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Empiricism: An Environment for Humanist Mathematics

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Synopsis

Humanists have extended some links between mathematics and the physical world, but most mathematicians still believe they operate in an immaterial realm of the mind, with unquestionable logic and abstract thought. By rehabilitating the empiricism of John Stuart Mill and combining it with growing knowledge of the character of the human mind, we can escape from the indefinable Platonic universe of immaterial consciousness and abandon the futile quest for certainty that has plagued philosophy since the time of the Greeks.

> *quid nobis certius ipsis sensibus esse potest, qui vera ac falsa notemus?* (What can we have more sure than the senses themselves, by which to discern the true and the false?)
>
> –Lucretius, de rerum natura, Book I, Lines 699-700 [16].

The humanist approach to mathematics is an admirable and successful movement to describe and analyze how mathematicians achieve what it is they achieve. On the way, a number of conventional assumptions have been challenged. One is that the practice of mathematics is simply the mechanical application of deductive formulas; instead, humanists claim, it involves intuition and value judgments that are uniquely human in character. Another is

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that mathematical proof is a formal process that can be completely and satisfactorily expounded to universal consensus. Humanists have demonstrated that in practice, proofs are necessarily abbreviated, and their acceptance depends on their ability to persuade the mathematical community that the omitted steps do not affect the validity of the conclusion.

Having proceeded thus far from the conventional view, humanists nevertheless have failed to persuade mathematicians to abandon the concept of mathematics as taking place in an immaterial world, involving immutable objects subject to laws that are true *a priori* and not subject to question.

In other words, most mathematicians work in Jonathan Swift's Laputa, the island in the air populated by speculative thinkers so focused on the abstract that they need to employ flappers to attract their attention to the physical world. Humanists have let down a few ladders from Laputa, but not many mathematicians have been lured down them to join the rest of humanity in the physical universe.

In a broader sense, however, most of humanity has lived in Laputa since the birth of consciousness. It is true that humanism originated in the Renaissance as a theme of focusing not on the life hereafter, but on the material world of the present. For half a millennium humanists have relished the here-and-now in the face of those who sought to assuage the woes of the present with the promises of the future.

Throughout this history of conflicting views, however, there has never been a time when humans, whether humanist or not, did not believe that there was an immaterial soul or spirit, a mind, a consciousness, directing the affairs of the crass material clay that we call the body. Even though Newton brought the heavens into synchronization with world of matter, and the Darwin brought humans into kingdom of the animals, it seemed inconceivable to most people that such a marvelous phenomenon as thinking could be anything but a miracle.

Durant's Dismay

There were doubters, of course, starting with Lucretius and continuing through Richard Dawkins. There were also believers who suffered doubts. One of the most poignant of these believers was the popular mid-20^{th} century writer Will Durant. In the 1920's, Durant wrote a wildly popular book, *The Story of Philosophy* [7]. It was lucid, balanced, particularly insightful in explaining the positions of individual philosophers, and conveyed a sense of the true worth of the subjects they dealt with.

He followed that success with another book on philosophy. His purpose, he said, was to focus not so much on the personalities as on the philosophy itself. But that was a disaster, for the philosophy at the time was a chaos of nonsense: Kantian idealism, taken into half a dozen wild worlds by Hegel, Schopenhauer, Nietzsche, and into existentialism; "analytical" philosophy by Moore and Russell and Ayer; logical positivism; Brouwer's intuitionism; Cantor and his infinitists. Durant pasted together a few pages out of these outré views of metaphysics and epistemology, and then declared the range of philosophy to include history, morality, religion, aesthetics, and politics. He devoted 80 pages to philosophy, and 600 pages to the other "mansions of philosophy" [6].

But in the process he laid out his dismay in the following paragraph:

> If there is any intelligence guiding this universe, philosophy wishes to know and understand it and reverently work with it; if there is none, philosophy wishes to know that also, and face it without fear. If the stars are but transient coagulations of haphazard nebulae, if life is a colloidal accident, impersonally permanent and individually fleeting, if man is only a compound of chemicals, destined to disintegrate and utterly disappear, if the creative ecstasy of art, and the gentle wisdom of the sage, and the willing martyrdom of saints are but bright incidents in the protoplasmic pullulation of the earth, and death is the answer to every problem and the destiny of every soul – then philosophy will face that too, and try to find within that narrowed circle some significance and nobility for man [6, page 21].

Philosophy will "face the reality" he defined so dramatically, he said. But philosophers did not, and Durant himself resigned from philosophy. He became instead a historian. A magnificent historian, a historian of civilization, including philosophy; there are few who have so comprehensively portrayed the "protoplasmic pullulation" (the word means "sproutings," and was no doubt as obscure to his first readers as it is today) of humanity with such verve and art. But his history ended with Napoleon – the last volume in his *Story of Civilization*, published in 1975 – and he never again dealt with modern philosophy.

Astonishing Hypothesis

Within the last few decades, the evidence has become overwhelming that "man is only a compound of chemicals, destined to disintegrate and utterly disappear," as Durant feared; that "death is ... the destiny of every soul." This dismaying conclusion, coming on top of the scientific evidence of the miniscule niche humans occupy in universal time and space, is still stoutly resisted by almost every consciousness. So strong is the reflexive rejection that Francis Crick, articulating the idea in 1994, described it as "The Astonishing Hypothesis," acknowledging its character as contrary to common belief. Nevertheless, Crick brought convincing evidence to support his hypothesis that "we are nothing but a pack of neurons"; that "our joys and sorrows, our memories and our ambitions, our sense of personal identity and free will, are in fact no more than the behavior of a vast assembly of nerve cells and their associated molecules" [3, page 1].

The reaction when one expounds the Crick hypothesis is uniformly astonishing. As Crick points out, most scientists and many other people "share the belief that the soul is a metaphor and that there is no personal life either before conception or after death." But when it comes to applying that view to themselves, there is a curious hesitation. Crick is jocular about it: "Whatever he says, Mabel, I know I'm in there somewhere, looking out on the world." But most people find it difficult to appreciate the irony. It does not entertain them to realize, to be conscious of the fact, that their consciousness of the world and of their own thoughts is a matter of electrons and synapses.

For the philosophically minded such as Durant, the loss of the immaterial soul is most distressing because it makes the ontological pursuit of final causes, especially for humanity, an impossible, insignificant endeavor. From the rational and spiritual master of the center of the universe that was the Greek philosopher, humanity has descended to a "colloidal accident." The query of poets and philosophers, "What's the meaning of life?", has lost its meaning, or at least its significance.

An Ordered Universe

Scientists in general have relinquished the search for ontological principles, at any rate in their professional activities. But for the scientific community – and I include mathematicians in this group – there is a further disquieting implication that some may draw from the Crick hypothesis, although Crick himself did not do so. It lies in the words Durant used in his lament: in the word "accident," in the word "haphazard." To the scientist, there is nothing haphazard or accidental about the universe we inhabit, no matter how tiny or evanescent may be the corner of it that we can observe. It is an ordered universe: a universe that behaves according to universal physical laws, some of which we have discovered. Does accepting the Crick hypothesis nullify this belief?

I would argue that it does not. Humans up to now have believed in the existence of an immaterial, spiritual world primarily because they found it inconceivable that the mind and its operations could be wholly grounded in physical reality. While accepting the Crick hypothesis does not deny that a spiritual universe exists, it does remove one of the most powerful reasons for believing that it must. However, the validity of the conclusions of consciousness does not depend on whether consciousness is an immaterial phenomenon or is part of the physical world. Humans with conscious minds have constructed a model of an ordered, logical universe. We do not need to abandon that model because we have changed our view of the engine that created it. That would be to throw out the baby with the bathwater.

The Quest for Certainty

However, I do think we need to revise our justification for believing our conclusions are true. And in the process we need to abandon an endeavor that has plagued philosophy since the Greeks first invented it: the quest for certainty.

As John Dewey [5] put it, the Greek philosophers undertook to replace religion with philosophy. In doing so, they "stripped away imaginative accretions" of myths and rituals. But they retained the principle of "immutable and necessary truth" that characterized belief in religious doctrines. Inevitably, this led to a downgrading of the kind of truth one obtains through physical observation, as compared to pure and uncontaminated thought, as exemplified by geometry.

The Greeks derived their view of the natural world on the principle of logic and necessary truth. Descartes, on the eve of the revolution that explained the physical world through the force of experimental science, decided that sense perceptions were unreliable. He created a dualistic universe: the external body that exists in the physical world, and the immaterial mind, where absolutely certain "clear and distinct ideas" can be found and declared. The same pursuit of certainty inspired the development of "analytic philosophy" in the late 19^{th} century, and continues to clog the structure of philosophy today [2].

As long as it was widely accepted that there exists an immaterial universe in which the mind operates (although contaminated to a greater or lesser degree by the emotions and other worldly events and interruptions), it was not too far a stretch to conclude that there was a way in which the mind might attain absolutely certain knowledge. In particular, it might have been valid to claim that the process of deductive logic leads to unchallengeable truth, and that the laws and theorems of mathematics, so derived, shared in that infallibility. The laws of the physical sciences also are derived by logic, but they are confirmed by observation of the physical universe. Mathematics, on the contrary, rested its claim to truth on mental operations that appeared to be independent of material contamination and the uncertainty of sense perceptions. It was not unreasonable to claim that mathematical knowledge was of a different sort from knowledge gained by observation of the physical world.

But the time has passed when such a distinction is valid. Because under the Crick hypothesis, ideas, thoughts, concepts, feelings, beliefs – all the evanescent phenomena that previously were thought to compose the mind, or the spirit, or the soul – are physical objects that exist in the physical world at a particular time. They are the chemical, electrical, and mechanical states of the neurons of human brains.

My philosopher friends will immediately pronounce that claim to be blatant reductionism, and I accept the charge. And I certainly am not suggesting that we can draw a circuit diagram of the brain of a human being as it thinks of the abstract number 5. It may never be possible to do that. The complexity of consciousness is probably as profound as any physical phenomenon we may encounter. We can reduce the weather into the collective action of atmospheric molecules that we hypothesize are following Newtonian laws of motion, but we can't draw a map of those molecules. In the same way, we can probably never map the action of the neurons of consciousness, even if we believe that they obey the physical and chemical laws that dictate the interactions of those physical objects.

But reduction is not the point here. By locating abstract ideas and concepts in the physical world, we can treat them as we treat other objects in the physical world. If we want to assert the truth of a proposition involving an abstract concept, we can use the same criterion that we apply to assertions of physical fact: did we observe it to be true? In fact, we can apply the principles of empiricism to the activities of the mind, including, significantly, mathematics. Specifically, it is time to rehabilitate the empiricism of John Stuart Mill.

Reenter John Stuart Mill

Mathematics (and formal logic, its close relative) occupy an anomalous place in today's materialistic world. Most mathematicians view the object of their profession as an abstract, ideal world with Platonic echoes. This immaterial character derives from the belief that mathematical propositions, unlike physical laws, are not testable by experiment in the material world, but depend for their proof on the interior logic of the system. And because logic appears to be more certain of producing true conclusions than observation of physical phenomena, there is a general belief that mathematical propositions, unlike scientific propositions, are not subject to doubt.

This common view of the distinctive character of mathematics is in sharp contrast with the empiricist philosophy that was prominent in Britain in the 18^{th} and early 19^{th} centuries. In particular, it was contradicted by the views of the 19^{th} century polymath, John Stuart Mill.

Mill was a major philosophical voice in Britain in the early Victorian era, and he remains well recognized and often cited. Today he is viewed almost entirely as a social and political philosopher. He developed and expanded the utilitarianism that he inherited from his father, James Mill, and Jeremy Bentham, and he played an important role in the changing British society as it adjusted to the challenges of the Industrial Revolution.

But Mill was equally renowned in his time as a pursuer of traditional philosophy and of the developing philosophy of science of the 19^{th} century. Building on the empiricism of John Locke and David Hume, Mill produced a massive *System of Logic, Ratiocinative and Inductive.*[2]

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[2] Until recently *The System of Logic* was available only as Volumes VII and VIII of an edition of *The Collected Works of John Stuart Mill* published by the University of Toronto

Mill's empiricism is very plainly stated: The only knowledge human beings can have is the physical observation of singular events. How, then, can we know the truth of generalities? Here is the key paragraph:

> Whence do we derive our knowledge ...? From observation. But we can only observe individual cases ... From instances we have observed, we feel warranted in concluding that what we found true in those instances holds in all similar ones, past, present, and future, however numerous they may be [17, Book II, Chapter iii, Section 3].

The key feature of Mill's epistemology is that he applies this principle to all knowledge, including mathematical propositions, and *even to the very process of logical deduction itself.* How do we know that if the premises of a valid syllogism are true, then the conclusion is true? Because we have observed many instances in which that process has led to the truth. We therefore "feel warranted in concluding" that it will lead to the truth the next time.

Note that Mill's principle does not specify the number of instances that are necessary to warrant our concluding a general principle. A simple sum, say five plus three equals eight, would probably not require more than a single counting on the fingers. In the case of Goldbach's conjecture, many individual instances have not been adequate for many mathematicians to satisfy the extension of inductive truth to the principle (in one of its forms) that "every even integer greater than 2 can be written as the sum of two primes."

Mill's *Logic* was a standard academic text in British universities for much of the 19^{th} century. From its first publication in 1843, it went through eight editions, the last in 1872. In the process of updating and revising the numerous editions, Mill engaged in lively debate over the principles he expounded about the source of human knowledge.

After his death in 1873, however, Mill's *Logic*, and the empiricism it expounded, lost favor. Through most of the 20^{th} century and into the present it

Press in 1973 [18]. *The Logic* is now in print in a 2002 paperback edition published by the University Press of the Pacific [17]. Of course since it is now in public domain, readers can also find the complete text online.

has been virtually ignored, both by philosophers and by scientists and mathematicians. This occurred primarily because of the difficulty mathematics presents to the empiricist model.

Historical Empiricism and Mathematics

The English empiricists of the 17th and 18th century found mathematics an embarrassment in their attempt to demonstrate their claim that all knowledge comes from the senses. Mathematics, especially geometry, seemed in its elements so unchallengeable, and at the same time so abstract from sensory data, that they could not bring themselves to class it with the other speculative mental exercises of Descartes which they rejected.

David Hume, it is true, refused to grant mathematical conclusions the quality of infallible truths. But he based their imperfections on the fallibility of mathematicians, not of mathematics. In "demonstrative sciences" (those involving number and geometry), he wrote, "the rules are certain and infallible, but when we apply them, our fallible and uncertain faculties are very apt to depart from them, and fall to error... By this means all knowledge degenerates into probability" ([13] as cited in [12, page 189]). The laws and theories of all other sciences are also only probably true, being based on observation and derived by applying the law of cause and effect, which is itself not deductively valid. But mathematical rules are "certain and infallible," and only the application of them, possibly erroneous, makes mathematical knowledge uncertain along with all other knowledge.

Hume's concession that mathematical rules are certain and infallible gave Immanuel Kant an opening to claim that other abstract principles of logic were also true *a priori*, exclusive of sensory input. In addition, Kant related arithmetic and geometry to human intuitions of time and space, and claimed that since those intuitions are the same for everyone, they are valid for every human mind.

Kant's "certain" knowledge was thus based on rather uncertain principles. Not only did he concede the impossibility of knowledge of the physical world except as observed via the senses. He also pinned the correctness of basic ideas to the structure of the human mind. This was a shift from the view of Descartes, who claimed his "clear and distinct ideas" were true because they were so obvious no one could argue with them. Kant demolished logical (and theological) assertions based on rationalistic arguments of this kind.

His own logic, and his mathematics, were true because the mind intuitively structured space and time on Euclidean principles. Since everyone has the same intuitive structure, logic and mathematics must be true, or at least valid, for everyone.[3]

Despite making these concessions to Hume's empiricism, Kant was able to structure an ideal world of reason by maintaining a strong Cartesian dualism separating mind and the physical universe. During the 19th century, Kantian idealism dominated Western philosophy, and the paradigm of the "ghost in the machine" structured most intellectual activity. Hume's empiricism was relegated to the level of a nuisance, and Kant had effectively made it irrelevant.

Mill: Intransigence and Disaster

By the mid-19th century, another British empiricist was ready to take a crack at mathematical and other "certain" knowledge, and John Stuart Mill was not making any concessions. Simply put, Mill argued that all human knowledge, general or specific, was derived inductively from observation of individual events, not deductively. In a supposedly deductive process, we would be moving from known premises to certain conclusions, but where would the premises come from? They, too, would have to come from induction.

Not content with basing experimental sciences on what we "feel warranted in concluding," Mill extended his principle into realms that had seemed certain and infallible. The axioms of geometry, the laws of arithmetic and mathematics, even the rules of logic and reason themselves, only appear certain because they have been observed to hold true up to now. They are certain, but their certainty derives from exactly the same process as any general conclusion about the physical world. The conclusion that the sun will rise in the east tomorrow is of the same quality, and very nearly the same certainty, that three plus five will equal eight the next time someone carries out that operation.

Mill's radical empiricism was widely accepted for many years, but after his death it lost favor. A major reason was that the loss of certainty implicit in it bothered mathematicians and philosophers, as it does to this day.

[3] [12] discusses Kant's mathematics at some length, see pages 129ff.

Bertrand Russell, who was probably the most influential destroyer of Mill as a philosopher, said, "What I most desired was to find some reason for supposing mathematics true." Mill's position, that the truth of all generalizations rests on inductive reasoning from single instances, was, in Russell's words, "very inadequate."

Another factor was probably a matter of style. Comparing Mill's verbose Victorian syntax and leisurely completeness with Russell's modern, vigorous, and entertaining prose, it is easy to see why the one has been unpublished for a century and the other widely excerpted and quoted.

But the primary cause of Mill's defeat, ironically, was the explosive creation of the natural sciences, which empiricism itself had fostered. The Scientific Revolution directed attention of "natural philosophers" toward the evidence of experiment as the source of knowledge, rather than relying on rationalistic speculation. By the 19th century the experimental sciences – physics, chemistry, geology, biology, even astronomy – were beginning an era of explosive and successful growth, which has been unabating since. All of their advances have been based on the principle that truth derives from observation of phenomena, as stimulated by experiment. Out of such observations scientists built theories to generalize the results and predict new ones, but always the proof of the proposition lies in verification by experiment and observation.

Thus the development of the natural sciences has rested on the very principle that Mill espoused: the truth of any proposition depends on observation, and general propositions are based inductively on observation of single instances.

The role of mathematics in this development was a curious one. The formulation, and even the understanding, of theories in the natural sciences have been and continue to be expressed mathematically, and mathematical ideas and concepts, developed independently, continue to find a role in various scientific environments. But mathematics seemed not to be a science like the others, based on observation and experiment. In mathematics, truth depends on logical proof, and logic, so it appeared, is a purely mental affair. What is there to observe, besides other mathematicians?

It is on this point that Mill met disaster. In *Logic*, Book II, Chapter 7, Section 2, Mill stated that all numbers "must be numbers of something: there are no such things as numbers in the abstract."

For such a remarkably sensible person, it was a remarkably foolish thing to say, and Mill himself knew it. He immediately started backtracking. Numerical and algebraic expressions are not necessarily attached to physical objects, he said; they "do not excite in our minds ideas of any things in particular." It was a futile maneuver, however. If numbers don't excite ideas of any particular thing, how can they be "numbers of something"? More importantly, if abstract numbers don't exist, how do we know that five plus three equals eight? We can show that five fingers and three fingers count up to eight fingers, but mathematicians generally don't count on their fingers. And no amount of finger counting will help anyone derive Euler's equation.

Why did Mill fall into this trap? The answer is, he saw no other way to deal with the problem that since abstract activities such as mathematics take place only in the mind, there appears nothing to observe. And at the time he was writing, indeed, in every period of history prior to the last few decades, mental activities were generally taken to operate in another universe. If Mill wanted to insist that our knowledge of numbers derived from observation of the physical world, he had to ground numbers in that world: numbers had to be numbers of something. Abstract numbers didn't exist – in the physical universe, that is. When we add three to five, we are adding three somethings to five somethings, and we observe that our result is eight of those things. But what do we get when we take the square root of eight somethings? In Mill's system, we get disaster.

Millsian Empiricism: Into Limbo

Mill's critics seized on the fallacy of claiming that there are no such things as abstract numbers. It might be impossible to say exactly where the numbers were, in a physical sense, given the general acceptance of the Cartesian separateness of the immaterial mind, but to say they did not exist was absurd. Gottlob Frege [9] complained that it restricted the scope of mathematics to "pebble arithmetic"; he might have called it finger counting. Bertrand Russell portrayed Mill as a bumbling, naive innocent whose heart was in the right place but who couldn't really understand what mathematics was all about.[4]

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[4]Russell's final assessment of Mill was that his achievement "depended more upon his moral elevation and his just estimate of the ends of life than upon any purely intellectual merits... Mill deserved the eminence which he enjoyed in his own day, not by his intellect but by his intellectual virtues" [19].

There were further grounds on which Russell and other anti-empiricists attacked Mill's logic. It was the implication, very important at the time, that empiricism implied psychologism. There are various definitions of psychologism, but in general it is described as the doctrine that reduces logical entities, such as propositions, universals, or numbers, to mental states or mental activities. There was a great deal of concern about psychologism in the late 19th century, particularly after the Darwinian revolution. Both Russell and G.E. Moore in England, and Frege on the continent, formulated their analytical philosophy largely in opposition to psychologism and naturalism; Moore and Russell held that even Kantian idealism was too contaminated with psychologist ideas to be acceptable.[5] The opposition to psychologism probably reflected the unease that now exists to a much greater degree about the Crick hypothesis, but its ostensible expression mainly was concerned with the implication that mind-dependent mathematics would lose its claim to absolute certainty. Mill's emphasis on physical observation as a source of truth was a major denial of that claim. It had to be destroyed completely as a philosophical force, and indeed it was.

Beyond Psychologism

The Crick hypothesis has brought us far beyond the tentative ideas of psychologism that so terrified Russell and Frege, but the issues remain the same. Mathematicians still reject the suggestion that their truths are no different from the truths the rest of us enjoy, and insist that, however much humans may contribute to the process of mathematizing, there must be something taking place beyond the physical activities of human brains and their internal synapses.

Despite such opposition, it is philosophically revealing to consider the implications of Crick's hypothesis. Because with it, Mill's logic becomes complete, and the haze of mystic conceptions of mind clears away. In combination with Crick's hypothesis, Mill's empiricism makes clear the nature of mathematical knowledge. Mathematics no longer needs to base its claim to truth on mere assertion, but can rely on the evidence other sciences have found so rewarding: the observation of physical fact.

[5] The article [10] discusses the attitude of Frege and Russell toward Mill, in particular the question of psychologism.

Nor is it necessary to deny, as Mill did, the existence of abstract ideas. Of course they exist. If the mind is entirely contained in the body, then the products of the mind – that is, thoughts – are physical objects. They are the mechanical, chemical and electrical states of neurons in the brains of human beings. No matter how abstract or fantastic these thoughts or concepts are, they possess the attributes of the physical world, having a specific location in space and time.

It might be argued that, if ideas and concepts exist in individual brains, there is no way we can know that a particular idea, such as the number five, in one brain is the same as it is in another brain. But this same doubt exists about all our knowledge of the existence and characteristics of all objects, whether they are located in an open field, or on a mountaintop, or in another human brain. Our knowledge depends on observation of individual instances. Whether those observations are accurate is beyond us. Solipsists argue that we cannot even know if other brains exist. We judge only by results.

Thus when we observe another human correctly adding five and three to get eight, we feel warranted in concluding that the abstract ideas held by that individual are the same as ours, and we record this as a physical phenomenon. If someone insisted on a different answer, we would suspect that that person's idea of the numbers involved, or the operation of addition, was different from ours. When we observe ourselves, or another person, deriving that sum from the definitions of abstract numbers and the axioms of the real number system, we are observing a physical phenomenon. And our knowledge that, in general, five plus three equals eight, is based on our observation of individual instances of that result, and our feeling that we are "warranted in concluding" that it is true in general.

It is important to distinguish this empirical approach from social constructivism. Constructivists focus on the process by which scientists and mathematicians "warrant" their conclusions. Frequently hinted at, but rarely explicitly stated, is the implication that in "constructing" scientific propositions, humans have the option of reaching different conclusions. But for the empiricist, humans observe physical phenomena, and if the phenomenon observed is the proof of an abstract mathematical theorem, well, that is physical too, because the neurons that are doing the abstracting and the proving are physical objects.

We do not "construct" mathematical truths, in the sense of willfully composing them. We observe the discovery that propositions are true by practicing the arts of ratiocination, which in the past we have found valid. That is why mathematicians all come up with the same answer when they derive Euler's equation:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

If they don't, they've made a mistake, just as they would if they concluded that the gravitational force varied with the cube of the distance rather than the square. Their associates will continue to press on them the correct answer, until they concede it or retreat to another calling.

It sometimes occurs, of course, that there is disagreement about whether a proposition is true or not. It is hardly likely that a simple statement, like three plus five equals eight, or even a well-established proposition like Euler's equation, will be challenged. Perhaps the centuries-long debate over the role of the infinitesimal in the calculus will illustrate how disagreements are resolved. Far from the rigid logicism that would seem demanded by an *a priori* set of unchallengeable principles, the debate was finally resolved by the mathematics community agreeing on an acceptable format and explanation. In other words, mathematicians "felt warranted to assume" that the problem was solved and the proposition true.

Empirical Knowledge: Good Enough?

Let me point out again that mathematics is unique in demanding absolute certainty. Humanist probing has established that mathematical proof is a matter of consensus, but that the consensus is often assumed to be based on agreement on the certain truth of its assertions.

The physical sciences also demand consensus. But in the case of physics, the force that underlies unanimity is the observation of phenomena. When everyone agrees on the same gravitational constant or the same charge on the electron, it is because they have repeatedly been observed to have those values and no others. However, a good part of mathematics is abstract and unrelated to physical observation. How much is subject to debate. We can confirm that three plus five is eight by counting on our fingers, but direct physical evidence of the truth of Euler's equation is hard to come by. Yet we all believe it, and at this moment, some mathematics student somewhere is probably deriving it or following somebody else's derivation.

Here is the problem for those who are dissatisfied with the need to assume that the laws of logic and mathematics are true *a priori*. If they are not, why is it that everyone agrees to use them in the same way and come up with the same answer?

I'm afraid the only solution to that problem is the crude empiricism of J. S. Mill. The only way we know something is by observing it. If we observe that the sun rises in the east, that as far as we know it always has, and that everyone we know says it does, then we say "The statement that the sun rises in the east is true." As Mill put it, "From instances which we have observed, we feel warranted in concluding that what we found true in those instances holds in all similar ones, past, present and future, however numerous they may be" [17]. That is why we believe the sun will rise in the east tomorrow.

Applying crude empiricism to mathematics – and that's exactly what Mill did – the reason we believe that Euler's equation is true is because ever since he first wrote it down, anybody who tried to derive it came up with the same answer. If not, the answer was labeled "wrong," and the person who produced the wrong answer either agreed to the correct one or took up politics or gastronomy.

Thus the empiricist argument turns the problem around. The Platonist asserts that Euler's equation is true *a priori*, because of the logical process by which it was derived. Since it is true, it is trivially unsurprising that everyone gets the same result. To the empiricist, on the other hand, following a logical sequence does not guarantee the truth of the result. The only source of truth in any activity is observation. It is the very observation that everyone gets the same answer that warrants our conclusion that Euler's equation is true. In fact, our feeling of certainty when we follow any sequence of logic derives from the same observation: that everyone else has followed the same sequence and got the same result. We are certain that the next person who follows it will also get that result.

I can feel rumblings of dismay at the crudity of this empiricism. In his review article in *The American Mathematical Monthly* of November 2006 [11], Charles Hampton pointed out that graduate mathematics students learn no philosophy, and generally receive no formal tuition in the foundations of mathematics. But one element of philosophy appears to be imparted informally and ingrained indelibly; that mathematical truths are somehow different, more certain, more immune to doubt, than mere observation of physical events can produce.

Stewart Shapiro, for instance, in his recent book *Thinking about Mathematics* [20], one of the books that Hampton reviewed in his *Monthly* article, puts it this way:

> Basic mathematical propositions do not seem to have the contingency of scientific propositions.

Here, in his introduction of the subject, Shapiro is diplomatically tentative, with that "seem to have." But there is no doubt in his mind. A little farther down the page, he says,

> Unlike science, mathematics proceeds via proof. A successful, correct proof eliminates all rational doubt, not just all reasonable doubt.

To have any doubt about it is not only unreasonable, it is irrational. And finally, in a grand climax:

> Basic mathematical propositions enjoy a high degree of certainty. How can they be false? How can they be doubted by any rational human being? Mathematics seems essential to any sort of reasoning at all. If we entertain doubts about basic mathematics, is it clear that we can go on to think at all?

Let me make it clear that I recognize mathematical propositions are different from what Shapiro calls "scientific" propositions. I've already noted that difference. But from an empiricist point of view, our *knowledge* of mathematical propositions – our *certainty* about their truth – rests on exactly the same grounds as so-called contingent propositions: it rests on inductive observation.

Does this approach truly threaten the basis of human thought? Am I striking at the heart of all knowledge? Nonsense. Any doubts I might have don't bar me from thinking; don't bar me from a really profound trust in logic.

Despite my crude empiricism, I am really, really certain that three plus five equals eight. My certainty rests on three points. First, other people have told me so. Second, every time I hold up five fingers in one hand and three in the other, and count, it come out eight.

Third, when I consider the definitions of numbers, and axioms of the real number system, I reach the logical conclusion that, if the definitions and axioms are true, then the sum of three and five is eight.

Despite its abstract character, or perhaps because of it, this third reason seems to be the most convincing. Let's look at it a little more deeply. Even this simple proposition is more complicated than it first appears. I'm frequently grumbled at, "It's true by definition," or even, "You can see that it's true." But it is not just a matter of primitive definition. We don't define eight as three plus five, we define eight as seven plus one. That's the way all the numbers are defined: the previous number plus one. And in addition to the definitions of the numbers we need some axioms. At a minimum we need the properties of equality.

Symmetry: if $a = b$, then $b = a$.

The transitive property: if $a = b$ and $b = c$, then $a = c$.

Addition: if $a = b$, then $a + c = b + c$.

And we need:

The associative field axiom: $a + (b + c) = (a + b) + c$.

Then, having defined the numbers two through eight, we can say:

$$\begin{aligned} 5+3 &= 5+(2+1) \\ &= (5+1)+2 \\ &= 6+2 \\ &= 6+(1+1) \\ &= (6+1)+1 \\ &= 7+1 \\ &= 8 \end{aligned}$$

Why is it that this rather complicated procedure is more convincing than the physical evidence of counting on my fingers? It may be the result of centuries of brain-washing, from Descartes and even back to Aristotle onward, of the superior reliability of reason over the faulty uncertainty of the senses.

Most probably it's the generality of the proposition; it applies not only to fingers but to spoons and toes and apples. It applies to everything, and to nothing. As Richard Feynman put it, "The GLORY of mathematics is that you don't know what you're talking about" [8].

But note that it depends not only on faith in the validity of deductive reasoning – the proof that Shapiro was trumpeting – but also on the belief in the specific axioms I cited, as well as any others I may have left out. To the apriorist, that is not a problem: deductive reasoning is valid, and the axioms of the real number system are true, because – well, because they are. As an empiricist, however, I say that our beliefs are based on the observation that everyone up to now has agreed that they are true, and on our assumption that everyone will continue to do so in the future.

Let's look at the so-called contingent question raised earlier: will the sun rise in the east tomorrow? I'm really, really, certain that it will. In fact, my certainty is based on the same kind of evidence that reassured me in the case of simple addition. First, I've been told so. Second, as far as I know it always has. Third ...

Third, if I assume that Newton's laws of motion and gravity are true, and follow a process of logical deduction based on a model of the planetary system of the sun, I will conclude that the earth will continue to rotate as it has, in the absence of an external force, of which there is no evidence. Having rotated far enough to obscure the sun this evening, it will continue to rotate until the sun reappears tomorrow morning. It will not stop rotating, or reverse its rotation so as to bring the sun into view again in the west.

As in the case of simple addition, the construction of a model of the planetary system and the exercise of deductive logic from basic principles is somehow, in this still-persisting Age of Reason, more significant than the simple observation of patterns, convincing though they may be. But the logical processes I followed in constructing my certainty about tomorrow's sunrise are exactly the same as those involved in summing five and three. I have constructed a model of a planetary system, just as I constructed a sequence of real numbers. I have applied basic axioms, in one case Newton's laws, in the other the axioms of the real number system. In each case I have to accept them as true, and I do, because everyone else does. I also follow the methods of deductive logic, for the same reason.

I would follow the same procedure if I set out to demonstrate the truth of Euler's equation. I am really, really sure that e to the i theta equals cosine theta plus i sine theta. But in the same way, and for the same reasons, I am really sure that the sun will rise in the east tomorrow morning, just as it did this morning.

Mathematics: A Science

The similarity in demonstrating the truth of these two propositions, the sum of three and five and the motion of the planets, suggests that the two disciplines they represent, mathematics and physics, are likewise similar. In fact, it makes it practical to assert that mathematics is itself a science, unique in its own way, but metaphysically and epistemologically like physics, chemistry, biology and other sciences that have made such great advances since they freed themselves from reliance on intuitive truths and focused on observation and experiment.

With the aid of Crick's hypothesis, mathematics can free itself from dependence on mystical concepts of a disembodied mind, and base its findings on the same Millsian empiricism that other sciences employ to investigate the ordered universe of the physical world: a universe that appears to operate according to natural laws which it is the goal of science to discover.

At the same time, by treating mathematics as a science, we can relieve the suspicion, raised by locating mathematics in the physical universe, that mathematical laws did not exist before humans thought of them. We do not claim that the law of gravity was invented by Newton; if we treat mathematics as a science, we do not need to claim that Euler's equation was invented by Euler, or that it did not exist before he did. We can treat mathematics as we treat the natural sciences: as models of an ordered universe which we discover by observing phenomena.

Advantages of Empiricist Philosophy

In exchange for acknowledging the possibility of some doubt of the truth of mathematical propositions – the *possibility*, not the existence, of doubt – we gain a number of advantages. The first of these is giving up the futile struggle to defend *a priori* truths.

Discussing Frege's criticism of Mill in his *Oxford History of Western Philosophy*, Anthony Kenny in 2000 summarized Frege's "brilliantly successful" attack in Kantian terms, and said: "No philosopher of mathematics today would defend" Mill's views [14, page 264]. But on analysis, Frege's argument seems less than brilliant, no matter how successful. As Kenny describes it,

> Kant had maintained that the truths of mathematics were synthetic *a priori* and that our knowledge of them depended neither on analysis nor on experience but on intuition. Mill took a quite opposite view: mathematical truths were a posteriori, empirical generalizations widely applicable and widely confirmed. Frege disagreed with both his predecessors: he maintained that the truths of arithmetic were not synthetic at all, [but] analytic.

The Cambridge Dictionary of Philosophy says that a statement is called analytic "if the predicate concept is contained in the subject concept"; otherwise it is synthetic [4, page 26]. The statement "All red roses are red" is analytic; "All roses are red" is synthetic. Any analytic statement, according to the Dictionary, is *a priori* – that is, it is knowable without empirical evidence – and necessary – i.e., something that could not be false.[6]

In other words, Frege's attack on Mill, as Kenny describes it, consisted of simply declaring that mathematics is knowable without empirical evidence, and something that could not be false.

Perhaps a more accessible criticism of Mill's position is one by A.J. Ayer, published in 1936. This was after Russell had demonstrated that he couldn't show Frege was right, having failed to derive mathematics from *a priori* propositions. Ayer asserted that *thought* is a source of knowledge, independent of, and more trustworthy than, *experience*. He conceded that mathematical laws were probably discovered by induction – that is, by experience. But he claimed that once discovered, they are so obviously true that they must hold true in every conceivable instance [1, page 722]. Note that Ayer's

[6]Philosophers make a distinction between a statement that is "certainly true" and one that is "necessarily true." If a statement is declared true *a priori*, it is assumed that its truth is demonstrable without empirical evidence. If it is "necessarily true," such as the sentence "a red rose is red," then it is true because it cannot be false. To an empiricist, however, the claim that "a red rose is red" cannot be false depends on the validity of the rules of logic, and those very rules are true only inductively: that is, they are valid because the repeated instances of their success leads us to "feel warranted in concluding" that they are true. Thus the distinction between "certainly true" and "necessarily true" is meaningless to an empiricist.

phraseology is almost identical to Mill's own description of induction: "We feel warranted in concluding that what we found true in those instances holds in all similar ones, past, present, and future, however numerous they may be." But Mill didn't claim that the inductive result was more certain than the original observations.

After Ayer's declaration, Mill's empiricism completely disappeared from the philosophical scene. The rest of the century was consumed with intricate discussions of the implications of *a priori* knowledge and the complications that arise from the concept. The futility of these efforts was illustrated in a patient and detailed analysis by Philip Kitcher in 1983 [15]. For anyone interested in the mental contortions of those trying to reconcile Platonic idealism with modern science, Kitcher's analysis of the philosophical difficulties of apriorism is exhaustive and cogent.

What Are We Talking About?

A second advantage to adopting empiricism is that mathematicians can clearly state what it is they are doing and where they are doing it. Without the empiricist grounding in the physical world, mathematics is located in a Platonic universe; it consists of manipulating immaterial objects on whose nature there is no agreement. All such speculation becomes irrelevant if we acknowledge that abstract objects – thoughts, ideas, fantasies, numbers – are physical objects that exist in a place, the human brain, and a time, the moment they are thought of.

It is important to graft the physicality of the Crick hypothesis to Mill's comprehensive empiricism in order to achieve this grounding. Without Crick, the Millsian concept that requires numbers to be numbers of something runs into immediate difficulties, particularly in the area of infinite sets. As one critic has noted, "There are not enough physical objects (or mental states) to account for $\mathbb{Z}$, much less $\mathbb{R}$." How can you attach an object to every real and complex number when there are an infinity of such numbers? Does that mean there are an infinity of physical objects? Where are they?

But if we agree that numbers, and mathematical concepts, have an existence of their own, independent of the objects they are applied to, this difficulty disappears. Rejecting Cartesian dualism, and regarding the mind and its activities as purely physical phenomena, we avoid the trap Mill fell into. Numbers and other abstract concepts can exist in the physical world without being attached to any physical objects other than the human brains that conceive them. Thoughts, even the abstract thoughts of pure mathematics, are physical objects: they are the physical states of human brains at the moment they are thinking them. They are physical phenomena, and as such can be studied scientifically. We do not need to posit an immaterial Platonic universe for them to exist in.

Furthermore, we have no difficulty in asserting the physical existence of $\mathbb{R}$, $\mathbb{Z}$, or the sum $\sum_{n=1}^{\infty} \frac{1}{n}$. They exist, and they exist only, as the physical states of human brains. (They also exist in the "external memory" of written records or other media, but external memory consists of marks on paper or other physical materials; it has significance only as sensory input to individuals' neuronal connections. As such, it is no different from a fossil skull to a paleontologist or a rock to a geologist.) Their existence does not depend on our ability to attach a physical object to each term in the sum, and if the sum converges (though obviously $\sum_{n=1}^{\infty} 1/n$ doesn't), we can use that fact without difficulty, just as we can count from one to two without enumerating all the real numbers that exist between them.

Dealing with the Doubtful: Navigating the Transfinite Universe

A further feature of empiricism is that it gives promise of some relief to the heartburn caused by considering some features of modern mathematics, particularly in the area of transfinite set theory. Through the consideration of infinity, beginning with the Zermelo axiom that it exists (whatever that means to a non-empiricist), paradoxes of various sorts appear to lurk, contradicting the stern belief that logical processes must necessarily lead to absolute truth.

It is beyond the scope of this article to plunge into the transfinite world; few practicing mathematicians do so. But its existence is a constant source of uneasiness for those who still quest for certainty. Perhaps an empiricist approach, which allows for doubt when consensus is impossible, will bring some relief to those who feel they must explore the infinite.

Philosophical Epilogue

This essay is not an exercise in academic philosophy, although I hope philosophers will encounter in it a reflection of the respect I feel for the Queen of the Sciences. I have carefully considered the major themes of contemporary philosophy and earnestly tried to deal accurately with philosophical concepts when my argument touches them. Given the nature of philosophy, I cannot expect to emerge without generating controversy, but I hope at least that my conclusions can be found philosophically palatable.

It is also not a defense of John Stuart Mill's mathematics; indeed, I specifically reject Mill's contention that there are no abstract numbers, an argument I declare to have been a disaster. I do argue that Mill's *epistemology*, his empiricist insistence that all our knowledge derives from observation of singular events, when combined with Crick's hypothesis that human consciousness and ratiocination are physical events in the brain, make it possible to view mathematics as a phenomenon taking place in the physical universe. Such a view can resolve many of the philosophical confusions that plague the profession at present.

This argument, to be effective, must be directed to mathematicians – those who are interested, to a greater or lesser degree, in the way their science relates to the rest of the sciences, and to the world they inhabit. The reason is that philosophers have universally accepted the view of mathematics that its own practitioners appear, frequently unconsciously, to hold; a view, as I have attempted to demonstrate, that is in many aspects wrong. From this derive most of the conundrums and controversies that frustrate philosophy of mathematics today, and have frustrated it from its very beginning. So the argument must be directed to the source of the frustration: to the mathematical community.

It is not an easy audience to approach. The philosophy of mathematics is not a formal part of the curriculum and is rarely touched on in any mathematics course. The principle reason given for this lack is the crowded agenda that mathematics majors face, along with the fact that it is perfectly possible to practice mathematics successfully without delving into questions of philosophy.

But the reason goes deeper. The fact is, even if there were an urge to explore philosophical questions with students, there is total disagreement on how such questions could be answered. The metaphysics of mathematics is in chaos. There is no widely accepted concept of what mathematical objects are, where they are located, or how they relate to the physical world. There is not even agreement on defining what mathematics is, or what its relationship to the physical sciences is.

Mathematicians do agree, almost universally, on the epistemology of mathematics. Students are not instructed in philosophy, but one philosophical principle is ingrained without question or doubt: the belief that mathematical propositions, once proved, are necessarily true, in contrast to propositions based on observation of the physical world.

In summary: There is chaos in the metaphysics of mathematics, and unity in its epistemology. Unfortunately, the almost unanimous view of mathematical necessity is wrong.

This article offers a simple answer to metaphysical confusion. It declares numbers, axioms, mathematical laws, theorems, and other abstract components of mathematical science, to be physical objects, along with all other abstract ideas and concepts. They are the physical state of the neurons in human brains. They exist – there is no need to axiomatize their existence – in specific times and places.

Assuming an ordered universe, in which physical events occur according to universal laws, the human mind when manipulating abstract concepts such as numbers will behave in a lawful way, in the same manner that other physical objects follow their own laws. We may observe such behavior and reach certain conclusions about the nature of those laws, just as we do for the rest of the physical universe.

The empiricist position on epistemology is equally simple. Contrary to the broadly accepted assertion that mathematical truth is a special kind, it holds that we know mathematical propositions are true in the same way that we know propositions in other sciences are true: by observation of singular instances which we generalize inductively into universal laws.

Though simple, these philosophical solutions are not easy to accept. As I have noted, adopting the Crick hypothesis means denying a view of human consciousness that has been almost unconsciously accepted throughout history. Equally difficult for mathematicians is abandoning faith in the necessary truth of mathematical propositions. I have tried to argue that putting mathematics on a level with physical sciences does not call for any significant degree of doubt about the truth of mathematical propositions. After all, if we were not firmly certain of the truth of Newtonian mechanics, we would not climb aboard airplanes. But it is certainly a hard sell.

Nevertheless, bringing mathematics into the family of material sciences should enable viewing it in an accurate light, opening new possibilities for its development and resolving many nagging quandaries that lurk on the outer fringes of its frontiers. And it certainly would remove a vast amount of confusion and error in the philosophy of mathematics.

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