ \sin x,\ \cos x\}$ is a linearly independent set of functions for all $x \in \mathbb{R}$, the general solution of the homogeneous part of (3.36) is

$$\phi_h(x; a_1, a_2, a_3) = a_1 e^{x} + a_2 \sin x + a_3 \cos x,$$

where $a_1,\ a_2$ and a_3 are arbitrary constants. Following Proposition 3.3, a particular solution for (3.36) is of the form

$$u_p(x) = e^{-ix}\left(\int e^{2ix} F_2(x)\, dx\right),$$

where

$$F_2(x) = \int e^{(1-i)x} F_1(x)\,dx, \qquad F_1(x) = \int e^x \cos x\,dx.$$

Calculating the integrals we obtain

$$F_1(x) = \frac{1}{2} e^x (\cos x + \sin x)$$
$$F_2(x) = \frac{1}{8} e^{(2-i)x} (\cos x + 2\sin x + i \sin x)$$

so that the particular solution becomes

$$u_p(x) = e^{-ix} \left[\int \frac{1}{8} e^{2x} e^{ix} (\cos x + 2\sin x + i\sin x)\; dx \right] = \frac{1}{8} e^{2x} \sin x.$$

The general solution of (3.36) is thus

$$u(x) = a_1 e^x + a_2 \sin x + a_3 \cos x + \frac{1}{8} e^{2x} \sin x.$$

3.3 Linear differential equations with nonconstant coefficients

First we consider second-order linear nonhomogeneous equations of the form

$$u'' + g(x)u' + h(x)u = f(x), \tag{3.37}$$

where f, g and h are differentiable functions on some common interval $I \subseteq \mathbb{R}$. Assume a factorization in terms of two linear first-order differential operators,

$$L_1 = D_x + q_1(x), \qquad L_2 = D_x + q_2(x), \tag{3.38}$$

such that (3.37) is equivalent to

$$L_1 \circ L_2\, u(x) = f(x). \tag{3.39}$$

Now (3.39) takes the form

$$u'' + (q_1 + q_2)u' + (q_1 q_2 + q_2')u = f(x) \tag{3.40}$$

and, comparing (3.40) to (3.37) leads to the condition

$$q_1 + q_2 = g(x), \qquad q_2' + q_1 q_2 = h(x),$$

or, equivalently,

$$q_2' = q_2^2 - g(x) q_2 + h(x), \qquad q_1(x) = g(x) - q_2(x).$$

To find the general solution of (3.39) we apply the corresponding integral operators $\hat{L}_1$ and $\hat{L}_2$, successively. This leads to Proposition 3.4.

Proposition 3.4. *The 2nd-order linear equation*

$$u'' + g(x)u' + h(x)u = f(x) \tag{3.41}$$

can be written in the factorized form

$$L_1 \circ L_2\, u(x) = f(x), \tag{3.42}$$

where $L_1 = D_x + q_1(x)$ *and* $L_2 = D_x + q_2(x)$*, if and only if* $q_2(x)$ *satisfies the Riccati equation*

$$q_2' = q_2^2 - g(x)q_2 + h(x). \tag{3.43}$$

Then $q_1(x) = g(x) - q_2(x)$*. Applying the corresponding integral operators,* $\hat{L}_1$ *and* $\hat{L}_2$ *successively on (3.42), leads to the general solution of (3.41) namely*

$$u(x) = c_1 e^{-\xi_2(x)} + c_2 e^{-\xi_2(x)} \int e^{\xi_2(x)} e^{-\xi_1(x)} dx + u_p(x) \tag{3.44}$$

where $u_p(x)$ *is a particular solution of (3.41) given by*

$$u_p(x) = e^{-\xi_2(x)} \int e^{\xi_2(x)} e^{-\xi_1(x)} F(x) dx. \tag{3.45}$$

Here c_1 *and* c_2 *are arbitrary constants and*

$$F(x) := \int f(x) e^{\xi_1(x)} dx, \quad \xi_1(x) := \int q_1(x) dx, \quad \xi_2(x) := \int q_2(x) dx.$$

Example 3.4. (Second-order Cauchy-Euler equation)
We consider the second-order nonhomogeneous Cauchy-Euler equation

$$ax^2u'' + bxu' + cu = f(x), \qquad x \neq 0, \tag{3.46}$$

where $a \neq 0$, b and c are real constants and f is a continuous function on some interval $I \subseteq \mathbb{R}$. Equation (3.46) can equivalently be presented in the form

$$u'' + \left(\frac{b}{ax}\right) u' + \left(\frac{c}{ax^2}\right) u = \frac{f(x)}{ax^2}. \tag{3.47}$$

Comparing (3.47) and (3.41) we identify

$$g(x) = \frac{b}{ax}, \qquad h(x) = \frac{c}{ax^2},$$

so that, by Proposition 3.4, equation (3.46) can be factorized in the form (3.42) if q_2 satisfies the following Riccati equation:

$$q_2' = q_2^2 - \left(\frac{b}{ax}\right) q_2 + \frac{c}{ax^2}. \tag{3.48}$$

A solution of (3.48) is of the form $q_2(x) = \alpha x^\beta$ with $\beta = -1$ and α satisfying the quadratic equation

$$\alpha^2 + \left(1 - \frac{b}{a}\right) \alpha + \frac{c}{a} = 0. \tag{3.49}$$

As an explicit example we consider $a = 1,\ b = -1$ and $c = 1$. This corresponds to the equation

$$x^2 u'' - xu' + u = x^3. \tag{3.50}$$

A solution for (3.49) is then $\alpha = -1$, so that

$$q_2(x) = -\frac{1}{x}, \quad q_1(x) = 0, \quad \xi_1(x) = 0, \quad \xi_2(x) = -\ln|x|.$$

Thus the equation

$$u'' - \left(\frac{1}{x}\right) u' + \left(\frac{1}{x^2}\right) u = x$$

factors in the form

$$\left(D_x\right) \circ \left(D_x - \frac{1}{x}\right) u(x) = x$$

so that, by the solution formula (3.44), the general solution of (3.50) becomes

$$u(x) = c_1 x + c_2 x \ln|x| + \frac{1}{4} x^3,$$

where c_1 and c_2 are arbitrary constants.

Example 3.5. Consider the equation

$$u'' + 2xu' + (x^2 + 1)u = e^{-x^2/2}. \tag{3.51}$$

Comparing (3.51) and (3.41) we identify

$$g(x) = 2x, \qquad h(x) = x^2 + 1,$$

so that, by Proposition 3.4, equation (3.51) can be factorized in the form (3.42) if q_2 satisfies the following Riccati equation:

$$q_2' = q_2^2 - 2xq_2 + x^2 + 1. \tag{3.52}$$

A solution for (3.52) is $q_2(x) = x$ so that $q_1(x) = x$. Hence (3.51) takes the factorized form

$$(D_x + x)(D_x + x)\, u(x) = e^{-x^2/2}$$

so that, by the solution formula (3.44), the general solution of (3.51) becomes

$$u(x) = c_1 e^{-x^2/2} + c_2 x e^{-x^2/2} + \frac{1}{2} x^2 e^{-x^2/2}$$

where c_1 and c_2 are arbitrary constants.

An extension to higher-order nonconstant-coefficient linear equations is possible, but the general condition for the factorization into first-order linear operators becomes rather complicated.

4 Conclusion

We have shown that the factorization of linear operators in terms of first-order linear operators can be applied to find the general solution of an nth-order linear differential equation. This provides an alternate approach to the ansatz-based methods that are often taught to students in an introductory course on ordinary differential equations. The method is based on the integral operator as defined in Definition 2.3. and also applies to linear differential equations with nonconstant coefficients.

This paper has been written on a basic level and with many examples to make it accessible to a broad audience.

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