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Flatterland: The Play

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Flatterland: The Play

based on *Flatterland: Like Flatland Only More So*

by Ian Stewart

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Abstract

This script is an adaptation of the popular science novel *Flatterland: Like Flatland, Only More So* by Ian Stewart. It breathes new life into mathematical ideas and topics. By bringing math to the stage, this script presents concepts in a more friendly and accessible manner. This play is intended to generate new interest in and expose new topics to an audience of non-mathematicians.

Preface

I was first introduced to *Flatterland* by Sue Buckwalter while I was at Phillips Academy. This book was much in alignment with her way of making math fun and engaging, occasionally sacrificing a little rigour in favor of giving an intuitive understanding of the subject. Though I had read other of Stewart's books, this one stood out to me for its playful nature and unconventional approach to explaining new topics in mathematics. To me, math has always been an exploration full of new discoveries, much as Vikki explores the worlds in the Mathiverse.

When I first read *Flatterland*, I would never have dreamed of creating a production based on it. The characters and topics seemed too unrealistic and in some cases abstract to become solid and materialize in this world. This book was the first time an explanation allowed me to visualize a fourth dimension and though now four, five, and six dimensions are more familiar to me, showing them to others is still a challenge. I loved this book and would always recommend it to someone looking for a good read, but generally put it aside for five years.

Five years later, in a Writing for Performance class at Pomona College, I began to explore settings that were somewhat surreal but still had their own logic. While looking for ideas, I remembered *Flatterland* and considered working with mathematical objects as characters. After the class ended, I was still interested in writing and understanding these characters and reexamined the book for ideas. I eventually decided to base a script on *Flatterland* to further explore this. This script contains only a selection of the wonderful worlds of *Flatterland* and only a tiny slice of the infinite worlds of the Mathiverse. I drew one scene—Pointland—from the original *Flatland* so that I could explore both the tiny and simple as well as the vast and complex portions of the Mathiverse. To my knowledge, this is the first adaptation for the stage of any of the books related to *Flatland*, including *Flatland* itself.

This script is intended to be performed on stage, though it may be in-

teresting to expore other media for it. For example, the Fractal Forest may be shown in much greater detail and overwhelming complexity through the use of computer graphics and animation. Topologica may be explored through a more mutable medium, such as computer or even clay animation. The two dimensional inhabitants of Flatland may be actually flat if portrayed using shadow puppets. This script, like Stewart's book, is intended to encourage curiosity and exploration for the actors, director, and designers as well as the audience. With these things, I hope to bring the numbers, shapes, and ideas to life.

Acknowledgments

To Sue Buckwalter, who first introduced me to *Flatterland* and who has been my inspiration and encouragement as I try to walk the line between art and mathematics.

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Chapter 1

Introduction

In 1884, Edwin A. Abbott's *Flatland: A Romance of Many Dimensions* introduced a revolutionary means of exploring mathematics. This book had two major agendas: social commentary and mathematical instruction. In this book, Abbott uses two-dimensional inhabitants of Flatland to explore and criticize the Victorian social hierarchy. Social ranking is highly dependent on the regularity and number of sides that an individual has; women are line segments and are thus unchangeably in the least powerful position.

The polygons who inhabit Flatland are of the firm belief that there is no such thing as a third dimension, until the narrator receives a visitation from a sphere, convincing him of the existence of a third, and possibly more dimensions. This idea challenges the long-held convictions of those at the top of the social hierarchy, who are strongly opposed to the formation of new ideas or change, and sentence him to life imprisonment for his heretical ideas.

Flatland has inspired many sequels by other authors including *Spaceland* (Rucker, 2003), *Sphereland* (Burger, 1983), *The Planiverse* (Dewdney, 2000), and *Flatterland* (Stewart, 2001), as well as an animated film, *Flatland: The Movie* (Abbott et al., 2007). Each of these attempts to bring life to more mathematical constructs or mathematical constructs into comfortable narratives. Though this play and *Flatland: The Movie* both serve to create a more visual and audible presentation of mathematical ideas, they have significant differences. Because this play is meant for the stage, some ideas that could be visually represented in animation require more imagination from the audience and skillful explanation from the actors. In this play, the characters are more than the talking polygons that appear in *Flatland: The Movie*. This is in part because *Flatterland* gives them additional depth

and in part because actors have greater challenges to be and explain their characters when their properties cannot all be shown as a colored polygon.

In 2001, Ian Stewart's *Flatterland: Like Flatland, Only More So* was published. Stewart has written many popular science books bringing life to various mathematical topics. His book *Flatterland* is the basis for this play. In this book, Stewart follows the adventures of a young line named Vikki, the great-great-granddaughter of the narrator of *Flatland*. For more information on her great-great-grandfather, see Appendix A.1. Vikki is guided through many worlds made unusual by their particularly visible mathematical qualities, with a focus on dimension and topology.

Flatterland differs somewhat in type of content from *Flatland*. It does not contain the social commentary present in *Flatland*, but instead focuses on exploring mathematical ideas that are, to many readers, more foreign than a fourth dimension. In this book, Vikki explores worlds made interesting by their unusual dimensional or topological qualities. The ideas in this book are rooted in geometry, as are those in *Flatland*. Additionally, Stewart draws on *Alice in Wonderland* to bring some more whimsical qualities to the unusual spaces that the characters explore. This creates a friendly introduction to the dimensional and topological ideas contained therein for a reader with a mathematical background through basic high school geometry. In this way, Stewart makes accessible topics generally not covered until later years of mathematical study to readers who may not gain exposure to them academically.

The topics, plot, and characters of this play are largely drawn from Stewart's text, with a couple ideas and a character from Abbott's. However, nearly all of the dialogue is original with the exception of a few particularly pungent lines from the book. In particular, the only large section of text quoted from *Flatterland* is the SPACE HOPPER's introduction to the Mathiverse. In order to accommodate an audience without a required background in mathematics or the time to sit through the exploration of every concept, multiple scenes have been cut from the original book.

The topics covered in this play include lower dimensions, higher dimensions, variables as dimensions, fractals and measurements of their dimensions, curved space (both spherical and hyperbolic), topological equivalence, and hyperbolic geometry. All of these topics are introduced qualitatively instead of quantitatively in order to provide more approachable scenes. This play should give the audience an intuitive understanding of these concepts, without the equations and numbers often present in presentations of mathematics.

This play is intended to be approachable for anyone with an under-

standing of basic high-school–level math. It is intended to demonstrate some unusual, interesting, and entertaining concepts in mathematics and generate new interest in math. It should also expose the audience to unfamiliar yet friendly topics, but is not intended to provide in-depth or rigorous discussions of the topics presented therein.

The Play

Cast of Characters

VIKKI	A young woman eager to explore the world, especially the parts her parents don't want her to know about. She, like all females in Flatland, is a line.
JUBILEE	Vikki's mother.
GROSVENOR	Vikki's father.
SPACE HOPPER	A transdimensional being who moves between spaces in the Mathiverse.
KING OF POINTLAND	The one and only almighty being that lives in, and is, Pointland.
BICIRCLIST	One of the many Peoples who inhabit Planiturth.
KOCH SNOWFLAKE	An inhabitant of the Fractal Forest who is not itself a fractal but whose boundary is.
RECTANGLE	Lives on a sphere and is forever painting it red, then white, then back to red again.
DOUGHMOUSE	A resident of Topologica. Carries a bag full of dough of various genera that can be continuously deformed to make any object he desires.
MUD HUTTER	An inhabitant of Topologica. Has tea with the Doughmouse and Harsh Mare. In his free time, makes shapes and huts from mud.

HARSH MARE	An angry equine inhabitant of Topologica. Has tea with the Doughmouse and Mud Hutter.
DUCK	Finds it necessary to intrude on the topologists' tea party and explain induction.
MOOBIUS	A nonorientable cow who provides milk for the topologists' tea party.
HEPTAGON	An inhabitant of Hyperbolica. This heptagon has the angles of a hexagon but manages to have seven sides due to the curvature of the space.

Act I

Scene 1

VIKKI offstage. JUBILEE is preparing dinner

VIKKI
No. . . Yes! No, no. . . This won't do at all. . . Ah! . . . no.

JUBILEE
Vikki dear, are you alright?

VIKKI
Yes, mom.

A small crash is heard.

JUBILEE
What are you doing, dear?

VIKKI
I need to find something interesting to wear out with my friends. All of my clothes are just too regular. They don't look quite right. . . I want something less normal.

JUBILEE
If you like, dear, you're welcome to look through my old clothes in the attic. Some of them should fit you nicely, though a few might have sheared over time. Hopefully none of them have gone concave. They're all in the attic.

VIKKI enters the attic. It is full of piles of numerous boxes and random geometric junk. A bookshelf coated in dust holds random odds and ends, along with a few books. As she enters, she knocks over a short stack of boxes.

JUBILEE
Vikki!

VIKKI
I'm all right, mom! Where in here?

JUBILEE
I think in one of the boxes! If not, they might be in one of the bags!

VIKKI
Bags?

JUBILEE
I think they're under some of the boxes! Look through the boxes first, though, they're probably in there!

VIKKI begins to search through the boxes. They are full of everything but what she is looking for. In an attempt to move to search boxes in a different corner of the room, VIKKI trips and nearly knocks over the bookshelf, causing it to spill half its contents.

JUBILEE
Vikki!

VIKKI
I'm fine!

JUBILEE
Just checking! Do be careful, dear!

VIKKI picks up and inspects a book that fell near her. It is old and difficult to decipher. She holds onto it and continues her search, eventually finding an interesting jacket. She puts it on and enters the main part of the house, pocketing the book.

JUBILEE
Find something, dear?

VIKKI
Yep, look!

JUBILEE
That looks nice on you.

VIKKI
Nice isn't exactly what I was going for, but I guess that's good. By the way, I found some really old diary or book sort of thing. It's starting to unravel. Flat—

GROSVENOR, VIKKI's father, enters, returning from work.

JUBILEE
Grosvenor!

GROSVENOR
Jubilee! And Vikki, how's your day been?

VIKKI
Good, dad! I found—

GROSVENOR
And how's school? Did you get your test back?

VIKKI
It's fine, I guess. And, yeah, I got my test back.

GROSVENOR
Well? Did my going over it with you help?

VIKKI
It's still a lot of equations to me. But, yeah, I guess so.

GROSVENOR
Good, good.

JUBILEE
Vikki, would you be a dear and set the table for me?

GROSVENOR
It looks like you've got everything all set here, so I'm going to go clean up.

(he exits)

JUBILEE
What were you saying you found, dear? A book?

VIKKI
Yeah, something about Flatland. I haven't really looked at it yet.

JUBILEE
It's probably just one of your father's old physics textbooks or something, dear. That room hasn't been cleaned out since your great-grandfather lived in this house.

VIKKI
It doesn't quite look like a textbook. It looks too old.

JUBILEE
Your father and I had books growing up. We're not that old. Books aren't exactly a recent technology.

VIKKI
I know. It's just... well... I don't know.

JUBILEE
Why don't you take that jacket off so it doesn't get food on it. Dinner will be on the table in just a few minutes.

VIKKI exits and returns without jacket or book.

JUBILEE
Now where's your father?

VIKKI
DAD! DINNER!

JUBILEE
You don't need to yell, I'm sure he can hear you.

GROSVENOR
COMING!

GROSVENOR enters. They sit down for dinner.

GROSVENOR
This smells delicious.

VIKKI
I found an old book in the attic. Mom said it might be yours?

GROSVENOR
What's the book?

VIKKI
I'm not sure, it looks really old. It said "Flatland" on the front and some other stuff I couldn't really make out.

GROSVENOR
Well that could be just about anything.

JUBILEE
I thought it might be one of your old physics texts.

GROSVENOR
Possibly, but I don't recall any by that name. There wasn't anything else you could read?

VIKKI
Yeah, it looked handwritten and was signed something like A. Square or something like that. It looked like it might have had some geometry in it.

GROSVENOR is severely startled by this and exchanges a significant glance with JUBILEE. After a lengthy, awkward pause…

GROSVENOR
Vikki, where is this book?

VIKKI
In my room, why?

GROSVENOR
Please get it and give it to me.

JUBILEE
Grosvenor, let her finish eating first.

GROSVENOR
But Jubilee, this is really—

JUBILEE
Grosvenor.

GROSVENOR
Fine, fine. Vikki, you need to give me that book as soon as you're done eating.

JUBILEE
Grosvenor.

GROSVENOR
Okay, okay! Vikki, just... you can't keep that book. It's not something you should be reading. So please, you need to give it to me. And don't read it in the meantime.

VIKKI
Oh. What is it?

GROSVENOR
Don't worry about that.

VIKKI
Is it full of secrets? Or maybe it's really shapist? It probably would have been written while the number of sides you had still determined your class.

GROSVENOR
Vikki.

VIKKI
Or maybe it's "adult" material. Is that what it is?

GROSVENOR
Vikki. I don't want you to worry about that book, it's not important.

VIKKI
But why?

GROSVENOR
It just isn't.

VIKKI
It just—

GROSVENOR
Vikki, drop it.

JUBILEE
Grosvenor, how are things going at work?

GROSVENOR
Oh, you know. The same, the same.

VIKKI
Dad, why—?

JUBILEE
Vikki.

They eat in silence.

VIKKI
Mom, I should go. I'm meeting Anne at the intersection a couple blocks north of here.

JUBILEE
Okay, have fun, dear!

GROSVENOR
Vikki—

VIKKI exits.

JUBILEE
Grosvenor, let her be.

GROSVENOR
I'll let her be when I get the book back. *Flatland*. Damn. I thought that'd been burned decades ago.

JUBILEE
Well it apparently wasn't.

GROSVENOR
Then we have to burn it. I'll go get—

JUBILEE
This is how you're going to deal with her finding this piece of family history?

GROSVENOR
'Lee, do you really think that's appropriate reading material for her?

JUBILEE
You can't keep it from her forever.

GROSVENOR
But I can keep it from her right now. She doesn't need to know about this. Not right now.

JUBILEE
Grosvenor it's—

GROSVENOR
No it is not okay! I am not going to have my daughter reading that filthy book!

JUBILEE
You know as well as I do that she's not just going to give up on it if you make her give you the book. She'll want to know why.

GROSVENOR
She doesn't need to know why.

JUBILEE
Grosvenor, that's only going to make it worse.

GROSVENOR
Worse than that completely inappropriate piece of trash?

JUBILEE
She's not going to stop asking.

GROSVENOR
And she's not going to get an answer.

JUBILEE
Do you really want her asking the neighbors if they know who A. Square was?

GROSVENOR
Okay.

GROSVENOR considers this carefully and quietly for some time. Eventually and resentfully...

Scene 2

VIKKI returns for the evening. JUBILEE and GROSVENOR are waiting for her.

JUBILEE
Welcome back, dear.

GROSVENOR
Vikki, sit down, we need to have a talk.

VIKKI
What? I didn't do anything.

GROSVENOR
Did you read any of the book that you found?

VIKKI
No, why does—

GROSVENOR
Good. Please go get it and bring it to me.

VIKKI
Right now? Why?

GROSVENOR
So I can burn it. No one should read that book.

VIKKI
What? Dad? No! Why would you. . . ? You can't do that!

GROSVENOR
You wouldn't want other people knowing about this. It should have been destroyed decades ago.

JUBILEE
Grosvenor, she doesn't know what it is. Vikki, that book was written by your great-great-grandfather. He wrote it while he was in prison.

VIKKI
He was an author? That's cool.

JUBILEE
Not in this case. He was imprisoned for life for attempting to spread blasphemous ideas—

GROSVENOR
And all of them are in that damned book you have.

JUBILEE
He lost his mind and claimed that he had communicated with a being from the Third Dimension—a serious offense. It still is.

GROSVENOR
Probably half the people you know wouldn't talk to you again.

JUBILEE
The family put in a lot of work trying to diffuse and cover up the scandal.

GROSVENOR
The idea that there could be more than two dimensions is completely ridiculous. Insane.

JUBILEE
It's something you were bound to find out eventually.

GROSVENOR
Crazy bastard. Vikki, get me that book.

VIKKI
What? No! It's family history! You can't burn it!

JUBILEE
Vikki, some things are best forgotten. Left in the past.

GROSVENOR
It's really not appropriate reading. Pure rubbish, actually.

VIKKI
Did he actually talk to a three-dimensional being?

GROSVENOR
Of course not. Don't say things like that, Vikki.

VIKKI
How do you know?

GROSVENOR
Because there is no third dimension. There's North–South and East–West and that's all. Of course there couldn't be another dimension.

VIKKI
But how do you know?

GROSVENOR
Don't be dense. Now please go get me that book before you get any ideas from it.

VIKKI
But—

JUBILEE
Do as your father says, Vikki.

VIKKI
Okay, fine. I'm not quite sure where I put it. I'll be back as soon as I find it.

VIKKI enters her room. The inside of her room is visible to the audience but not to her parents. She rummages around a bit then pulls out a camera. She hurriedly takes pictures of the entirety of the book then hides her camera in her bed. She returns to the main room with the book.

VIKKI
Here it is.

GROSVENOR takes the book and pulls a lighter from his pocket. He catches the corner on fire and once he's satisfied that the flame will not go out, he tosses it into the fireplace. VIKKI watches sadly.

GROSVENOR
There we go.

VIKKI
Did you really have to do that?

GROSVENOR
Yes. And for your own good, too. You should be thanking me.

VIKKI
But it was just a book! What could be the harm in that?

GROSVENOR
It got your great-great-grandfather put away for life.

VIKKI
But that was years ago! Centuries!

JUBILEE
Vikki, now that you know what that book was, I think we all really ought to be going to bed. It's getting pretty late.

GROSVENOR
Your mother's right, Vikki. Have a good night.

JUBILEE
Good night, dear.

VIKKI
Night, mom, dad.

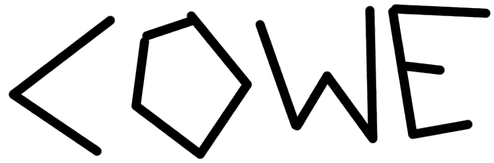


Figure 1.1 VIKKI's signs to the SPACE HOPPER.

JUBILEE and GROSVENOR exit. VIKKI retires to her room. She retrieves her camera and starts looking through the pictures. Eventually she comes upon a diagram and, inspecting it, begins haphazardly arranging objects in her room into a messy form (Figure 1.1).

After completing it, she waits for a second, expectantly. After nothing happens, she gives up and climbs into bed, turning out the light.

A dim glow appears in the room. Barely visible are bits and pieces that seem to move as one but are not connected. They slowly swell to become the body of the SPACE HOPPER.

SPACE HOPPER
Someone called? Hello, is someone here?

VIKKI
Uh... hello... sir? My name is Vikki and I want to meet a being from the Third Dimension.

SPACE HOPPER
Only the third?

VIKKI
Are, um, are you the sphere, sir?

SPACE HOPPER
Ha, no, I am not a sphere. Nothing so prosaic!

VIKKI
Then what are you?

SPACE HOPPER
I am a Space Hopper!

VIKKI
That's a funny name.

SPACE HOPPER
Not at all! It makes perfect sense!

VIKKI
How so?

SPACE HOPPER
Because I hop between spaces!

VIKKI
Is that how you got into my room?

SPACE HOPPER
Yes, you could say that. I saw this from above—

(gesturing at her construction)

VIKKI
You can see it from outside? You mean Mum and Dad could see... no that doesn't make any sense, you can't see through the walls.

SPACE HOPPER
Oh no, you can't see through the walls. You can see, hm, you can see around the walls.

VIKKI
So... you hopped around the walls from outside to here?

SPACE HOPPER
Outside the space, yes.

VIKKI
You mean like outer plane?

SPACE HOPPER
Nope, outside like above. It's quite easy when you have an extra dimension or two.

VIKKI
So. . . you came—you hopped into—around the walls to my room from the north?

SPACE HOPPER
Not from the north, from the above. From a different space.

VIKKI
Um?

SPACE HOPPER
From the Third Dimension.

VIKKI
Oh! Oh, where is it?

SPACE HOPPER
It is up from here. And down from here. Your plane could be considered to be just a tiny slice through it. Flatland is in it. The Third Dimension is a space. Flatland is a space.

VIKKI
Are the spaces just different. . . places? Just. . . spaced out? Can you take me there?

Barely visible, JUBILEE moves to right outside VIKKI's closed door, curious about the noise.

SPACE HOPPER
I never thought you'd ask!

(producing a contraption for Vikki to wear)

SPACE HOPPER
This is a Virtual Unreality Engine. A VUE. It can take you to new places. Change your point of view.

The SPACE HOPPER connects the VUE onto, into, through, and around VIKKI.

SPACE HOPPER
Now, this will be a bit disorienting at first, but you've got to trust me and try to follow. This button here—no, don't press it just yet—frees you from the dimension you're in.

VIKKI
And you're sure this won't turn me inside out or something like that?

SPACE HOPPER
Of course. Most of these things only exist as theoretical constructs in the Mathiverse. Okay, are you ready? It'll feel a bit weird at first, that's perfectly normal. On the count of three, you just have to go for it. One... two...

JUBILEE throws open the door.

JUBILEE
Vikki!

SPACE HOPPER
Three!

The SPACE HOPPER and VIKKI disappear, leaving JUBILEE alone in the room. VIKKI's voice can be faintly heard.

VIKKI
Don't worry, mother, I'll be fine!

Act II

Scene 1

VIKKI and the SPACE HOPPER are in some space in the mathiverse.

VIKKI
Where are we? Is this the Mathiverse? You mentioned the Mathiverse—what is it?

SPACE HOPPER
Ah, yes. The Mathiverse is a universe that we are about to explore. It transcends Time and Space, Intelligence and Extelligence, Thought… it transcends Transcendence itself. Within it are not only all Spaces and Times that have existed or will exist, not even only all that could exist, but all Spaces and Times that could not exist. The Mathiverse contains all numbers. It contains all shapes, all geometries. The Mathiverse contains all vectors, matrices, permutations, combinations, integrations, separations, projections, injections, surjections, bijections, semigroups, transformations, relations, functions, functors, functionals, algebraic group schemes, supermanifolds, K-theories, M-theories, M-sets, power sets, subsets, supersets. All data structures, processes. All formal descriptions of logical structures and all informal descriptions of illogical structures. If someone were to invent something never seen before that fit in the same category as these, it already would have existed in the Mathiverse.

VIKKI
…oh. Where are we going?

SPACE HOPPER
I'm going to take you to the most basic of the spaces in the Mathiverse. It's called Pointland.

VIKKI
Pointland? Do they have points instead of shapes?

SPACE HOPPER
You could say that. Pointland is zero-dimensional.

VIKKI
Zero? How does that work?

SPACE HOPPER
Well, what does dimension mean? Why is Flatland two-dimensional?

VIKKI
We have two dimensions because there are two directions. There's North–South and East–West. And that's all. You still haven't shown me how there can be more than two and it doesn't seem like less than two would even work.

SPACE HOPPER
Why not?

VIKKI
Well, for one if you only had one dimension, creatures wouldn't even be able to move around each other. Their ends would just bump into each other and that's all. They couldn't move anywhere or do anything,

SPACE HOPPER
And that's exactly how Lineland works. But we're going one fewer!

VIKKI
Yes and that shouldn't work at all! If there aren't any directions nothing can move at all and then everything would all be stuck on top of the same place. In the same place, I suppose. There is no top. But in any case that makes no sense. There can't be creatures there if they're all in the same place.

SPACE HOPPER
Let me introduce you to the King of Pointland.

Scene 2

In Pointland. All is dark. For further explanation of the zero-dimensional world, see Appendix A.3

SPACE HOPPER
The Abyss of No Dimensions, as the Sphere explained it to your great-great-grandfather.

VIKKI
What are we, er, looking at?

SPACE HOPPER
A point.

KING OF POINTLAND
...oh wonderful existence! It is everything and everything is It. Blessed is It and there is none besides It. Its existence glorifies It and It is all. It contains all and...

VIKKI
A point? A point is doing all that talking?

SPACE HOPPER
The point is Pointland. The King of Pointland is talking.

KING OF POINTLAND
...all of Space. It fills completely and what It fills, It is. It is meaning and...

VIKKI
But he can't possibly fit in there. There's no room. Even if he is just a point, Pointland is just a point.

SPACE HOPPER
So?

VIKKI
So he would take up all of the space. Which there isn't any of in the first place. He can't be inside the point. Points don't have insides.

SPACE HOPPER
He is the point.

VIKKI
He is the point?

SPACE HOPPER
Yes.

VIKKI
But then he is the entire space.

SPACE HOPPER
Exactly.

VIKKI
Oh.

KING OF POINTLAND
... It is the beginning and It is the end. It is the whole. It praises It for Its great capacities for all...

VIKKI
And so "it" is...?

SPACE HOPPER
It is it. The space. The King of Pointland. The point. His entire existence is this point.

VIKKI
But doesn't he realize there isn't anything in there?

SPACE HOPPER
Actually, he can't perceive anything outside of his point.

KING OF POINTLAND
...It is the Whole. It is the Thinker and Perceiver, Utterer and Listener, It is the Word and the World. It thinks and It speaks. It hears as It speaks and It speaks existence...

VIKKI
But there actually isn't room for all that stuff to be in there. Or for any of it to be in there. There is no "in there."

SPACE HOPPER
You're exactly right, Vikki. They aren't in it, they are it.

VIKKI
So the King is the space and the existence and whatever else he's just said?

SPACE HOPPER
You got it.

VIKKI
That is very strange.

SPACE HOPPER
It's perfectly normal to him.

VIKKI
What if you explain to him about other dimensions? Then would he realize that there could be at least one more? I could show him that I am a line.

SPACE HOPPER
He can't see, Vikki, but go ahead and try.

VIKKI
Um, King, Sir?

KING OF POINTLAND
. . . It is almighty. As It is Space and Existence, It is infinite.

VIKKI
Now that can't be right.

KING OF POINTLAND
It is filled with It and It fills. . .

VIKKI
King, Sir? Please. You have to listen, I'm trying to explain something important.

KING OF POINTLAND
Ah, magnificent thought! It speaks as It thinks and It hears as It thinks! It produces all..

VIKKI
No. Here, I am out here. You can't see me right now but I am.

KING OF POINTLAND
It speaks of It and It feels Its presence encompassing the whole. As It thinks It understands all…

VIKKI
No, King, Sir, whatever your name is, there is space that is not inside your little dot.

KING OF POINTLAND
It describes the infinitude of existence and It knows…

VIKKI
No, I am not you! I am me.

KING OF POINTLAND
It is all and all is It. The Speaker is the Listener is the Thinker. All one…

VIKKI
No! That isn't—!

SPACE HOPPER
Vikki, it's okay. The King of Pointland can't comprehend existence beyond his own.

VIKKI
But he can hear me!

SPACE HOPPER
Yes and no. He hears you in the same way that he hears himself think. He is the entirety of his existence; he is all that he can understand.

VIKKI
But if I explain how there are directions that space could extend in—

SPACE HOPPER
Perhaps we should go. You will not be able to convince him of your presence, let alone the manner in which you can take up space.

VIKKI
I… uh… okay.

SPACE HOPPER
Next we'll go somewhere much more interesting, full of new creatures!

VIKKI
They're not all going to be like Pointland, are they? That space seemed very… limited.

SPACE HOPPER
I think you'll find that in many lands the inhabitants are convinced that their view of space is uniquely correct and universal.

VIKKI
Like us Flatlanders can't be convinced of a third dimension?

SPACE HOPPER
Most of you Flatties can't.

VIKKI
Oh, oh! Are we going to go to the Third Dimension? This will be so cool! I haven't been able for the life of me to figure out where another dimension would go! There are the usual two, but then… they just run out of ways. And you can't just get a direction in between those because then it would just be some combination of those two directions. Are we going to visit there?

SPACE HOPPER
Actually, I have somewhere even more interesting in mind.

VIKKI
Oh. It'll at least have more than two dimensions, right?

SPACE HOPPER
Of course.

VIKKI
Oh good!

SPACE HOPPER
Now, when your great great grandfather was visited by the sphere, it appeared to him as many many circles stacked on top of each other. "Top" is your new direction, incidentally.

VIKKI
"Top"…?

SPACE HOPPER
Top–Bottom. Or really Up–Down.

VIKKI
Where did those names come from? And why are there two?

SPACE HOPPER
They came from our next destination. Planiturth. As for why there are two, Planiturth is inhabited by a creature called a People. Lots and lots of Peoples. And Peoples are a very confused creature.

VIKKI
Is Planiturth the Third Dimension?

SPACE HOPPER
It has a third dimension. That is, it has a third direction. Actually, Planiturth is just a roughly sphere-shaped structure in a much larger universe. It's where the Peoples live.

VIKKI
So, it's three-dimensional.

SPACE HOPPER
For a very long time the Planiturthians thought it was. However, some argued that there was a fourth—

VIKKI
Wait, can we talk about what the third dimension—and where the third dimension is—before you go on adding another one?

SPACE HOPPER
Yes of course! Now, here, first show me where your first two are.

VIKKI
Here. . . and here.

SPACE HOPPER
May I draw that for you?

VIKKI
Um, sure?

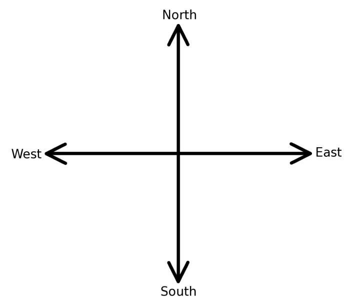


Figure 1.2 The SPACE HOPPER draws a pair of axes.

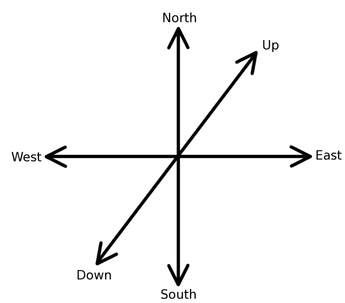


Figure 1.3 The SPACE HOPPER draws a set of axes in three dimensions.

The SPACE HOPPER draws a set of axes (Figure 1.2).

SPACE HOPPER
Here… and here.

VIKKI
Yeah.

SPACE HOPPER
Now suppose I were to draw this direction.

The SPACE HOPPER draws a diagonal line through the other pair of lines (Figure 1.3).

VIKKI
No, that's just a combination of the two. That's not really a new direction.

SPACE HOPPER
Good. But the Peoples draw three dimensions that way.

VIKKI
That's very odd.

SPACE HOPPER
No, see.

The axes shift. A third dimension is added. VIKKI inspects the new axis, now vertical. A projection of the axes remains on the floor.

VIKKI
Woah... no... that... woah.

She continues to inspect it and touches it, making the whole set of axes rotate.

VIKKI
That's so weird.

SPACE HOPPER
Up... and Down.

VIKKI
Up and Down. Like North and South. But kind of... twisted.

SPACE HOPPER
So do you like it?

VIKKI
No. I mean, yes, but it's all very weird. That direction is, like, between, no, in, on top of, inside the other directions? That... huh.

SPACE HOPPER
Now suppose we were to add another direction. Call this one the Chalk–Cheese direction

The SPACE HOPPER draws an additional axis on the set of axes on the floor (Figure 1.4).

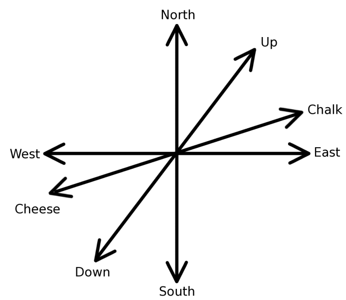


Figure 1.4 The SPACE HOPPER draws a set of axes in four dimensions.

SPACE HOPPER
Chalk… and Cheese.

VIKKI
Oh now that's even worse.

SPACE HOPPER
But you can see where this comes from.

VIKKI
Sure, you just added another line.

SPACE HOPPER
Which is another pair of directions.

The set of axes shifts to three dimensions such that there are three perpendicular lines and one pointing diagonally through them. A shadow remains on the ground as well as a copy of the original set of three axes and its shadow.

VIKKI
Sure, but now that one is just a combination of… oh…

SPACE HOPPER
There is a particular species of the Peoples known as Mathematicians. Some say they study the Mathiverse. Some say they created it. But in any case, they like to be able to write these down, so they made a shorthand for it. They just call these directions by letters. So this would be the x-direction. This would be the y-direction. And th z-direction. And this is the w-direction, because "w" is the letter that comes after z in the Mathematician's alphabet.

VIKKI
We have that in Flatland in our math classes, but there are only x and y.

SPACE HOPPER
We could go on adding more directions, w, v—

VIKKI
Oh, please don't! The more you draw the higher dimensions in two dimensions, the less sense I can make of them.

SPACE HOPPER
All right.

VIKKI
You said the Peoples weren't sure how many dimensions they had. Couldn't they just count how many lines they could draw without making one just a combination of the other ones?

SPACE HOPPER
Well, they can't actually see what some of them argue is a fourth dimension.

VIKKI
Then why would they think they have one?

SPACE HOPPER
A group of Mathematicians and Physicists argued that time was the fourth dimension. Other Peoples didn't believe them. This second group was right, but for the wrong reason.

VIKKI
How were they right but for the wrong reason?

SPACE HOPPER
Time is a fourth dimension, but it isn't the fourth dimension. The first People to join their three directions with time was called Hermannminkowski in 1908. The Peoples now call these four dimensions space-time and it is very closely connected to a field of Physics called special relativity, developed by the famous Alberteinstein around the same time.

VIKKI
So they do have four dimensions.

SPACE HOPPER
Four, five, as many as you like! But for the most part they are convinced that there are only three and there can be no more.

(drawing on floor)

VIKKI
But you just said—

SPACE HOPPER
That a small subset of the Peoples think of time as a fourth dimension.

VIKKI
But if they live in a five or whatever dimensional space, then shouldn't it be obvious?

SPACE HOPPER
Not at all.

VIKKI
You're not telling me they have more invisible dimensions!

SPACE HOPPER
No, not like time.

VIKKI
Well then it should be obvious.

SPACE HOPPER
Let's go talk to them and see, shall we?

Scene 3

On a dirt path on Planiturth. A People on a bicircle approaches.

SPACE HOPPER
We're very fortunate. You don't see these often.

VIKKI
See what?

SPACE HOPPER
See that. It's a bicirclist.

VIKKI
The creature with two circles for feet?

SPACE HOPPER
No, it's a machine he's riding.

VIKKI
What does it do?

SPACE HOPPER
It just makes him go faster.

As the BICIRCLIST nears them, the SPACE HOPPER tosses a stick in his path, causing him to lose balance and fall off his bicircle.

BICIRCLIST
What the—you—what the hell did you do that for? You can't just go—you can't—that's—you... Why are you dressed up all funny?

SPACE HOPPER
This is a People.

(whispering to VIKKI)

BICIRCLIST
How did you even get into those costumes? That square thing looks almost invisible from the side it's so skinny. Are you advertising something or something like that? It's very impressive.

SPACE HOPPER
We are visitors from another dimension!

BICIRCLIST
Haha. Really, now? I'd say you're a few months late for Halloween. Still, very impressive. What exactly are you supposed to be?

VIKKI
No, we really are. I'm from Flatland and he's from... er... well, I'm not really sure where he's from, but that's why it looks like I disappear from one side because Flatland is two dimensional.

SPACE HOPPER
Well, kind of.

VIKKI
Oh, not this again.

SPACE HOPPER
No, not this again.

BICIRCLIST
What?

SPACE HOPPER
What what?

BICIRCLIST
You can't actually be from other dimensions. How did you get here?

VIKKI
He's a space hopper.

SPACE HOPPER
I'm a space hopper. We hopped over to this space from another dimension.

BICIRCLIST
No, really. How did you make it look like you're so flat? Is it some kind of illusion?

SPACE HOPPER
Not at all.

BICIRCLIST
Okay, then how did you really get here?

SPACE HOPPER
I just told you.

BICIRCLIST
Nonsense.

SPACE HOPPER
Why?

BICIRCLIST
Because there are no other dimensions.

VIKKI
How do you know?

BICIRCLIST
Do you see any other dimensions? There's this way, this way, and this way, and that's all.

SPACE HOPPER
What if your three dimensions were just a slice through a four dimensional space?

BICIRCLIST
How can it be a slice?

SPACE HOPPER
It would be one very very thin layer in it.

BICIRCLIST
I don't follow.

SPACE HOPPER
Let me tell you about a place called Flatland. Flatland only has two directions: North–South and East–West. It's just one infinite plane. So you can imagine how that plane could be just one slice through a three dimensional space.

BICIRCLIST
Sure, but that's because planes are really thin.

SPACE HOPPER
What do you mean by "thin"?

BICIRCLIST
They, well, you know, they're really thin. They don't have any thickness.

SPACE HOPPER
What is thickness?

BICIRCLIST
It's, I don't know, it's what things that aren't two dimensional have! What are you getting at?!

VIKKI
So, a third dimension?

BICIRCLIST
Yeah, sure.

SPACE HOPPER
So a plane is thin along, say, the Up–Down direction?

BICIRCLIST
Yeah.

SPACE HOPPER
So, what if I were to add another direction.

The SPACE HOPPER draws on the ground with a stick as he explains (Figures 1.2, 1.3, and 1.4).

SPACE HOPPER
Now, this here, call this North–South. And if we draw this one East–West, that's all we can get on the ground because it's basically two-dimensional. But if we draw this direction—this is how you usually do a third direction, right?—that's what you're calling three dimensional. Up–Down.

BICIRCLIST
Sure.

SPACE HOPPER
And now if I were to say there were another set of directions, say the Chalk–Cheese axis, it would be four. Like this.

BICIRCLIST
But that line doesn't go anywhere. It just kind of crosses all the other ones where they cross.

SPACE HOPPER
Yes, but isn't that what the Up–Down line does too?

BICIRCLIST
I guess. But you can kind of imagine where it would go.

SPACE HOPPER
Well, just do that with the Chalk–Cheese line!

BICIRCLIST
That doesn't make any sense. There isn't another direction there.

VIKKI
Oh! Here. Yes it is rather hard to see. You can't really do that on a two dimensional surface very convincingly. But the Chalk–Cheese direction is the one that you don't have any of, right?

BICIRCLIST
Yeah?

VIKKI
And so your space is thin along the Chalk–Cheese dimension like mine is thin along the Up–Down direction!

BICIRCLIST
Yeah but yours actually is thin. Space here isn't thin.

VIKKI
No, but it is! Just it's thin along the Chalk–Cheese dimension and you can't see the Chalk–Cheese dimension.

BICIRCLIST
Yes, but… well… but…

SPACE HOPPER
What if we replace "Chalk" with "Past" and "Cheese" with "Future"? Each space is a tiny slice through time.

BICIRCLIST
I… well, okay.

SPACE HOPPER
So then the fourth dimension would be…?

BICIRCLIST
Time. But time can't be a dimension like the other ones, otherwise you should be able to move through it the same way you move through space. So there's still really only three dimensions.

SPACE HOPPER
But you can move through it. You're in a different slice of space-time from a second ago.

BICIRCLIST
No, but you can't actually move anywhere in it.

SPACE HOPPER
Sure you can. You're just stuck going one direction at a constant speed right now. But that doesn't mean it's not possible to travel through time.

BICIRCLIST
Time travelers don't actually exist, though. That's all just a bunch of science fiction. There was a book—*The Time Machine*—Wells. It's not as if it's real though.

SPACE HOPPER
Did you know that soon after Wells published *The Time Machine*, a People called Alberteinstien invented the theory of relativity and another People called Hermannminkowski formulated this theory in terms of four-dimensional space-time?

BICIRCLIST
I did not.

SPACE HOPPER
Well, after that a lot of Peoples got it into their heads that time was the fourth dimension. The truth is, though, that there are a lot more dimensions than that on Planiturth. Your world has four, five, a hundred, or even an infinite number of dimensions and none of those have to count time!

BICIRCLIST
That's nonsense. I suppose I can see your argument for time as a dimension, but there certainly aren't more than four, if there even are that!

SPACE HOPPER
Oh no, there certainly are!

BICIRCLIST
Fine, then. Show me some.

SPACE HOPPER
Take, for instance, this bicircle of yours. It has seven dimensions.

For further reading, see Appendix A.4.

BICIRCLIST
Okay, one, two, three. Where's the other four?

(turning the handlebars)

SPACE HOPPER
Here's one.

(turning the pedals)

SPACE HOPPER
And here's another.

BICIRCLIST
But those aren't dimensions.

SPACE HOPPER
And there's one more for the front circle and one more for the back circle. That's seven.

BICIRCLIST
You didn't actually move it through any other dimensions, you just moved it in 3-D.

SPACE HOPPER
Yes, but it changed shape.

BICIRCLIST
So?

SPACE HOPPER
So it moved through another dimension.

VIKKI
It did change shape, though! You moved the bit with horns on it and now it doesn't fit back into the same exact space it was before, right? So that movement was through another dimension. If it wasn't, you could just shift it back into its original position through rigid motions in 3-D.

BICIRCLIST
Okay. . . but then where are these dimensions? What are they?

SPACE HOPPER
The Turn-The-Handlebars direction, the Turn-The-Pedals direction, the Turn-The-Front-Circle direction and the Turn-The-Back-Circle direction!

(pointing in directions)

BICIRCLIST
But the wheels and the handlebars don't go in directions. They go in circles.

SPACE HOPPER
They don't go in the North–South or East–West direction, no. But you'll agree that they do turn and you can measure how far they've turned.

BICIRCLIST
Sure.

SPACE HOPPER
Now, as Vikki said, when you rotate the handlebars or the wheels, it changes something about the bicircle. It's true that when you turn the wheels entirely around that it's the same as it started, but that just means that the space is finite in that direction, then it repeats itself. Anyway, these direction or dimensions describe how your bicircle is. How and where.

BICIRCLIST
What do you mean how?

SPACE HOPPER
I mean that you could measure the angle at which the handlebar is twisted and that would tell you how the handlebar is sitting. Similarly, you can measure how far the wheels have rotated.

VIKKI
So you could tell me that your bicircle is six meters North, one meter down, twelve meters to the East, fifteen degrees in the Turn-The-Handlebars direction, negative twenty degrees in the Turn-The-Front-Circle direction, one hundred degrees in the Turn-The-Back-Circle direction, and zero degrees in the Turn-The-Pedals direction, I would know where your bicircle is and how all its bits were turned.

SPACE HOPPER
Exactly. And so you can think of a direction as something that you can measure about an object. You could also, for example, say where this rock is in the color direction. Then if I gave you its coordinates you would know that about it as well.

BICIRCLIST
I suppose you could call those dimensions. It's not the same as the three dimensions that you can go in here, though.

SPACE HOPPER
That's because when you describe a location in three dimensions here, all you're describing is where it is.

BICIRCLIST
Yes, but that's because that's what dimensions describe.

SPACE HOPPER
Not at all! Anything you can measure can be a direction. They don't have to be a visible direction.

BICIRCLIST
Hm, hm. Okay. And is that how you're claiming time to be a dimension?

SPACE HOPPER
Yes! Exactly!

(The SPACE HOPPER bounces excitedly.)

BICIRCLIST
So if I have something that is this big by this big by this big and it's three seconds, uh, long. . . that means what?

SPACE HOPPER
That means the amount of time it takes up is three seconds, like the amount of space in this direction it takes up is one meter.

BICIRCLIST
So it has a start and end time and then it just disappears?

SPACE HOPPER
Yep. Just like your imaginary box disappears if you go too far in this direction.

BICIRCLIST
You mean that that's where the end is.

SPACE HOPPER
Right. And the end in time means that after you go past the end there's no more box there.

The BICIRCLIST stands skeptically puzzling over this.

Act III

Scene 1

VIKKI and the SPACE HOPPER become visible through a mass of intricate shadows of leaves and plants. The shapes in the shadows repeat themselves endlessly in smaller and smaller patterns. They are in the Fractal Forest.

SPACE HOPPER
—call them variables.

VIKKI
So it's basically the number of things you can change. Right?

SPACE HOPPER
Sure, you could think of it like that. But there are other ways you could measure dimension. Like this here has more than one dimension but less than two.

The SPACE HOPPER indicates one of the fractal plants. For further information on fractals, see Appendix A.5.

VIKKI
What? But look, this has at least two because there are two directions. It can't move or anything so I'm not sure how it would have more, but I certainly can't see how you could make it have less than two.

SPACE HOPPER
There's two directions if you're looking at the plane containing it, but not if you stay inside of it.

VIKKI
What do you mean, can't you just... what exactly is this thing?

SPACE HOPPER
It's a fern.

VIKKI
Yeah but it's all very... really... it gets really tiny on the ends.

SPACE HOPPER
It's a fractal. They're very intricate.

VIKKI
What's that?

SPACE HOPPER
Look closer at the fern.

VIKKI
It's got smaller and smaller copies of the same bits on it... do they go on like that forever?

SPACE HOPPER
Yes they do.

VIKKI
So, what makes it a fractal?

SPACE HOPPER
Each of the little bits you were talking about earlier are actually just copies of the entire frond.

VIKKI
And so no matter how close you look at it, it will look exactly the same?

SPACE HOPPER
Exactly.

VIKKI
Okay, so this frond is just made with copies of itself stuck on in place of leaves.

SPACE HOPPER
Yes! And the trees—

VIKKI
Have copies of the trees for branches.

(VIKKI pauses to inspect the trees)

VIKKI
But what does that have to do with dimension?

SPACE HOPPER
Well, dimension doesn't have to be defined by the number of directions or ways you can manipulate an object. And if you look at this fern, it's actually just a bunch of line segments attached at points. And line segments are only one dimensional.

VIKKI
Okay, but then shouldn't sticking a bunch of them together make something that's still one dimensional? Or two dimensional? You can't have one and a half dimensions... each dimension should be a direction or a rotation or something. You can't have half of a direction.

SPACE HOPPER
Half a direction, no. Half a dimension, yes.

VIKKI
That doesn't make any sense.

SPACE HOPPER
I actually have someone I want you to meet here.

VIKKI
Here?

SPACE HOPPER
He's a Koch Snowflake. He isn't a fractal, but his boundary is.

VIKKI
Is everything here a fractal?

SPACE HOPPER
This is the Fractal Forest.

VIKKI
Ah.

SPACE HOPPER
Over here.

VIKKI
Do all of these fractals have a weird number of dimensions?

SPACE HOPPER
Well, many of them have a number you wouldn't immediately expect. Ah! Hello!

KOCH SNOWFLAKE
You! Here! How are you?

SPACE HOPPER
Vikki, this is the Koch Snowflake. Snowflake, this is Vikki.

KOCH SNOWFLAKE
Hello!

VIKKI
Nice to meet you.

KOCH SNOWFLAKE
So where did you pick this one up? She looks rather, um. . . ?

SPACE HOPPER
She's a flatty.

KOCH SNOWFLAKE
Ah, that would make sense.

VIKKI
What?

KOCH SNOWFLAKE
Nothing.

SPACE HOPPER
Anyway, this is Vikki's first visit to the Fractal Forest.

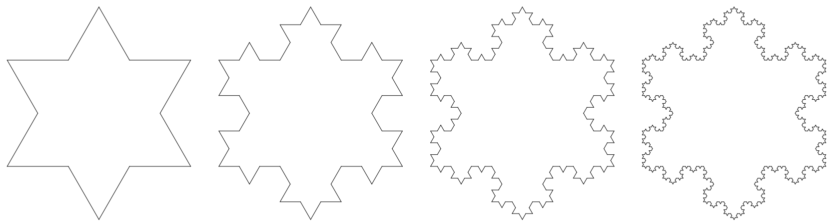


Figure 1.5 A Koch snowflake may be constructed iteratively.

KOCH SNOWFLAKE
How do you like it?

VIKKI
It's very... intricate.

KOCH SNOWFLAKE
That it is.

VIKKI
If you don't mind my asking, what are you?

KOCH SNOWFLAKE
Like he said, I'm a Koch Snowflake. If you take an equilateral triangle, then stick equilateral triangles in the middle thirds of the edges, then even smaller ones in the middle thirds of the resulting edges... ad infinitum, you get this shape.

As he speaks, he demonstrates the triangles he describes (Figure 1.5).

VIKKI
But you're not actually made up of a bunch of copies of yourself.

KOCH SNOWFLAKE
Right. And so I'm not actually a fractal. But my boundary is.

VIKKI
How's that?

KOCH SNOWFLAKE
Here, if you look at just this edge you can see.

SPACE HOPPER
His edge is made of four parts, here, here, here, and here. Each of those parts—

VIKKI
Is a scaled down copy of the whole edge!

SPACE HOPPER
Correct.

KOCH SNOWFLAKE
Yeah!

SPACE HOPPER
His area is two-dimensional, but his boundary actually is dimension about one and a quarter.

For further reading on fractal dimensions, see Appendix A.6.

KOCH SNOWFLAKE
Roughly.

SPACE HOPPER
Roughly.

VIKKI
Okay, so show me how that works.

SPACE HOPPER
In order to count dimensions like that, you need to come up with a new way to define them. You can start with things you already know, then build on it and generalize.

KOCH SNOWFLAKE
That's the interesting part.

SPACE HOPPER
So, let's start with one dimension, because you're not going to get anything interesting out of zero.

VIKKI
Sure.

SPACE HOPPER
Say I start with a 1-D shape, a line, and I want to make it twice as big. I need how many copies?

VIKKI
Two.

SPACE HOPPER
Three times?

VIKKI
Three.

SPACE HOPPER
Right. Now, say I have a 2-D shape, a square, and I want to make it twice as big. That is, I want a new square that is both twice as tall and twice as wide. How many?

VIKKI
Four.

SPACE HOPPER
And three times?

VIKKI
Nine.

SPACE HOPPER
Okay, now this one's a bit more difficult. 3-D, a cube, and I want it twice as big in each direction.

VIKKI
That's, hm, eight?

SPACE HOPPER
Yep. Three times?

VIKKI
Um. . . a lot more than that.

SPACE HOPPER
Twenty seven. If you think about it, your new shape will have three of the original one across, three of the original one tall, and three of the original one back.

VIKKI
Sure.

SPACE HOPPER
Now think about it like this. For the 2-D shapes, you have one—the original, four—the one twice as big, and nine— the one three times as big. The next one would be sixteen.

VIKKI
Square numbers.

SPACE HOPPER
And for the cube?

VIKKI
One, eight, twenty seven... they're numbers cubed. One cubed, two cubed, three cubed.

SPACE HOPPER
So you can say that, in general, the number of copies of the original that you need is the number of times bigger you want it to the power of the dimension.

VIKKI
Um...

SPACE HOPPER
You want a square three times as big, so you have a 2-D shape and three to the second power is nine. You need nine squares.

VIKKI
Right.

SPACE HOPPER
It works with other shapes like triangles too, but those might be a little harder to see.

VIKKI
Sure.

SPACE HOPPER
Now, take this side of the Koch Snowflake. You see it's made up of four smaller copies, right?

VIKKI
Right.

SPACE HOPPER
But it's only three times bigger than the little copies.

VIKKI
Hm. Yes, that's right.

SPACE HOPPER
So how many dimensions does that make it?

VIKKI
Well, it's three times as big, so three to the number of dimensions should be. . . four?

SPACE HOPPER
Exactly.

VIKKI
Wait, but that makes it—

SPACE HOPPER
About one point two six.

VIKKI
But what does that even mean?

SPACE HOPPER
Well, you can see where we got the numbers that—

VIKKI
Yeah, but how is a number like that even useful? It doesn't say how many dimensions or variables or whatever you have. What would that even tell you about something?

SPACE HOPPER
You can think of it as a measurement of how well a shape fills space. For instance, a line segment doesn't fill a lot of space, it's one dimensional, and a square fills a lot more space than the line. It's two dimensional.

VIKKI
Sure.

SPACE HOPPER
Now, if you look at this edge here, you can kind of travel along it like a line. And if you move away from it, like if you move away from a line, you're no longer in the boundary anymore.

VIKKI
So it's kind of like a line. But?

SPACE HOPPER
But it's very, very, extremely, well, crinkly. In fact, if you take any two points on this curve, the distance between the points along the curve is infinite. It fits a lot more stuff in there than a line does.

VIKKI
Hm.

SPACE HOPPER
Okay, here. Look at this. This is called a Sierpinski Triangle. It's built with three copies of itself, like so. Now, if you wanted to construct it, one way you could do that is take an equilateral triangle, cut out the middle, cut out the middle of the remaining triangles, cut out the middles of the triangles left after that, et cetera. The original triangle is a two-dimensional shape, but after you've taken out all of those triangles, you're left with something with zero area.

VIKKI
So it has dimension. . . less than two?

SPACE HOPPER
Right. About one point five nine.

VIKKI
Right. And you are?

KOCH SNOWFLAKE
I am 2-D. But my boundary is about one point two six.

VIKKI
Because it's really crinkly.

KOCH SNOWFLAKE
Exactly.

SPACE HOPPER
There's actually another interesting fractal-like shape called the Mandelbrot Set. He doesn't quite meet some of the definitions of a fractal, but he certainly looks interesting. He's made of a set of points in the complex plane. Weirdly enough, the boundary of the Mandelbrot Set is two dimensional.

VIKKI
But if it's on a plane, wouldn't he be two dimensional? And so shouldn't be bounded by a two-dimensional boundary?

SPACE HOPPER
It's an interesting question. It's all about how you define "dimension"...

Scene 2

A RECTANGLE, roughly the shape of a huge playing card, stands on a dimly lit floor. He has a paint roller on long pole and is painting in circles around a red circle on the floor, making the spot larger and larger. As he paints, he hums and sings. VIKKI sits off in the distance, in a different space, watching sleepily.

As the circle grows, the edge begins to flatten out until it is nearly a straight line. As the RECTANGLE continues to paint, it seems to become flat, then suddenly it becomes apparent that he is not actually painting a circle; he is painting himself into a circle. See Appendix A.7

The RECTANGLE paints around himself until he is standing in a tiny white circle on a solid red floor. He looks over it, surveying his work, then draws a fan out of his pocket, opens it, and fans at the stage impatiently, drying it. Once it is dry, he steps out of the white circle and turns his paint roller around, painting white around the circle, creating a bigger and bigger circle. He continues as he was previously and once he finishes the

white floor, he stands and waits for it to dry so that he can paint it red again.

Suddenly, VIKKI is instead onstage with a small sphere. She builds around the sphere, making it larger and larger. As she works, she hums the tune of the RECTANGLE. The surface goes from small and round to huge and flat as if she is simply making thicker a huge wall. The surface goes from flat to concave and eventually surrounds her in a shrinking spherical cave. Suddenly, VIKKI realizes that she is trapped and stops building and panics. She wakes up.

VIKKI
Ah! What! I. . . !

SPACE HOPPER
Ah what?

VIKKI
I. . . um. . . nothing. I just had a weird dream.

SPACE HOPPER
It happens. All this space hopping will put weird concepts in your mind.

VIKKI
Can I ask you a question?

SPACE HOPPER
Of course!

VIKKI
I had this dream and a rectangle was painting around a circle so that it got bigger and bigger and bigger and eventually it got so big that it surrounded him and trapped him in the inside of a little circle. He stopped when there was just a tiny circle around him so he could barely move. But that doesn't make any sense. Circles don't turn inside out when the get really big.

SPACE HOPPER
Ah, yes. But the circle didn't really turn inside out, did it? The rectangle didn't end up inside the circle he was painting, did he?

VIKKI
No, I suppose not. It's just that the edge of the circle got concave, which edges of circles are never supposed to do because they're always convex and that doesn't make any sense. And besides, if it's concave, the middle must be all weird... I'm not even sure what that would mean because the middle would have been everything outside of the little circle he was in which... I don't get it.

SPACE HOPPER
And what would have happened if he had painted even the circle he was standing in?

VIKKI
Uh...

SPACE HOPPER
Would there be anything left unpainted?

VIKKI
No! And that doesn't make any sense either! The plane is infinite! Everyone knows that!

SPACE HOPPER
Flatland is infinite, Vikki. That doesn't mean all spaces are.

VIKKI
But how can it not be infinite? It's not like it just goes off in one direction for a hundred or so yards and then it just ends. If it ends somewhere there's got to be something on the other side of where it ends. If I go this way until I get to where the space ends, that doesn't mean when I get there that direction ceases to exist! You can't just run out of a direction! And it's not like he painted until he got to a wall and couldn't paint anymore.

SPACE HOPPER
That's not the only way a space can be finite.

VIKKI
It isn't?

The SPACE HOPPER draws a large white sphere in the air.

SPACE HOPPER
What if the rectangle was on here?

VIKKI
You mean like on the surface of that? I... hm.

The SPACE HOPPER pokes the top of the sphere and a red dot appears. The circle slowly extends over the surface until only a small white circle remains at the bottom. The SPACE HOPPER turns the sphere to display the spot on the bottom.

VIKKI
Oh! So then, after that I was building around a sphere until I was stuck inside—that was on the surface of a, um, sphere?

SPACE HOPPER
A 3-sphere, exactly. The "three" indicates the number of dimensions on the surface, so this here is a 2-sphere.

VIKKI puzzles over this for a minute. She draws her own 2-sphere in mid air, then, tries to extend it into the fourth dimension. It starts to expand, faster and faster, consuming the room and exploding in a flash of light.

VIKKI
Uh...?

(the SPACE HOPPER chuckles as VIKKI puzzles over it)

VIKKI
So... The reason that the plane wasn't infinite was that it was on a sphere and what looked flat was actually curved. And then the space that I was in when I got caught was also curved and finite?

SPACE HOPPER
Very good!

Scene 3

VIKKI and the SPACE HOPPER enter to a long, empty table. Asleep underneath the table lies the DOUGHMOUSE, clutching a large lumpy sack.

VIKKI
What's this for?

SPACE HOPPER
They must be around somewhere…

VIKKI
Who?

SPACE HOPPER
This… is Topologica. Now, where have they gone off to?

See Appendix A.8.

VIKKI
What?

SPACE HOPPER
Ah!

The MUD HUTTER and the HARSH MARE enter. The MUD HUTTER is covered in mud from building huts. The HARSH MARE is an angry horse.

SPACE HOPPER
Here! This, my friends, is Vikki Line. She's a flatty.

HARSH MARE
What'd you bring her here for? They're all close-minded and pretty two dimensional. If that. This one looks like she doesn't even have the two.

SPACE HOPPER
Now, now.

VIKKI moves to take a seat at the table.

HARSH MARE
Hey now, you can't sit there!

VIKKI
Oh, I'm sorry! Um?

MUD HUTTER
Why not? It's not like you're sitting there.

HARSH MARE
You can't just go sitting down at our tea party. You haven't been invited!

MUD HUTTER
Would you two care to join us for tea?

HARSH MARE
Shut up!

SPACE HOPPER
We'd be delighted.

They all sit in a row, the HARSH MARE at the right end of the row.

HARSH MARE
Hey! Where's my tea?

MUD HUTTER
Now just a second.

HARSH MARE
Hey Doughmouse! Where's my tea?

The DOUGHMOUSE awakes with a start. He sleepily and trepidatiously peers up at the table above him, inspects the visible portions of those present at the table and yawns.

MUD HUTTER
Give him a minute!

The HARSH MARE kicks the DOUGHMOUSE. The DOUGHMOUSE rises.

DOUGHMOUSE
Hm? Yes? What? What do you need?

HARSH MARE
Tea! I asked for it an hour ago!

SPACE HOPPER
I think you'll find—

HARSH MARE
Tea!

DOUGHMOUSE
All right, all right.

The DOUGHMOUSE pulls a donut shape out of his bag and timidly offers it to the HARSH MARE who throws it onto the table in disgust. It deforms and flattens onto the table into a plate-like ring with a large hole in the middle.

(banging on the table)

HARSH MARE
Move over, move over.

VIKKI
What?

HARSH MARE
Move over a seat. I don't want to sit here.

They do so. The DOUGHMOUSE removes four spheres from his bag and flattens them into saucers, which he places before each of them.

HARSH MARE
This one's dirty. Move over!

They do so. The DOUGHMOUSE continues making saucers. The DOUGHMOUSE takes out donut shapes and forms them into teacups, squishing

the hole off to the side to become the handle and forming a depression in the center, then makes a sugar bowl. He then takes out a donut with two holes in it.

VIKKI
What is that?

DOUGHMOUSE
It's a teapot.

VIKKI
A tea pot?

DOUGHMOUSE
Precisely.

VIKKI
That can't be a tea pot!

DOUGHMOUSE
Why not?

The DOUGHMOUSE begins to deform the two-holed doughnut.

SPACE HOPPER
Vikki, all of these transformations are topological transformations.

DOUGHMOUSE
Right!

SPACE HOPPER
So this—

(he takes a normal doughnut)

SPACE HOPPER
—and this—

(he takes a teacup)

SPACE HOPPER
—are topologically identical. You can continuously deform one shape into the other without poking any holes in it or sticking any parts together.

(he takes a bite out of the doughnut)

VIKKI
Okay, well, sure, but that's not the same as a tea pot! It has two holes. Tea pots don't have holes.

HARSH MARE
Look at this! Tea pots don't have holes! How do expect to pick the thing up?

VIKKI
Well, okay, I guess it's got a handle like the tea cups, but there aren't any other holes through it. The tea would all spill out!

HARSH MARE
Really. How are you going to get the tea through the pot?

VIKKI
Through? Oh, I suppose there's a hole in the top as well.

DOUGHMOUSE
That's not a hole!

VIKKI
Of course it's a hole! It's where you put the tea in!

DOUGHMOUSE
But that hole only goes in. It doesn't come out somewhere else like a proper hole.

VIKKI
A hole can go in and not come out again. Then it's a tunnel.

DOUGHMOUSE
Terminology, terminology. It's a hole.

VIKKI
So it's a hole then.

DOUGHMOUSE
In Topologica, a hole has to go in then come back out somewhere else for it to be a hole. So tell me, how does your tea get out of the pot?

VIKKI
Through the spout.

DOUGHMOUSE
Then that's the other end of your hole.

VIKKI
But that's not really a hole at all. It's more of a tube thing.

DOUGHMOUSE
A tube is just a long, thin hole. Here "long" and "thin" don't matter. Everything can be stretched or bent or compressed to be whatever shape you like as long as you don't make any new holes or close any existing ones.

VIKKI
So the hole in the top and the hole that comes out through the spout—

HARSH MARE
Is a hole! Can we get on with our tea?

The DOUGHMOUSE obliges and finishes constructing the tea set.

VIKKI
OH! They're two ends of the same hole?

The DOUGHMOUSE nods. VIKKI curiously picks up a tea cup and bites it; it crumbles into her mouth as if made of doughnut. The DOUGHMOUSE pours her tea into the bitten cup and she stares at it, expecting it to collapse.

SPACE HOPPER
In Topologica, all objects have multiple existences. They are identical to anything they can topologically transform into. Their use is entirely context dependent.

VIKKI
What makes two shapes topologically equivalent?

DOUGHMOUSE
Two objects are the same if you can transform one into the other without making any new holes or closing up any holes that are already there. And you can't glue parts together, either.

SPACE HOPPER
You can stretch and compress it as much as you like, but the surface stays the same.

MUD HUTTER
Pass me the sugar, would you?

VIKKI does so.

DOUGHMOUSE
Me too.

The sugar is passed around.

HARSH MARE
Over here.

MUD HUTTER
Can I get a bit more?

The SPACE HOPPER motions for the MUD HUTTER to pass it to him.

VIKKI
I would like some too please.

DOUGHMOUSE
Pass it back.

At this point the DOUGHMOUSE's cup is overflowing with sugar, but he continues to add more. The DOUGHMOUSE passes the sugar back to the SPACE HOPPER, who passes it on to the HARSH MARE. A DUCK quietly walks on stage unnoticed. This passing back and forth of the sugar is actually a graph theory problem. Graph theory is introduced in Appendix A.9.

VIKKI
If you're going to add that much sugar, why don't you just add it all at the beginning?

DOUGHMOUSE
Oh but I like asking for the sugar!

MUD HUTTER
He does rather like it when we pass the sugar around.

VIKKI
How many times do you usually pass it around before you can drink?

MUD HUTTER
Well, that depends on how many people we have for tea with us.

VIKKI
How so?

DOUGHMOUSE
Well, the sugar only gets passed between each pair of people one time, so it is just how many different possible pairs of people you have.

MUD HUTTER
You use something called the choose function.

DOUGHMOUSE
It tells you how many ways you can choose a group of some size—

MUD HUTTER
From a larger group of people.

VIKKI
And here you want a group of two people?

MUD HUTTER
Exactly. So if we have x people here, the sugar can get passed x choose two times between unique pairs of people.

VIKKI
Okay, but what is x choose two?

DOUGHMOUSE
That's simple. It's x times x minus one divided by two.

VIKKI
What? Why?

DUCK
Inducktion! Pass the sugar, by the way.

VIKKI
What?

DUCK
You induck! First you show that it's true for the smallest number of people, then you can show that it's true for more people! Sugar, anyone?

The DUCK passes the sugar back to the others, then continues passing it back and forth until he has passed it to or from everybody. It may be beneficial for one of the characters to write what the duck is explaining on the wall as the duck talks.

DUCK
In the smallest case, you definitely need two people to pass the sugar between, so x equals two. Then if you do x times x minus one divided by two you get two times one divided by two which is one which is of course the total possible number of pairings between the two people there.

VIKKI
Right. . .

DUCK
Now for the inducktive step! You assume that that formula is true for if we have x people—

VIKKI
But how do you know that? There's no reason it being true for two people should make it true for more.

DOUGHMOUSE
Shh, he's getting there!

DUCK
—and now we add an x plus oneth person. That x plus oneth person adds x new pairs—one with each of the people who was already there. Then the total number of pairings is x times x minus one divided by two plus x.

VIKKI
Uh—

He writes it out.

DUCK
Which you can see simplifies to be x plus one times x divided by two.

VIKKI
Sure.

DUCK
Which means that the formula still works for x plus one people, right? We just added one to all of the places we had x before because now we have x plus one.

VIKKI
Right. . .

DUCK
Now that means that it works for any number of people!

VIKKI
Hold on, I'm not quite sure about that. You showed that it worked for two people and that if it works for some number of people it works for one more than that number of people.

DUCK
Precisely.

VIKKI
So. . .

DUCK
If it works for two people, it works for three. If it works for three, it works for four. If it works for four, it works for five. If it works for five, it works for six. If it—

VIKKI
Oh! I see! That's very clever of you!

DUCK
Not at all.

VIKKI
So we had ten pairs, then you came along and now we have fifteen? Have you got any milk?

HARSH MARE
Picky, picky! Have we got any milk, indeed! Flatties!

The DUCK exits unnoticed.

MUD HUTTER
Here, Vikki, let me introduce you to someone. He supplies our milk.

The MUD HUTTER walks off and returns with a very flat cow with a twist in its tail that bends over to connect to its nose. MOOBIUS is a Möbius strip, explored further in Appendix A.10.

SPACE HOPPER
That's an interesting beast. What's its name?

HARSH MARE
Moobius.

SPACE HOPPER
And his milk comes in—?

HARSH MARE
Klein bottles.

SPACE HOPPER
I was afraid so.

MOOBIUS approaches VIKKI and begins to circle her and the SPACE HOPPER. VIKKI turns around to watch MOOBIUS but eventually gets dizzy.

VIKKI
Can you stop that? It's terribly disorienting!

MOOBIUS
I suppose so.

(he stands and lets her stare at him)

HARSH MARE
What are you doing? Give the lady her milk so we can get back to our tea party and leave!

MOOBIUS
Not yet! She's looking at me!

HARSH MARE
No she isn't! She's just waiting for her milk!

MOOBIUS
She is!

HARSH MARE
Isn't!

MOOBIUS
Is!

HARSH MARE
Isn't!

MOOBIUS
Is!

HARSH MARE
Isn't!

MOOBIUS
Isn't!

HARSH MARE
Is!

MOOBIUS
Is!

VIKKI
Wait, what did you just say?

HARSH MARE
I said that you were waiting for your milk, then he said that you were looking at him, then I said you weren't, then he said that you were, then I said that you—

VIKKI
I know that! That end bit, though. Moobius changed his mind and agreed with you, but then you changed your mind and agreed with Moobius' original argument, then he changed his mind again—

HARSH MARE
Yes, so?

VIKKI
Do you even know what side you're on?

DOUGHMOUSE
I know what side I'm on.

VIKKI
What?

DOUGHMOUSE
The inside!

HARSH MARE
And you are on the outside.

VIKKI
Uh. . . but what sides are they on?

MOOBIUS
Both sides! Or, really there only is one side. And it is on me.

VIKKI
What are you talking about?

MOOBIUS
I am only on one side. There aren't multiple sides to be had!

VIKKI
What do you mean, like a front and a back?

MOOBIUS
Just that.

VIKKI
But you clearly have a front and a back. There's this side, and there's that side.

MOOBIUS
Those are the same side.

VIKKI
No they aren't! They can't be! Look, this side is over here, then there's an edge, then there's the other side.

MOOBIUS
Only locally. Here, Doughmouse, can you get me a hose?

DOUGHMOUSE
On it.

The DOUGHMOUSE exits and returns with a hose.

MOOBIUS
Here, Vikki. Hose down one side of me so we can tell the two sides apart.

VIKKI
Okay. . .

VIKKI hoses down MOOBIUS, starting at his face. When she reaches his tail, she follows it around, over the twist and continues to his face and the rest of his body. When she is finished, she stands back to look. MOOBIUS is entirely wet on all sides.

MOOBIUS
See?

VIKKI
How did you do that! You cheated somehow!

MOOBIUS
Not at all. You watched what you were doing.

VIKKI
But how… Oh, the twist in your tail! It connects the sides.

MOOBIUS
Exactly.

VIKKI
And so… you only have one side. But… you do still have a front and a back. This side is the front, and this is the back!

MOOBIUS
Nope, because they're the same side.

HARSH MARE
All right, all right, if you want to talk about nonorientable surfaces and all that, get out and leave us to our tea!

The HARSH MARE takes one side of the table and compresses it until it can easily be carried, ignoring the tea cups and saucers that fall to the floor and bounce back to their original donut shapes. He picks the table up with one hand and the DOUGH-MOUSE with the other and walks off huffily.

SPACE HOPPER
Well, Vikki, I guess we'd be best be off. Pleasure meeting you, Moobius.

MOOBIUS
You too.

VIKKI
All right.

Scene 4

VIKKI and the SPACE HOPPER are positioned above a giant disk, looking down at it. This is Hyperbolica, and giant sets of train tracks can be seen running parallel to each other, eventually meeting at the edges of the disk. See Figure A.5.

VIKKI
What are those? And where are we?

SPACE HOPPER
That is Hyperbolica. And those are train tracks.

VIKKI
Train tracks?

SPACE HOPPER
Yep.

VIKKI
But they look odd. The angles are weird and they curve.

SPACE HOPPER
No they aren't.

VIKKI
No, but they do! They very clearly do! That seems terribly inefficient, though. Wouldn't you want trains to go along straight lines? So that they take the shortest distance?

SPACE HOPPER
The tracks look perfectly straight to me.

VIKKI
Straight? Those look like perfect arcs of circles! And shouldn't they be parallel? The pairs of tracks don't look parallel at all.

SPACE HOPPER
Well, sure. They are equidistant—

VIKKI
No, look, they are all going to meet at the edges!

SPACE HOPPER
—it all depends on what you mean by “straight” and “parallel.”

VIKKI
What do you mean “what you mean”? Those lines are curvy. And curvy lines are not straight! And they get closer together at the ends, so they can’t be parallel!

SPACE HOPPER
Yes and no. Some of the tracks aren’t straight, but each pair of the tracks is equidistant.

VIKKI
That’s ridiculous.

SPACE HOPPER
They aren’t really parallel because one of each pair isn’t quite straight.

VIKKI
So, one of the curved lines is straight and the other is curved and they are equidistant but they meet at the edge?

SPACE HOPPER
Edge? What edge?

VIKKI
The edge of the disk… plate… thing.

SPACE HOPPER
Oh… oh, no. No, no, that’s not an edge at all. Hyperbolica has no edge.

VIKKI
But then… what’s the… edge? That you can see? The circle that encloses it all?

SPACE HOPPER
Well, see, that looks like an edge, but you can’t actually get there.

VIKKI
What do you mean?

SPACE HOPPER
Look at the tracks.

VIKKI
They're equidistant. Okay, sure. Except that they look like they get closer as they go towards this nonexistant edge.

SPACE HOPPER
So what does that mean for the trains?

VIKKI
They'd have to shrink. Which also makes no sense, by the way.

SPACE HOPPER
They do?

VIKKI
Yes, they do! The trains that are closer to the edge are smaller than the ones in the middle! You can see it! There!

SPACE HOPPER
You shouldn't believe everything your eyes tell you. They can trick you, you know. Why don't we go into Hyperbolica and meet some of its inhabitants? That might help.

VIKKI
Are they just as nonsensical?

Scene 5

In Hyperbolica. Everything is almost normal, but slightly curved. Normally straight lines curve towards the center in a way that is barely noticeable yet disconcerting. There is no edge of the world visible. They watch a train pass. As it goes off into the distance, the SPACE HOPPER peers after it. Hyperbolic space is explained further in Appendix A.11.

SPACE HOPPER
You're right! See, they do get smaller as they get further away!

VIKKI
Don't be ridiculous. Things just look smaller when they get further away. Like that. That's just perspective.

SPACE HOPPER
Are you sure?

VIKKI
Yes!

SPACE HOPPER
That's not what you said when we were looking at the space from outside.

VIKKI
But—oh. This is really the same space?

SPACE HOPPER
It really is. Let's go see if we can find some of its inhabitants.

VIKKI
Sure. Oh! It's a squarrel!

The SPACE HOPPER momentarily leaves into a third dimension, then returns right by the squarrel, catching it.

SPACE HOPPER
Are you sure?

VIKKI
Of course. It's got right angles and all.

SPACE HOPPER
Have you counted the sides?

VIKKI
No… But it's got to have four sides. You can't get more than four with right angles.

SPACE HOPPER
Are you sure?

VIKKI
Of course I'm sure! What are you getting at?

SPACE HOPPER
Indulge me. Count the sides while I spin it. You can tell when you've seen them all when I get back to its face.

VIKKI
Fine. One. Two. Three. Four. Five? What? A square can't have five sides!

SPACE HOPPER
Maybe it's a right-angle pentagon.

VIKKI
Nonsense! Pentagons can't have right angles!

SPACE HOPPER
This squarrel begs to differ.

VIKKI
But... oh. It's got to do with all those curvy straight lines we saw, doesn't it? That's crazy.

SPACE HOPPER
Actually, let's see... this plant has triangular leaves. Equilateral triangles, in fact.

VIKKI
Sure. Ooh, that's a bit off too, isn't it. The angles are a bit too small.

SPACE HOPPER
There must be a bigger one somewhere... Aha. I'll make it out of these sticks. See, these three are all the same length.

VIKKI
Sure.

SPACE HOPPER
But when I lay them out like this...

VIKKI
That's not right at all! All of the angles are too small! Don't they all have to add up to one eighty degrees?

SPACE HOPPER
Not here. In fact, none of the triangles here have that property. And the bigger they are, the worse it is.

VIKKI
That's so weird! But it's because all of these lines are curved, right?

SPACE HOPPER
Oh, no, no, not at all. That's a very Euclidean idea.

VIKKI
Euclidean?

SPACE HOPPER
Geometry on Euclidean planes. Like Flatland. Not curved.

VIKKI
What does. . . ? Never mind. How can the train tracks be equidistant but get closer to each other at the ends?

SPACE HOPPER
Think of it this way. What if you assume that things can, in fact, shrink? They don't, but assume that's the case.

VIKKI
They shrink as they get closer to the edge?

SPACE HOPPER
Sure.

VIKKI
By how much?

SPACE HOPPER
It doesn't really matter, but they shrink by two to the power of the distance from the edge. Assuming this disk we looked at had radius one. The important part is that they disappear when they actually get to the edge. They are zero times their original size.

VIKKI
Okay. . .

SPACE HOPPER
So, how far is it for a creature to go from here to the edge?

VIKKI
Just the radius… wait. You're saying distances change?

SPACE HOPPER
Yep.

VIKKI
Hm. I'm not sure, then. Okay, let's say there's some creature that can take huge steps and it starts in the middle. All of its steps are the same length and the first one takes it halfway to the edge. Then it steps again, which takes it…

SPACE HOPPER
Halfway between where it is and the edge.

VIKKI
Sure. So after n steps, the distance, if you're looking from above, between it and the edge is… one over two to the n?

SPACE HOPPER
Exactly. It keeps dividing the distance in half with each step.

VIKKI
But it never gets there.

SPACE HOPPER
Right.

VIKKI
So this edge is infinitely far away?

SPACE HOPPER
Right.

VIKKI
So no one can ever get there.

SPACE HOPPER
It's not much of an edge, then, is it?

VIKKI
I suppose not. . .

SPACE HOPPER
It's hard to get a feel that it's infinite from outside the space, isn't it?

VIKKI
Yes. Very. Still, I could see an edge! It's just. . . infinitely far away.

SPACE HOPPER
Not a very good edge, then, is it? Hello, sir!

Enter a 120-degree angled HEPTAGON.

HEPTAGON
Sir!

SPACE HOPPER
How do you do?

HEPTAGON
Very good, sir.

SPACE HOPPER
I am introducing this young lady to your fine land here and I think she may have a few questions. Do you mind?

HEPTAGON
Very good, sir. My lady?

VIKKI
Well, um, I was just wondering if you ever notice things getting smaller.

HEPTAGON
Smaller?

VIKKI
You know, shrinking? When you get closer to the edge?

HEPTAGON
What edge is this?

VIKKI
You know, the edge of the world?

HEPTAGON
Ah, that! Ha, there's no edge. I don't know where you've been hearing those stories, but the world doesn't got an edge!

VIKKI
No, but there is!

HEPTAGON
Sorry, young lady, that's just a story parents tell their children. Be good or we'll throw you off the edge. There ain't no real edge nowhere.

VIKKI
But if you go really far west... or any direction, really...

HEPTAGON
It just keeps going. You never reach no edge.

VIKKI
Are you sure?

HEPTAGON
Sure as I am I have got seven sides. Ain't that right, sir?

SPACE HOPPER
Yes, Vikki, he does know what he's talking about.

HEPTAGON
So sorry to leave you in such a rush, but I've got somewhere I've got to be. Enjoy your visit! Sir, madam.

SPACE HOPPER
Good bye, sir!

Exit HEPTAGON.

SPACE HOPPER
Well, you see?

VIKKI
Just because he can't see the edge doesn't mean there can't be one.

SPACE HOPPER
Have you ever seen an optical illusion?

VIKKI
Well, sure, but. . . oh.

SPACE HOPPER
You can’t always trust what you see. Just because you saw an edge doesn’t mean there has to be one.

VIKKI
But that wasn’t an illusion! The whole world is clearly contained in a giant circle!

SPACE HOPPER
And where is that circle? Go bring me to it.

VIKKI
I. . . well, that’s difficult.

SPACE HOPPER
Then how do you know it’s there?

VIKKI
Because I saw it!

SPACE HOPPER
What if I say I didn’t see it? Then how do we know?

VIKKI
But you had to have seen it too!

SPACE HOPPER
I saw no such thing.

VIKKI
Ohhh, you’re just being difficult.

SPACE HOPPER
Not at all.

VIKKI
This place makes no sense. It has curvy lines that are actually straight, and lines next to those lines that actually are curvy, it has parallel lines that aren't equidistant, it has an edge that isn't actually an edge at all, and the trains don't even take the shortest distances between places!

SPACE HOPPER
Shortest distances? Why wouldn't they?

VIKKI
Because they go in curves and the shortest distance between two points is a straight line!

SPACE HOPPER
But they do go in straight lines.

VIKKI
You—but—they don't look like it!

SPACE HOPPER
Sure they do.

VIKKI
And the squarrel! With five sides! And polygons change their angles if you make them bigger or smaller. How could one possibly live in a place like this without going mad?

SPACE HOPPER
You're the one getting mad, Vikki.

VIKKI
But you can't—! Sorry. This whole place is just a bit much.

SPACE HOPPER
Do you want to go back to Flatland?

VIKKI
Yes, and no. I do miss my parents. They must be terribly worried about me. And things make a lot more sense there. But exploring all of this was very interesting.

SPACE HOPPER
I can take you back, but there's something I want to show you first. . .

Act IV

Scene 1

VIKKI and the SPACE HOPPER hover above Flatland. They can see small polygons going about their daily life below.

VIKKI
Wow. You really can see everything from here. You can see around all the walls.

SPACE HOPPER
That's the magic of a third dimension.

VIKKI
And there's my house! They must be so worried about me. I wonder what they're talking about.

SPACE HOPPER
Before you go back, there's something you have to see.

The SPACE HOPPER produces a mirror and shows it to her.

VIKKI
What's—? Oh, wow. You can see a third dimension in this mirror... That's very weird.

SPACE HOPPER
You're only a line in the plane that Flatland sits in. All of the women are.

VIKKI
I never imagined I was a pentagon.

SPACE HOPPER
A very nice pentagon at that.

VIKKI continues to stare into the mirror.

SPACE HOPPER
I can take you back now if you like. This one is your room, right?

VIKKI
Yes, but—

VIKKI slips back into the plane of Flatland. The SPACE HOPPER, hovering above the plane is no longer visible. She is in her bed, somewhat disoriented.

VIKKI
Space Hopper? Are you there?

After taking a minute to compose herself, VIKKI quietly exits her room into the living room, where both of her parents are sitting, clearly worried. When she appears, GROSVENOR's worry turns to rage.

VIKKI
Uh, hello mother—

GROSVENOR
Victoria Line! Your mother and I are very disappointed, very disappointed in you! You have no right to go running off with some random stranger who you just met! And to not even tell us where you were going! The nerve!

JUBILEE
Vikki, dear—

GROSVENOR
You are a disrespectful and irresponsible daughter and I am ashamed to have had you living in this house! Get out!

VIKKI
I'm sorry, I didn't—

GROSVENOR
I don't care what excuses you have, young lady, that was unacceptable behavior and you know it!

JUBILEE
Grosvenor, please!

GROSVENOR
I will not tolerate such immaturity from someone of your age in this house. You know better than to act that way!

VIKKI
Dad, I'm really sorry—

GROSVENOR
Victoria Line, I do not believe that—

JUBILEE
Grosvenor! Vikki, perhaps it's best if you go to your room for a little while.

GROSVENOR
I am not done with her!

VIKKI returns to her room. The sounds of GROSVENOR and JUBILEE arguing can still be heard. VIKKI sits on the bed uncomfortably and tries to look up but finds that there is no up to look towards. Beginning to wonder if there ever really was a third direction, or a fourth or fifth, she goes to her desk and takes out a pen and paper.

VIKKI
Dear diary. I had the most incredible adventure. You'll never believe where I went. . .

Appendix A

Glossary

A.1 A. Square

A. Square (his first name is never given) is the main character and narrator of *Flatland: A Romance of Many Dimensions*. He is a square inhabitant of Flatland and though initially skeptical of the existence of higher dimensions, the Sphere brings him above Flatland and convinces him of the existence of a third.

Upon his return, he attempts to convince others of the existence of higher dimensions, but is imprisoned for life for heresy. His belief in higher dimensions yet inability to convince the other inhabitants of Flatland of them results in his being shunned and considered an embarrassment.

He is Victoria Line's great-great-grandfather.

A.2 Dimension

The dimension of an object is a measure of topological properties. It is most commonly understood to be the number of directions an object can be measured in. For example, a point is zero-dimensional, a line is one-dimensional, a square is two-dimensional, and a cube is three-dimensional. Objects of greater than three dimensions are often referred to as "hyper-" such as a hypersphere or a hypercube.

To create an object of a higher dimension, one can expand a lower dimensional object in a new direction, such as a line segment extended perpendicular to itself becomes a square, a square extended vertically becomes a cube, and so on to increase dimension arbitrarily.

There are other ways to define "dimension," as in Sections A.4 and A.6.

A.3 Zero Dimensions

In a zero-dimensional universe, there are no directions. As such, the universe is a single point. That point takes up all of the horizontal, vertical, and any other possible space that exists and thus there is nothing in this universe that is not the point. The King of Pointland, in order to exist somewhere, must exist at this point and thus he is the point. Because the universe contains nothing that is not the point, he is everything he experiences.

A.4 The Bicirclist's Dimensions

The seven dimensions that the Space Hopper shows on the bicircle are clearly not all directions that can be traveled in. Here, dimensions measure things that can change and be measured. For example, suppose that we have a plane that can vary in temperature. To give all possible information about a point, we would have to give not only its x-coordinates and y-coordinates, but also its temperature. These three dimensions are not all spatial. We can visualize this as a three-dimensional map, where each (x, y) point has a height, corresponding to its temperature.

Though it is significantly more difficult (and not necessary!) to visualize this in higher dimensions, the connection still applies. For instance, if we had a space with three directions with temperature and humidity, we would have five numbers describing every point (i.e., x, y, z, temperature, humidity) and a five-dimensional space. In this sense, dimensions are essentially variables, and the number of variable qualities that an object or space has determines its dimension.

The Bicirclist's bicircle is unusual in that many of its dimensions seem to run in circles. While this is the case, it does not affect the number of dimensions. It simply means that a particular direction is finite—it can only be moved a certain amount in that direction before it gets back to having been moved not at all in that direction. In fact, we could say that the space (don't try to visualize it as dimensions being directions) is finite in that dimension. For example, our humid room is finite in the humidity direction—at some point it can't be any more (or less) humid.

For another example, we can consider a light. We obviously have the three spatial coordinates or directions. We additionally have the vertical and horizontal tilt of the light, two more dimensions the light may be moved in. We also have the wavelength of the color light that the fixture is

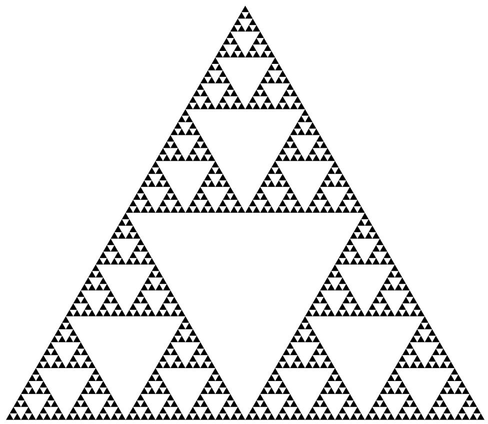
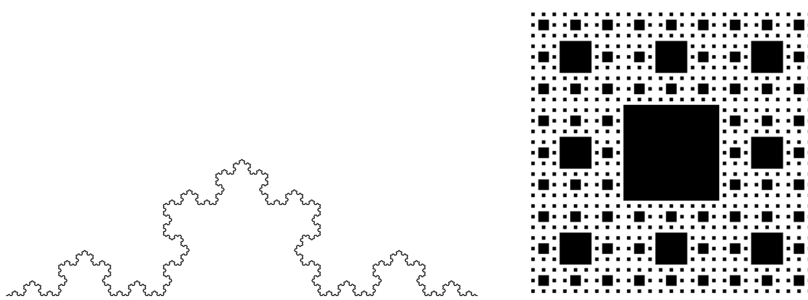


Figure A.1 The Sierpinski Triangle is an example of a fractal.



a. The Koch Curve **b.** The Sierpinski Carpet

Figure A.2 The Koch Curve and the Sierpinski Carpet also show self-similarity.

producing as well as its intensity. As such this light has seven dimensions. Note that given numbers for all of these variables, we know everything there is to know about the light.

A.5 Fractal

A fractal is an object that is self-similar. This means that a small part of the object looks just like the whole object. For example, the Sierpinski Triangle is made of three copies of itself which, as you can see, are themselves made of three copies of the large triangle (Figure A.1).

Some other examples are the Koch Curve (Figure A.2a) and the Sierpinski Carpet (Figure A.2b).

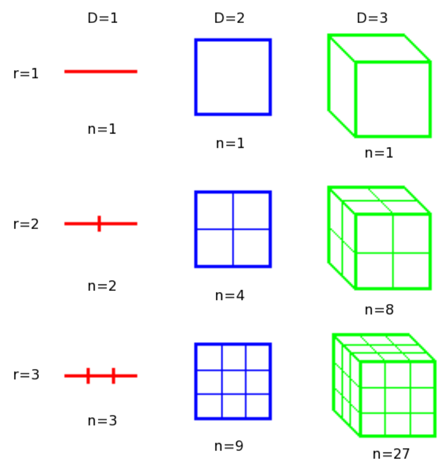


Figure A.3 Hausdorff dimensions of basic shapes may be calculated by inspection.

A.6 Hausdorff Dimension

Fractals are characterized by their Hausdorff dimension—a dimension measurement that is often the same as the topological dimension in well-behaved objects, but can yield unusual numbers when applied to objects like fractals. Most basically, the Hausdorff dimension is a measurement of how well the object fills space. For example, we can think of a square filling more space than a line, but a cube filling more space than a square. All three of these things have Hausdorff dimension equal to their topological dimension.

Fractals, however, can have noninteger Hausdorff dimension. Consider this new way of understanding dimension:

Given an object, if we want to make it r times larger, we need n copies of it, as in Figure A.3.

Then the number of copies of the original object that we need can be described by the relationship. When we apply this to fractals, we can get some surprising results. For example, the Koch curve in Figure A.2a is made of four copies of itself. However, it is only three times as big. So, if $n = 4$ and $r = 3$, then $r^D = 4$. Solving for D we find that this fractal has Hausdorff dimension 1.2619.

The other fractals shown have dimensions 1.5849 (Sierpinski triangle; Figure A.1) and 1.8928 (Sierpinski carpet; Figure A.2b).

How does this relate to space filling? If we consider the Sierpinski triangle and Sierpinski gasket, they both have area zero. Thus they are filling noticeably less space than a normal two-dimensional object. However, they are more than a line or a curve. The Koch curve seems like it should be one-dimensional—it is simply a curve. However, it has the unusual property that any section of it, no matter how small, has infinite length. So perhaps it is more than a one-dimensional line.

The Cantor set, which is created by repeatedly removing the middle third of any line segments in it, has length zero, but it has an infinite number of points. Thus it can be thought of to fill more than zero dimensions but less than one.

More formally, Hausdorff dimension is defined as

$$\inf \left\{ \sum_i r_i^d : \text{there is a cover of } S \text{ by balls with radii } r_i > 0 \right\}.$$

This yields interesting measurements of dimension for nonfractal shapes as well: the boundary of the Mandelbrot set has Hausdorff dimension two, despite the set itself being two-dimensional.

A.7 Curved Space—the Ant's Sphere

The ant painting the two-dimensional space until it is fully covered may seem nonsensical initially, but more logical once we see that the space is actually the surface of a sphere. Because it is on the surface of a sphere, the space is actually finite, even though locally it is like a plane. As the surface of a sphere, the space has a positive curvature, which affects geometry on it. For example, a circle can grow to become a straight line then eventually shrink as it moves past the largest part of the sphere.

Spaces of higher dimensions can be curved as well. Though difficult to visualize, the ball that Vikki creates is actually the surface of a hypersphere (four dimensional). That is, her ball grows until it reaches the widest part of the hypersphere, becomes flat, and eventually shrinks again with her inside of it. Here we can't see any curvature, but it is present in a fourth dimension.

A.8 Topology

Topology is the study of properties of objects that are preserved through various transformations. (Topology may also be defined in terms of set op-

erations and open sets, but we will not be exploring that here.) In order to be topologically equivalent, two objects must be able to be continually deformed to become the same object. That is, they can be deformed, twisted, and stretched, but cannot be torn or have parts connected.

Often when we discuss topological objects, one important property is their genus. The genus of a surface is the number of holes in it (more formally, it is the number of nonintersecting simple closed curves that can be drawn on the surface without separating it). If two objects have different genus, they cannot be topologically equivalent, as we cannot increase or reduce the number of holes in a surface through continuous transformations. However, if two objects have the same genus, this does not necessarily make them topologically equivalent; for example, both the Klein bottle and the torus have genus one.

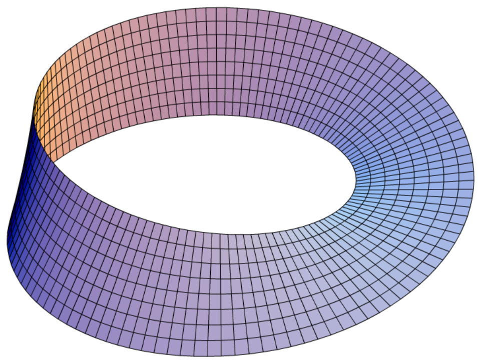
A.9 Graph Theory

Graph theory is the study of the properties of graphs: objects comprised of a set of vertices and a set of edges between them. Graph theory is used briefly at the topological tea party to determine the number of edges possible between n vertices, where n is the number of people present. Determining this number is relatively straightforward as done in the text: a graph on one vertex contains no edges; each time we add a vertex it can be connected to all of the previously present vertices and as such there are $n-1$ more edges in the complete (containing all possible edges) graph on n vertices than there were in the complete graph on $n-1$ vertices. Using this, we can come up with an expression for the number of edges in a complete graph on n vertices: the sum from one to n of $i-1$, or $n(n-1)/2$.

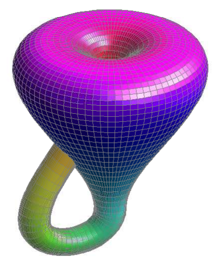
A.10 Möbius Strip

The Mobius strip is a one-sided surface that can be created by taking a long rectangle, twisting one end one half turn, then attaching the two ends, forming the object in Figure A.4a.

This object also only has one edge and is said to be nonorientable. Nonorientable surfaces do not have a front and back, inside and outside, or other "sides" that are generally present in most objects. Another example of this is the Klein bottle. The Klein bottle only has one side, so for it the inside is the same as the outside. In Figure A.4b, the Klein bottle appears to



a. This is an example of Möbius strip.



b. This shows one half of a Klein bottle embedded in 3-space. This cross section shows the "interior" and "exterior" and how they are connected to be the same space.

Figure A.4 The Möbius strip and Klein bottle are examples of nonorientable surfaces.

intersect itself, but this "intersection" actually utilizes a fourth dimension and thus the Klein bottle does not actually intersect itself.

A.11 Hyperbolic Plane

The hyperbolic plane is another example of a curved space. Hyperbolic spaces have negative curvature. Unlike with the sphere, the curvature of this space does not make it finite.

The Poincaré hyperbolic disk is a common representation of the hyperbolic plane and the representation that Vikki and the Space Hopper see when outside the space. Figure A.5 is an ideal tiling of the hyperbolic plane with triangles. In this model, straight lines appear to be curved. The edges of all of these triangles are parallel (and straight lines) and hence the triangles have a sum of interior angles equal to zero degrees.

Geometry on hyperbolic planes is unusual due to the fact that Euclid's fifth postulate (the parallel postulate) no longer holds and given a line and

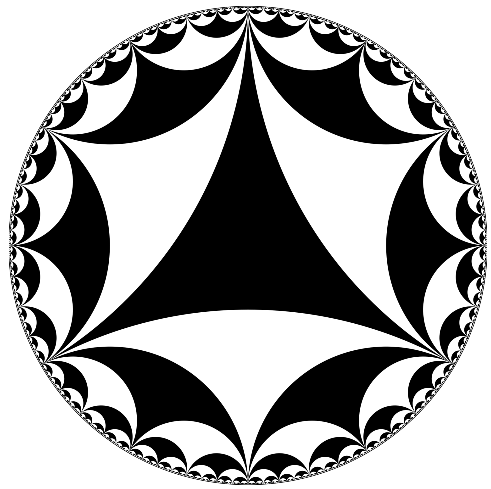


Figure A.5 This shows the ideal triangular tiling of the hyperbolic plane. All sides of the triangles are straight, parallel lines that extend toward infinity. The triangles each have a sum of interior angles equal to zero.

a point not on the line, there are an infinite number of lines through that point that are parallel to the original line. All of these lines diverge from the original line in at least one direction—two of them will converge toward the original line but not intersect it (two of the sides of a triangle above are parallel and do not intersect, but converge to the same point).

The study of hyperbolic geometry is very applicable to physics. Specifically, when black holes bend space around them, they cause the space to be hyperbolic.

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