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On Doing Mathematics

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Synopsis

Who is a mathematician? What does it mean to do mathematics? I discuss my process in solving a math problem, and what it meant to me.

Recently I ran across the Mathematics Genealogy Project,[1] and looked up my mentor, Gisela Ahlbrandt. I wanted her to have at least one mathematical "descendant", so I wrote to the people who run that site and asked what the requirements were for identifying oneself as a mathematician. Their rule is that you need a PhD in math, so I don't qualify. But Gisela, I want you to know that I am doing math and having fun with it, and *I* consider myself a mathematician.

I've seen much discussion lately of the fact that, even in graduate school, students go for years before working on any original problems. The problem I'll describe in this paper is not original, but it's probably the first problem I approached on my own, just because I wanted to understand it – probably the closest I've come to really "doing mathematics". I have found my thinking about what it means to "do mathematics" evolving some, and would like to explore that in the context of this problem.

I'm a math teacher more than a mathematician. I often discuss teaching with a good friend who used to teach Waldorf kindergarten. A basic tenet of the Waldorf philosophy is that teachers should keep learning new skills. This helps them stay in touch with the joys, frustrations, and confusion inherent in learning something new, and helps students see that learning is a process. She said, "You want to bring the material out of yourself, not a textbook,

[1] http://genealogy.math.ndsu.nodak.edu, accessed July 27, 2011.

so there's life in it." I think this is a good idea for all teachers. I often tell my students about how slow a learner I am when it comes to music. And sometimes I'm able to share with them how excited I am by some new mathematical insight I've gained.

I love playing around with math. I know that teaching math has deepened my understanding of much of the mathematics I learned before I entered college. Problem-solving is what math is all about to me, but it's mysterious to me how I can teach that. I've read good ideas on this from George Polya, Paul Zeitz, Alan Schoenfeld, Stephen Brown, and others,[2] and hope I've helped my students approach math in a healthier way. On an online math discussion list,[3] we recently began discussing how useful it would be to have narratives of different people's thoughts as they solve a problem. I loved that idea, and recorded my thinking process as I solved a counting problem I'd recently encountered.[4] I also remembered that I'd written up the problem I describe here, and decided it would be a good description of my problem-solving process. At the time I first wrote this problem up, I was trying to make my thinking clear to two other people, so I made the effort to capture as much of my thinking in writing as possible.

I was first introduced to this problem back in the '70s, most likely in Harold Jacobs' *Mathematics: A Human Endeavor*.[5] No answer was given, so I've returned to it now and again over the years.

> On a circle, put some points. Connect each point to every other point with straight lines. How many regions are created for n points? (Check your prediction by working out the circle with 6 points.)

If you've never tried this problem, you'll get more pleasure out of this essay if you try it now, before you move to the next page.

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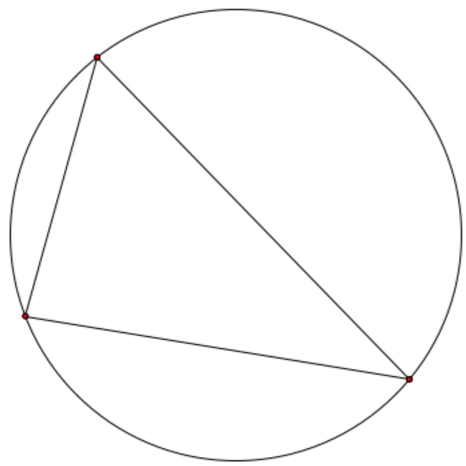
[2] George Polya, *How To Solve It*, 1945, Princeton University Press, 204 pages.
Paul Zeitz, *The Art and Craft of Problem Solving*, 1999, John Wiley, 334 pages.
Alan Schoenfeld, *Mathematical Problem Solving*, 1985, Academic Press, 409 pages.
Stephen I. Brown, Marion I. Walter, *The Art of Problem Posing*, 1983, Franklin Institute Press, 147 pages.

[3] `http://groups.google.com/group/mathfuture` accessed July 27, 2011.

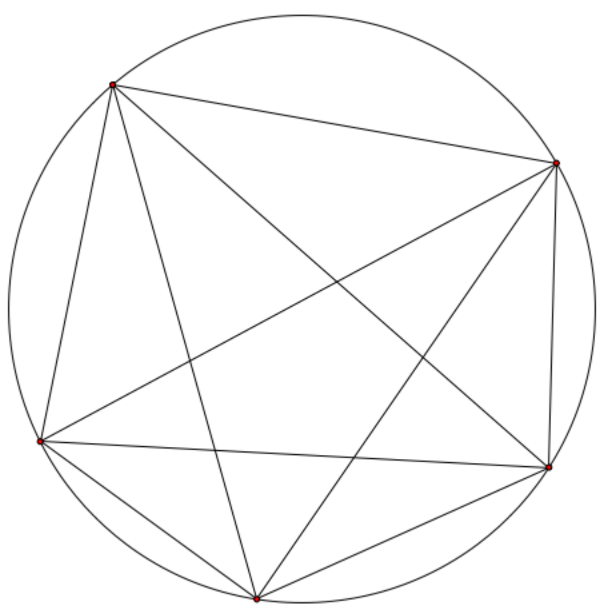
[4] `http://mathmamawrites.blogspot.com/2011/02/tin-ceilings-triangles-and-loving-math.html` accessed July 27, 2011.

[5] Harold Jacobs, *Mathematics: A Human Endeavor*, 1970, W.H. Freeman, 527 pages.

We begin drawing for small n. 3 points makes 4 regions:

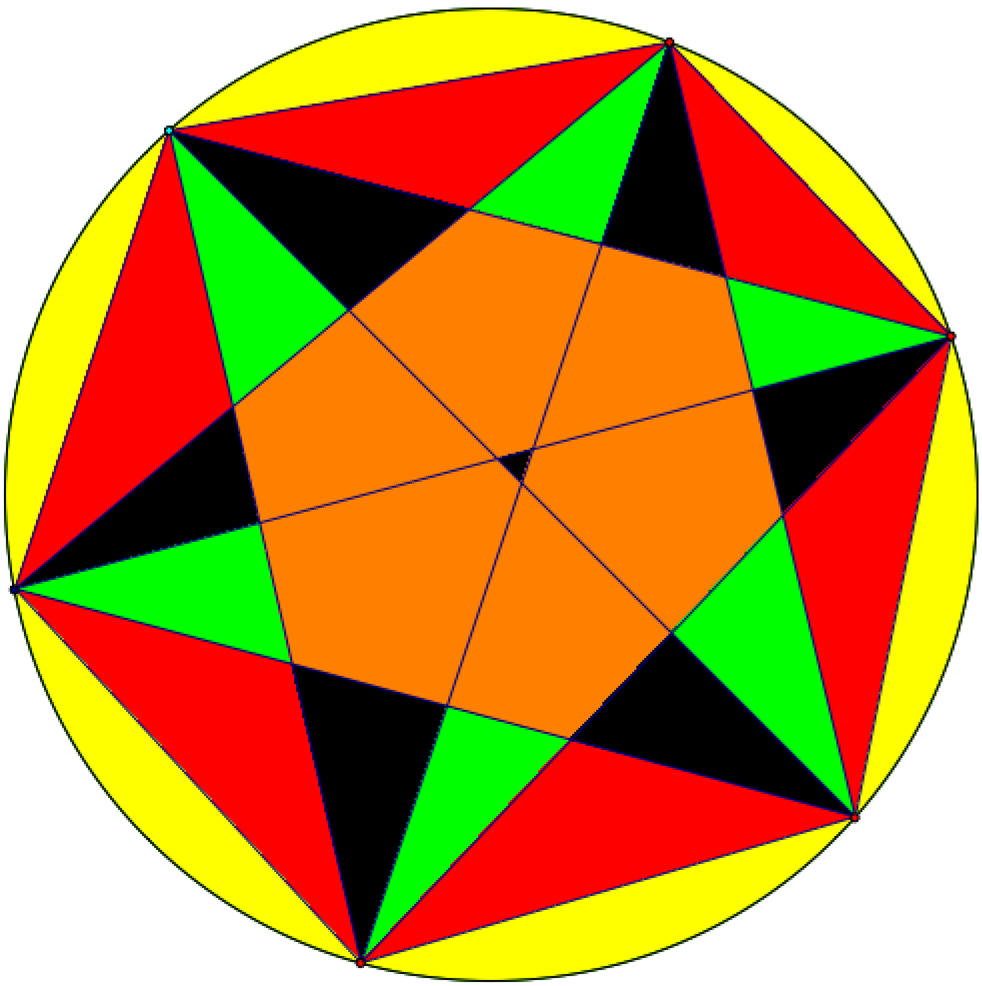


And 5 points makes 16 regions.



After drawing, counting, and thinking some more, see the next page.

The first time I encountered this problem was when I was in my teens. I played with it for a long time and saw that the original pattern of doubling didn't hold. (Did you figure that out yourself?) The number of regions doubles from 1 to 2 to 4 to 8 to 16, but then, for 6 points, there are only 31 regions![6]



[6]Actually, if the points are evenly spaced, there are 30 regions – the center region disappears where three lines cross. This can happen with any $n > 5$, so we redefine the problem as finding the most possible regions using n points, and note the new problem of exploring how many times we could cross 3 lines in one place.

I tried to look at the differences between terms. But I saw no pattern and eventually gave up. I played with this problem a number of times over the years, and never got much further.

In early 2008 I was facilitating a group of people online who were studying Jacobs' text. When we began the chapter on inductive reasoning, this problem was my favorite choice for showing that an obvious pattern isn't necessarily the true one.[7] But I figured I should know the true pattern, since I was planning to facilitate a discussion on it.

I worked on this problem with more diligence than I had in the past. I also had more tools in my head than I had in the past. As a college math teacher, I've been explaining combinatorics in my statistics classes and factoring of higher order polynomials in my pre-calculus classes semester after semester, both of which came up as I solved this. So this time I got much further. I still didn't quite crack the nut, though, and finally I cheated and looked online. I loved the elegance of the solutions I saw.

Fast forward about a year. I now conduct a math salon at my home, and at one meeting I gave this problem to Chris and Sharon, the two biggest math enthusiasts in the bunch. They made some headway, but were stuck. I had had enough time to forget all the details (the joys of a bad memory!), so the three of us struggled with it. We had to let it go when other commitments called us. But I woke up the next morning hungry to solve it. Although I had forgotten the details of the elegant method, I had some vague notion that it involved combinatorics. What we had gotten the previous day was:

1. that n points make $\frac{n(n-1)}{2}$ lines (each of the n points will have a line connecting it to each of the other $n-1$ points, but that would count each line twice, so divide by 2), and
2. that each line makes a new region each time it crosses another line and when it reaches its destination.

Where we were stuck was in figuring out how many times each line crossed another, since some didn't cross others, and some did.

We figured that it was some sort of combinatorics thing, and that it would involve expressions like $n(n-1)(n-2)\cdots$. That was my starting point in

[7] Ben Blum-Smith has compiled a list of such problems at:
`http://researchinpractice.wordpress.com/2010/05/07/pattern-breaking/`
accessed July 27, 2011.

the morning. Since we thought it would be somehow combinatorial, that would mean any formula for it was likely to be polynomial. I drew pictures for n up to 8, counted regions, and looked at the differences in number of regions between n and $n+1$ points. I still didn't see a pattern so I looked at 2^{nd} differences (i.e., differences between the 1^{st} differences). Still nothing. The 3^{rd} differences, though, were $1, 2, 3, 4, \ldots$, which made 4^{th} differences constant! (I have never worked with n^{th} differences in any classes I've taken or taught; I must have picked this technique up from a recreational math book.)

Here's what I got:

the number n of points	the number r of regions	1^{st} diff	2^{nd} diff	3^{rd} diff	4^{ht} diff
1	1				
2	2	1			
3	4	2	1		
4	8	4	2	1	
5	16	8	4	2	1
6	31	15	7	3	1
7	57	26	11	4	1
8	99	42	16	5	1

The 4^{th} difference is sort of like the 4^{th} derivative, which is constant for a 4^{th} degree polynomial, so I decided to go with that. I decided the formula should be of the form

$$r = an^4 + bn^3 + cn^2 + dn + e.$$

Now if we think of the a, b, c, d and e as our variables, and put our first five pieces of data in for n and r, we get:

(1,1) gives $a + b + c + d + e = 1$;
(2,2) gives $16a + 8b + 4c + 2d + e = 2$;
(3,4) gives $81a + 27b + 9c + 3d + e = 4$;
(4,8) gives $256a + 64b + 16c + 4d + e = 8$;
(5,16) gives $625a + 125b + 25c + 5d + e = 16$.

I knew that, although all this data looked exponential, it could also be fit by a 4^{th} degree polynomial. Given any m points with all different x coordinates, I knew this method would allow me to find a polynomial of degree $m-1$ (or lower) that fit them.

This is a system of 5 equations in 5 variables, and should have a solution. So I put it in my TI calculator as a matrix and had it row reduce.

The answer looked very ugly, but after converting it to fractions, I got

$$a = \frac{1}{24}, \quad b = \frac{-1}{4}, \quad c = \frac{23}{24}, \quad d = \frac{-3}{4}, \quad \text{and} \quad e = 1.$$

Still too ugly for me, but ... the formula now looked like this:

$$r = \frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{23}{24}n^2 - \frac{3}{4}n + 1. \tag{1}$$

I think I had gotten this far in the past, but it had seemed too ugly to make any kind of sense. I'd always been stumped at this point. But knowing that the solution had something to do with combinatorics (and knowing I'd seen and forgotten a simple solution online) gave me more stamina this time. At this point, I still had the attitude of a student, needing the security of knowing a solution is out there to give me the will to keep working. (I don't know if I've moved beyond that stance yet.)

From my discussion with Chris and Sharon the evening before, I knew we could pull out the 1 (for the first region in the circle, before any lines add more). I also pulled out a 1/24 and an n. (Note to self: $24 = 4!$ This looks good!!!) I got:

$$r = 1 + \frac{n}{24}\left(n^3 - 6n^2 + 23n - 18\right). \tag{2}$$

Now I wanted to factor that cubic. I needed some of the theory I teach in pre-calculus for that. If there's a rational factor, it will come from factors of 18 over factors of 1 (the coefficient of n^3). I was ready to try $n \pm 1$, $n \pm 2$, $n \pm 3$, $n \pm 6$, $n \pm 9$, but when I used polynomial division, my first guess of $n - 1$ worked. Factoring out the $n - 1$, we have:

$$r = 1 + \frac{n}{4!}(n-1)\left(n^2 - 5n + 18\right). \tag{3}$$

The last bit doesn't factor, but if we're thinking combinatorics, we might think we want $n-2$ and $n-3$ in there. $(n-2)(n-3) = n^2 - 5n + 6$, so we might think of this as $[(n-2)(n-3)+12]$, but where does that $+12$ come from???

That's how far I got before I felt stuck and started typing an email message to Chris and Sharon to tell them about my progress. As I finished the message, I thought I might have something else to try ... I went back to paper and pen ...

Aha! I thought I liked 4!, but if I go back to $n(n-1)$ as the number of lines, then I'd want it over 2. Now I get:

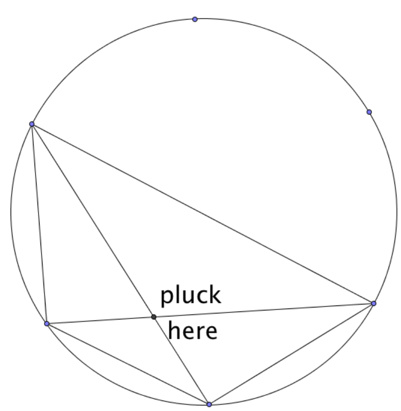
$$r = 1 + \frac{n(n-1)}{2}\left(\frac{(n-2)(n-3)+12}{12}\right) \quad \text{or } \ldots$$

$$r = 1 + \frac{n(n-1)}{2}\left(\frac{(n-2)(n-3)}{12} + 1\right). \tag{4}$$

And I can almost see it! Now I just need to figure out where the $\frac{(n-2)(n-3)}{12}$ part comes from. I already know the inside $+1$ comes from there being one more region for each line than the number of other lines it crosses.

We saw that the number of crossings was different for each line. Perhaps this tells us something like ... on average, the number of crossings is $\frac{(n-2)(n-3)}{12}$. Why would that be??? Stuck again, I sent the message, and went on to other things.

A few hours later: I've got it!! I kept looking at one crossing, because I didn't like that notion of average. Each crossing happens because of 4 points that make two lines that cross one time. (And every time I said that to myself, trying to wrap my brain around it, I imagined pulling the lines up as if they were strings in a game of cat's cradle.)



Every collection of 4 points will produce 6 lines that cross one time. How many collections of 4 points are there in n points?

$$_nC_4 = \frac{n!}{4!(n-4)!}$$

So ... I get to have my cake (4!) and eat it too (number of lines). If we distribute the $\frac{n(n-1)}{2}$, we get:

$$r = 1 + \frac{n(n-1)(n-2)(n-3)}{24} + \frac{n(n-1)}{2} \quad \text{or } \ldots$$

$$\boxed{r = 1 + {}_nC_4 + {}_nC_2.} \tag{5}$$

Elegance from ugliness! We have 1 for the region that's there to begin with, $_nC_4$ for the number of crossings, and $_nC_2$ for the extra region each line produces as it reaches its ending point. Yeay!!!

Although I was thrilled to find this, I felt like I had hit the problem over the head with a hammer, using brute force for something that could have come more easily with a more delicate touch. I was distressed at my lack of insight. But isn't it always like that? We see things so hazily at first, and then when they become clear we can't imagine why it was hard to see them in the first place. Now I can admire the way my method is like doing science or detective work. I'm trusting that the pattern is there, and I'm using data to look for the pattern. After I've found the formula, I can use it to get to the insight. Nice!

As a teacher, I've spent most of my time over the years thinking about how to help students, and not so much time doing mathematics. But a few years ago I discovered the communities of people online who are discussing math and how to help students learn it, and in the process I began to explore more and more mathematics on my own.

As I work on these problems, many recreational, I have become more comfortable with my own methods, and less self-conscious about my lack of insight as I start on a problem. Although I haven't solved any open problems, I have made up a few puzzles of my own, and have begun to feel like a creator of math. It's a good feeling.