2012

Counting Vertices in Isohedral Tilings

John Choi
Harvey Mudd College

Counting Vertices in Isohedral Tilings

John Choi

Nicholas Pippenger, Advisor

Arthur Benjamin, Reader

May, 2012

HARVEY MUDD
COLLEGE
Department of Mathematics

Abstract

An *isohedral tiling* is a tiling of congruent polygons that are also transitive, which is to say the configuration of degrees of vertices around each face is identical. Regular tessellations, or tilings of congruent regular polygons, are a special case of isohedral tilings. Viewing these tilings as graphs in planes, both Euclidean and non-Euclidean, it is possible to pose various problems of enumeration on the respective graphs. In this paper, we investigate some near-regular isohedral tilings of triangles and quadrilaterals in the hyperbolic plane. For these tilings we enumerate vertices as classified by number of edges in the shortest path to a given origin, by combinatorially deriving their respective generating functions.

Contents

Abstract **iii**

Acknowledgments **ix**

1 Introduction **1**
1.1 Regular Tessellations and Isohedral Tilings 1
1.2 Hyperbolicity 2

2 Isohedral Tilings of Triangles **5**
2.1 Feasibility 5
2.2 Case Study: $\{3, [x, y, y]\}$ 5

3 Isohedral Tilings of Quadrilaterals **11**
3.1 Feasibility 11
3.2 Case Study: $\{4, [x, w, y, w]\}$ 12

4 Future Work **17**

Bibliography **19**

List of Figures

1.1 M.C. Escher, *Circle Limit IV* 2

2.1 A $\{3, [7, 6, 6]\}$ tessellation 9

3.1 A $\{4, [5, 4, 4, 4]\}$ tessellation 15

Acknowledgments

I would like to thank my advisor, Professor Nicholas Pippenger, and my second reader, Professor Arthur Benjamin. I would also like to thank Claire Connelly, the Mathematics Systems Administrator, for her technical work and overall patience.

Chapter 1

Introduction

A tiling of polygons is said to be *isohedral* if all of its faces are congruent and also have the same configuration of degrees in the vertices around them. A special case for isohedral tiling is regular tessellations, or tilings of congruent regular polygons. Examples of regular tessellations include the Platonic solids and tilings of equilateral triangles in the Euclidean plane.

1.1 Regular Tessellations and Isohedral Tilings

Coxeter and Moser (1980) and Paul and Pippenger (2011) denoted regular tessellations with the notation $\{p, q\}$, which represents a tiling of p-gons with q of them meeting at each vertex, for $p, q \geq 3$. Different pairs $\{p, q\}$ correspond to tessellations in different kinds of planes. If $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$, we have the Platonic solids, which can be seen as tessellations on a spherical plane; if $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, we have regular tessellations in the Euclidean plane, which involve regular triangles, squares, or hexagons; for $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, we have regular tessellations in the hyperbolic plane, in which all edges have the same hyperbolic length and all faces have the same hyperbolic area (Brannan et al., 1999).

In particular, geometries of the hyperbolic plane can be modeled using the Poincaré disk model. This model depicts the hyperbolic plane in a circle on the Euclidean plane, where hyperbolically congruent figures look smaller the farther away they are from the center of the circle. An example of such a depiction is shown in Figure 1.1.

Extending the notation by Coxeter, we define the notation to be used in

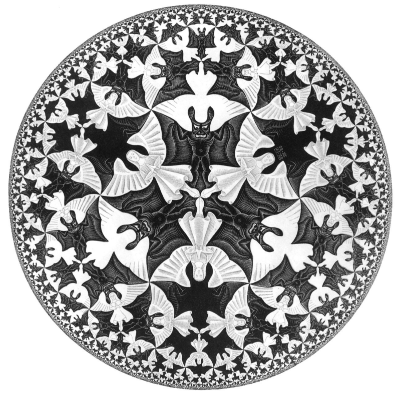

Figure 1.1 M.C. Escher, *Circle Limit IV*, 1960 (Ernst, 2007).

this paper for isohedral tilings of p-gons. This notation shall be

$$\{p, [q_1, \ldots, q_p]\},$$

which indicates a tiling of congruent p-gons where the vertices of each p-gon have degrees $q_1, q_2, \ldots, q_p$, starting at a suitable vertex and going around counterclockwise. Under this notation, we can consider Figure 1.1 to be an artistic representation of a tiling of the form $\{3, [6, 8, 8]\}$. We note that when $q_1 = q_2 = \cdots = q_p$, the regular tessellation $\{p, q_1\}$ arises as a special case.

Paul and Pippenger derived generating functions that enumerate the number of vertices according to shortest distance, defined by number of edges in the shortest path from a given origin, for regular tessellations in Euclidean and hyperbolic geometries. I seek to generalize this result by looking at isohedral tilings of p-gons, denoted by $\{p, [q_1, \ldots, q_p]\}$, with a vertex of degree q_1 chosen as the origin without loss of generality. As a starting point, we analyze cases when $p = 3$ and $p = 4$.

1.2 Hyperbolicity

One useful question to answer going into this investigation is "When is $\{p, [q_1, \ldots, q_p]\}$ hyperbolic?" For the purposes of this paper, it will suffice to propose the condition in which this happens without providing a rigorous geometric proof. This condition is as follows: if the internal angles $\frac{2\pi}{q_i}$ add up to less than $\pi(p-2)$, the expected sum of internal angles in a

p-gon in the Euclidean plane, the geometry must be hyperbolic. Tweaking the inequality a bit, we conclude that $\{p, [q_1, \ldots, q_p]\}$ is hyperbolic if and only if $\frac{1}{q_1} + \cdots + \frac{1}{q_p} < \frac{p-2}{2}$. Note that when $q_1 = q_2 = \cdots = q_p$, this reduces to the condition $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$ for regular tessellations in the hyperbolic plane. Finally, we note that as with regular tessellations, tilings will be on the sphere if the inequality is reversed and on the Euclidean plane if it is an equality.

Chapter 2

Isohedral Tilings of Triangles

Consider when $p = 3$. Then we have a $\{3, [q_1, q_2, q_3]\}$ tiling.

2.1 Feasibility

We note that when $q_1 = q_2 = q_3$, we get the special case of regular hyperbolic tessellations, where q_1 can be any integer greater than 6.

Suppose that $q_i \neq q_j$ for distinct $i, j \in \{1, 2, 3\}$. Call the remaining degree q_k. Consider triangles meeting at a vertex of degree q_k. We see that going around, the degrees of the perimetral vertices of these triangles must alternate between q_i and q_j. This means that q_k must be even for $\{3, [q_1, q_2, q_3]\}$ to be a feasible tiling. If q_k were odd, the alternating behavior would force one of the vertices to simultaneously have degrees q_i and q_j, which is a contradiction since vertices should have unique degree. This holds true for any pair (i, j).

In terms of distinctness and parity, this is a necessary condition for $\{3, [q_1, q_2, q_3]\}$ to denote a feasible tessellation. Thus we are able to disregard tilings which do not fit this condition, as they will be unfeasible.

2.2 Case Study: $\{3, [x, y, y]\}$

We consider a special case, when $q_2 = q_3$, and derive generating functions for vertices by generation from an origin of degree q_1. Note that if $q_1 \neq q_2$, then q_2 must be even; however, if $q_1 = q_2$, no such condition is necessary and the tiling is simply a regular tessellation. For simplicity in argument, we assume that $q_1, q_2, q_3 \geq 6$. Take some tiling $\{3, [x, y, y]\}$ where x and y satisfy these conditions.

2.2.1 Derivation

To derive the generating function that counts vertices of this tessellation by generations, we follow the strategy outlined in Paul and Pippenger. First, we define some terms that will be useful: if a vertex A in generation n is adjacent to vertex B in generation $n-1$, A is B's *child*; B is A's *parent*. Vertices in the same generation are *siblings*. This being a tiling of triangles, each vertex in any generation greater than 0 is connected by edges to two of its siblings (Paul and Pippenger, 2011). Call such edges between siblings *fraternal edges*. Since each vertex in a generation has two fraternal edges, we can deduce that these edges form a cycle around the origin, and the cycles created this way are nested—the further down the generation, the longer the cycle, and each subsequent cycle encircles previous cycles. Thus the generations in this type of tessellation can be completely represented by these cycles, which makes them easier to visualize.

Paul and Pippenger proved that in a tessellation where each vertex has at least four edges that are not fraternal edges, no vertex will have three or more distinct parents. Since we assumed that $q_1, q_2, q_3 \geq 6$, taking out the fraternal edges still leaves four or more edges at every vertex. Thus, each vertex can have up to two parents, and since there are two possible degrees for the vertex in $\{3, [x, y, y]\}$, we can classify four types of vertices with regard to number of parents and degree: $(1, y)$, $(2, y)$, $(1, x)$, $(2, x)$. Note that each of the children of these vertices must also be one of these four types, so one might imagine that there is an underlying recurrence relation to be found through looking at the configuration of types of children connected to a generic vertex of each type.

It turns out that there are two varieties of the $(1, y)$ vertex; a $(1, y)$ vertex with a parent of degree y is associated with a different configuration of children from that of a $(1, y)$ vertex with a degree x parent. As for the siblings of the remaining three types of vertices, there is only one variety each: the $(2, y)$ type vertex will have a degree x sibling on one side and a degree y sibling on the other, and since we can consider mirror images of configurations, the orientation of the siblings does not matter; the $(1, x)$ type vertex will have a degree y vertex as a sibling on either side; and the $(2, x)$ vertex will also have a degree y vertex as a sibling on either side.

For convenience, let us call $(1, y)$ vertices with degree x parent type A, $(1, y)$ vertices with degree y parent type B, $(2, y)$ vertices type C, $(1, x)$

vertices type D, and $(2, x)$ vertices type E.

We can deduce the types of the child vertices for each type of parent. For a given parent vertex X, its two children on the far left and far right must have two parents. Consider its leftmost child L, without loss of generality. Call the sibling of X in the same direction as this child Y. Since there is an edge between X and L, and also an edge between X and Y, there must be an edge between L and Y, since all faces are triangles and the existence of a vertex between L and Y also connected to X would contradict either that L is the leftmost child or that Y is a sibling of X. Finally, since Y is in the same generation of X which is a parent of L, Y must also be a parent of L. Moreover, all the other children of X must have one parent, since they must have siblings on either side that are also connected to X in the cycle of their generation. If a child that is not on the far left or far right has a parent other than X, a crossing over of edges will have to occur, which is not possible in a planar graph.

Furthermore, looking at the triangles formed by the edges that comprise the cycle of each generation, we can deduce the degree of each child vertex. If the parent is of degree x, all of the children will have degree y; if the parent is of degree y, the degrees of the children will alternate between x and y, the orientation of the alternation depending on the degrees of the parent's left and right siblings. Following this logic, it is possible to completely determine the types of vertices of the children of each type of parent vertex. The result of this is shown below.

A	$E, B, (D, B), E$	the sequence (D, B) repeated $\frac{y-6}{2}$ times
B	$C, D, (B, D), C$	the sequence (B, D) repeated $\frac{y-6}{2}$ times
C	$C, (D, B), E$	the sequence (D, B) repeated $\frac{y-6}{2}$ times
D	$C, (A), C$	the type (A) repeated $x - 5$ times
E	$C, (A), C$	the type (A) repeated $x - 6$ times.

From these relations, and the fact that generation 1 has x type A vertices, we can deduce the recurrence relation and initial condition among the types of vertices in $\{3, [x, y, y]\}$. For each type, we look at which types can be its parent and how frequently they can be so. For the types with two parents, we halve the number of parents to account for double-counting. Then, if we let $A(n)$ be the number of type A vertices in generation n, we get

$$A(1) = x;$$
$$A(n) = (x - 5)D(n - 1) + (x - 6)E(n - 1);$$

$$B(n) = \left(\frac{y-4}{2}\right) A(n-1) + \left(\frac{y-6}{2}\right) B(n-1) + \left(\frac{y-6}{2}\right) C(n-1);$$
$$C(n) = \frac{1}{2}\left(2B(n-1) + C(n-1) + 2D(n-1) + 2E(n-1)\right);$$
$$D(n) = \left(\frac{y-6}{2}\right) A(n-1) + \left(\frac{y-4}{2}\right) B(n-1) + \left(\frac{y-6}{2}\right) C(n-1);$$
$$E(n) = \frac{1}{2}\left(2A(n-1) + C(n-1)\right).$$

From these recurrences, we can obtain a system of generating functions $A(z)$, $B(z)$, $C(z)$, $D(z)$, $E(z)$, where z is a placeholder unknown. We transform the above recurrence relation to a system of generating functions, also considering the initial condition:

$$A(z) = (x-5)zD(z) + (x-6)zE(z) + xz;$$
$$B(z) = \left(\frac{y-4}{2}\right) zA(z) + \left(\frac{y-6}{2}\right) zB(z) + \left(\frac{y-6}{2}\right) zC(z);$$
$$C(z) = \frac{1}{2}\left(2zB(z) + zC(z) + 2zD(z) + 2zE(z)\right);$$
$$D(z) = \left(\frac{y-6}{2}\right) zA(z) + \left(\frac{y-4}{2}\right) zB(z) + \left(\frac{y-6}{2}\right) zC(z);$$
$$E(z) = \frac{1}{2}\left(2zA(z) + zC(z)\right).$$

Solving this system, we get

$$A(z) = \frac{-2xz + (-4x+xy)z^2 + (-6x+xy)z^3}{-2 + (-4+y)z + (12-4x-4y+xy)z^2 + (-4+y)z^3 - 2z^4};$$
$$B(z) = \frac{(4x-xy)z^2 + (-4x+xy)z^3}{-2 + (-4+y)z + (12-4x-4y+xy)z^2 + (-4+y)z^3 - 2z^4};$$
$$C(z) = \frac{(8x-2xy)z^3}{-2 + (-4+y)z + (12-4x-4y+xy)z^2 + (-4+y)z^3 - 2z^4};$$
$$D(z) = \frac{(-6x+xy)z^2 + (-4x+xy)z^3 - 2xz^4}{-2 + (-4+y)z + (12-4x-4y+xy)z^2 + (-4+y)z^3 - 2z^4};$$
$$E(z) = \frac{-2xz^2 + (-4x+xy)z^3 - 2xz^4}{-2 + (-4+y)z + (12-4x-4y+xy)z^2 + (-4+y)z^3 - 2z^4}.$$

Then, computing $V(z) = 1 + A(z) + B(z) + C(z) + D(z) + E(z)$ yields the final generating function for $\{3, [x, y, y]\}$:

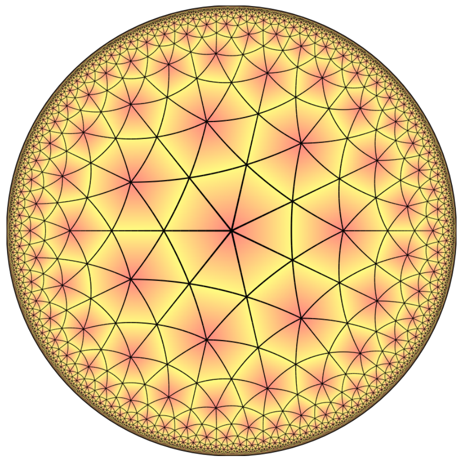

Figure 2.1 A $\{3, [7, 6, 6]\}$ tessellation (Rocchini, 2007b).

$$
\begin{aligned}
V(z) &= \frac{-2 + (-4 - 2x + y)z + (12 - 4y)z^2 + (-4 - 2x + y)z^3 - 2z^4}{-2 + (-4 + y)z + (12 - 4x - 4y + xy)z^2 + (-4 + y)z^3 - 2z^4} \\
&= 1 + xz + (-4x + xy)z^2 + \frac{1}{2}\left(30x - 4x^2 - 12xy + x^2y + xy^2\right)z^3 \\
&\quad + \frac{1}{4}\left(-224x + 48x^2 + 136xy - 24x^2y - 24xy^2 + 3x^2y^2 + xy^3\right)z^4 \\
&\quad + \cdots .
\end{aligned}
$$

2.2.2 Example: $p = 3$, $q_1 = 7$, $q_2 = q_3 = 6$

We illustrate the accuracy of the derived generating function with an example. Consider the tiling $\{3, [7, 6, 6]\}$, seen in Figure 2.1.

In this case, it is easiest to view the vertex in the center of the circle as the origin to which distances are measured.

Substituting $x = 7$ and $y = 6$ into our generating function, we get

$$
\begin{aligned}
V(z) &= \frac{1 + 6z + 6z^2 + 6z^3 + z^4}{1 - z - z^2 - z^3 + z^4} \\
&= 1 + 7z + 14z^2 + 28z^3 + 49z^4 + \cdots .
\end{aligned}
$$

The coefficients of the first few terms in fact match the numbers of vertices in the first few generations when counted by hand.

Chapter 3

Isohedral Tilings of Quadrilaterals

Consider when $p = 4$. Then we have a $\{4, [q_1, q_2, q_3, q_4]\}$ tiling.

3.1 Feasibility

We note that when $q_1 = q_2 = q_3 = q_4$, we get the special case of regular hyperbolic tessellations, where q_1 can be any integer greater than 4.

Suppose that $q_i \neq q_k$ for (i, k) being either $(1, 3)$ or $(2, 4)$, meaning vertices opposite each other. Call the remaining degrees q_j and q_l (note that due to symmetry, it does not matter which is which). Consider quadrilaterals meeting at a vertex J of degree q_j. We see that going around, the degrees of the vertices of these quadrilaterals that are adjacent to J must alternate between q_i and q_k. This means that q_j must be even for $\{4, [q_i, q_j, q_k, q_l]\}$ to be a feasible tiling. If q_j were odd, the alternating behavior would force one of the vertices to simultaneously have degrees q_i and q_k, which is a contradiction since vertices should have unique degree. This holds true for both $(1, 3)$ and $(2, 4)$. Thus, when two vertices across from each other are identical, the other two vertices must be even for the tiling to be feasible.

In terms of distinctness and parity, this is a necessary condition for $\{4, [q_1, q_2, q_3, q_4]\}$ to denote a feasible tessellation. Thus we are able to disregard tilings which do not fit this condition, as they will be unfeasible.

3.2 Case Study: $\{4, [x, w, y, w]\}$

We consider a special case, when $q_2 = q_4$, and derive generating functions for vertices by generation from an origin of degree q_1. Note that if $q_1 \neq q_3$, then q_2 must be even. For simplicity in argument, assume that $q_1, q_2, q_3 \geq 4$. Take some tiling $\{4, [x, w, y, w]\}$ where x, y, and w satisfy these conditions.

3.2.1 Derivation

As before, we consider the relationships between generations in terms of edges connecting "parent" and "child" nodes. Since we have a tiling in which every vertex has degree at least 4, no vertex will have three or more distinct parents (Paul and Pippenger, 2011). Thus, each vertex can have up to two parents, and since there are two possible degrees for the vertex in $\{4, [x, w, y, w]\}$, we can classify six types of vertices with regard to number of parents and degree: $(1, w)$, $(2, w)$, $(1, y)$, $(2, y)$, $(1, x)$, $(2, x)$. Note that each of the children of these vertices must also be one of these six types, so one might imagine that there is an underlying recurrence relation to be found through looking at the configuration of types of children connected to a generic vertex of each type. It is also worth noting that for tilings of quadrilaterals, every edge connects a parent to one of its children (Paul and Pippenger, 2011).

This time, it turns out that there are two varieties of the $(1, w)$ vertex; its parent may be of degree y or degree x. As for the remaining five types of vertices, there is only one variety each: the $(2, w)$ type vertex will have a degree x parent on one far side and a degree y parent on the other, with mirror images considered equivalent; the other four types will always have parent(s) of degree w and thus only one distinguishable configuration of child types.

For convenience, let us call $(1, w)$ vertices with degree x parent type A, $(1, w)$ vertices with degree y parent type B, $(2, w)$ vertices type C, $(1, y)$ vertices type D, $(2, y)$ vertices type E, $(1, x)$ vertices type F, and $(2, x)$ vertices type G.

As in the $p = 3$ case, the children on either the far left or far right of any type vertex will have two parents. This can be deduced (without loss of generality) by looking at the quadrilateral on the left side of the edge between the parent and its leftmost child. Suppose we have a vertex X and its leftmost child L. X and L are two of four vertices that form a quadrilateral that the edge between X and L borders. Call the other vertex M which is adjacent to L, and call the remaining vertex W. Thus we are now looking at quadrilateral $XLMW$. If we denote the generation of X to be n, L must be of generation $n+1$. Since W is to the left of L, which was said to be X's leftmost child, it must be a parent of X, and thus in generation $n-1$. This leaves M to be adjacent to a vertex of generation $n+1$ and one of $n-1$, and since every edge connects an earlier generation to a later generation, it follows that M must be in generation n and thus be a parent of L.

Again, any child of a vertex that is not a far left child or far right child will have only one parent, since this tiling is a planar graph.

With these facts about the number of parents of each child, as well as deducing the degree of each child by looking at the quadrilaterals that the edges connecting them are part of, we can obtain the configurations of degrees of children for the seven types of vertices introduced:

A	$E, F, (D, F), E$	the sequence (D, F) repeated $\frac{w-4}{2}$ times
B	$G, D, (F, D), G$	the sequence (F, D) repeated $\frac{w-4}{2}$ times
C	$E, (F, D), G$	the sequence (F, D) repeated $\frac{w-4}{2}$ times
D	$C, (B), C$	the type (B) repeated $y-3$ times
E	$C, (B), C$	the type (B) repeated $y-4$ times
F	$C, (A), C$	the type (A) repeated $x-3$ times
G	$C, (A), C$	the type (A) repeated $x-4$ times.

From these relations, and the fact that generation 1 has x type A vertices, we can deduce the recurrence relation among the types of vertices in $\{4,[x,w,y,w]\}$. For each type, we look at which types can be its parent and how frequently they can be so. For the types with two parents, we halve the number of parents to account for double-counting. Then, if we let $A(n)$ be the number of type A vertices in generation n, we get

$$
\begin{aligned}
A(1) &= x;\\
A(n) &= (x-3)F(n-1) + (x-4)G(n-1);\\
B(n) &= (y-3)D(n-1) + (y-4)E(n-1);\\
C(n) &= D(n-1) + E(n-1) + F(n-1) + G(n-1);\\
D(n) &= \left(\frac{w-4}{2}\right) A(n-1) + \left(\frac{w-2}{2}\right) B(n-1) + \left(\frac{w-4}{2}\right) C(n-1);\\
E(n) &= \frac{1}{2}\left(2A(n-1) + C(n-1)\right);\\
F(n) &= \left(\frac{w-2}{2}\right) A(n-1) + \left(\frac{w-4}{2}\right) B(n-1) + \left(\frac{w-4}{2}\right) C(n-1);\\
G(n) &= \frac{1}{2}\left(2B(n-1) + C(n-1)\right).
\end{aligned}
$$

From these recurrences, we can obtain a system of generating functions $A(z), B(z), C(z), D(z), E(z), F(z), G(z)$, where z is a placeholder unknown. We transform the above recurrence relation to a system of generating functions, also considering the initial condition:

$$\begin{aligned}
A(z) &= (x-3)zF(z) + (x-4)zG(z) + xz;\\
B(z) &= (y-3)zD(z) + (y-4)zE(z);\\
C(z) &= zD(z) + zE(z) + zF(z) + zG(z);\\
D(z) &= \left(\frac{w-4}{2}\right) zA(z) + \left(\frac{w-2}{2}\right) zB(z) + \left(\frac{w-4}{2}\right) zC(z);\\
E(z) &= \frac{1}{2}\left(2zA(z) + zC(z)\right);\\
F(z) &= \left(\frac{w-2}{2}\right) zA(z) + \left(\frac{w-4}{2}\right) zB(z) + \left(\frac{w-4}{2}\right) zC(z);\\
G(z) &= \frac{1}{2}\left(2zB(z) + zC(z)\right).
\end{aligned}$$

Solving this system, we get

$$\begin{aligned}
A(z) &= \frac{2xz + (wx + 2xy - wxy)z^3 + (-2x + wx)z^5}{P(z)};\\
B(z) &= \frac{(4x - 3wx - 2xy + wxy)z^3 + (-4x + wx)z^5}{P(z)};\\
C(z) &= \frac{(-4x + 2wx)z^3 - (-4x + 2wx)z^5}{P(z)};\\
D(z) &= \frac{(-4x + wx)z^2 + (4x - 3wx - 2xy + wxy)z^4}{P(z)};\\
E(z) &= \frac{-2xz^2 + (2x - 2wx - 2xy + wxy)z^4}{P(z)};\\
F(z) &= \frac{(-2x + wx)z^2 + (wx + 2xy - wxy)z^4 + 2xz^6}{P(z)};\\
G(z) &= \frac{(-2x + 2wx + 2xy - wxy)z^4 + 2xz^6}{P(z)};
\end{aligned}$$

where $P(z) = -2 + (6 - 4w - 2x - 2y + wx + wy)z^2 + (-6 + 4w + 2x + 2y - wx - wy)z^4 + 2z^6$.

Then, computing $V(z) = 1 + A(z) + B(z) + C(z) + D(z) + E(z) + F(z)$

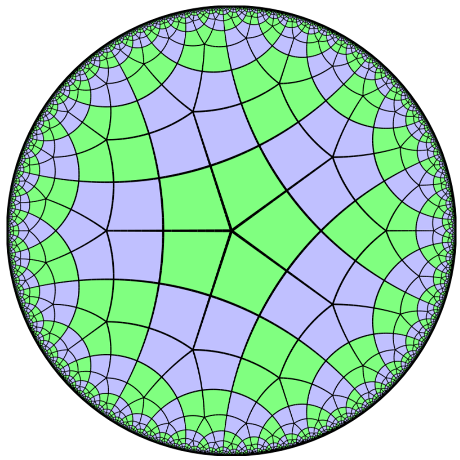

Figure 3.1 A $\{4, [5, 4, 4, 4]\}$ tessellation (Rocchini, 2007a).

$+\,G(z)$ yields the final generating function for $\{4, [x, w, y, w]\}$:

$$\begin{aligned}
V(z) &= \frac{-2 - 2xz + (4 - 4w + 2x - 2y - wx + wy)z^2 - 2xz^3 - 2z^4}{-2 + (4 - 4w - 2x - 2y + wx + wy)z^2 - 2z^4} \\
&= 1 + xz + (-2x + wx)z^2 + \frac{1}{2}(6x - 4wx - 2x^2 + wx^2 - 2xy \\
&\quad + wxy)z^3 + \frac{1}{2}(-8x + 12wx - 4w^2x + 4x^2 - 4wx^2 + w^2x^2 \\
&\quad + 4xy - 4wxy + w^2xy)z^4 + \cdots .
\end{aligned}$$

3.2.2 Example: $p = 4$, $q_1 = 5$, $q_2 = q_3 = q_4 = 4$

We illustrate the accuracy of the derived generating function with an example. Consider the tiling $\{4, [5, 4, 4, 4]\}$, seen in Figure 3.1.

Again, it is easiest to view the central vertex as the origin to which distances are measured.

Substituting $x = 5$, $y = 4$ and $w = 4$ into our generating function, we get

$$\begin{aligned}
V(z) &= \frac{1 + 5z + 7z^2 + 5z^3 + z^4}{1 - 3z^2 + z^4} \\
&= 1 + 5z + 10z^2 + 20z^3 + 30z^4 + 55z^5 + 80z^6 + \cdots .
\end{aligned}$$

The coefficients of the first few terms in fact match the numbers of vertices in the first few generations when counted by hand.

Chapter 4

Future Work

Further research on this topic could be done in several directions. One way would be to look for special-case solutions to larger or more general values of p. Another would be to look at tilings of triangles and of quadrilaterals with no constraining conditions at all. Finally, one could attempt to answer different enumerative questions regarding these tilings, possibly involving faces and edges, for which the current results could be helpful. For example, a natural question one could ask when looking at *Circle Limit IV* (Figure 1.1) is "How many angels and devils are there?" There should be certain questions for which the corresponding sequence, or the generation function that contains such information, should be easily derivable from my current results.

Bibliography

Brannan, D. A., M. F. Esplen, and J. J. Gray. 1999. *Geometry*. Cambridge University Press.

Coxeter, H. S. M., and W. O. J. Moser. 1980. *Generators and Relations for Discrete Groups*. Springer-Verlag.

Ernst, Bruno. 2007. *The Magic Mirror of M.C. Escher*. Taschen, 2nd ed.

Paul, A., and N. Pippenger. 2011. A census of vertices by generations in regular tessellations of the plane. *Electronic Journal of Combinatorics* 18:87.

Rocchini, Claudio. 2007a. Deltoidal tetrapentagonal tiling. Wikipedia, the free encyclopedia. URL http://en.wikipedia.org/wiki/File:Deltoidal_tetrapentagonal_tiling.png.

———. 2007b. Order-3 heptakis heptagonal tiling. Wikipedia, the free encyclopedia. URL http://en.wikipedia.org/wiki/File:Order3_heptakis_heptagonal_til.png.