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Gromov–Witten Theory of Blowups of Toric Threefolds

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May, 2012

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Abstract

We use toric symmetry and blowups to study relationships in the Gromov–Witten theories of $\mathbb{P}^3$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. These two spaces are birationally equivalent via the common blowup space, the permutohedral variety. We prove an equivalence of certain invariants on blowups at only points of $\mathbb{P}^3$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by showing that these invariants descend from the blowup. Further, the permutohedral variety has nontrivial automorphisms of its cohomology coming from toric symmetry. These symmetries can be forced to descend to the blowups at just points of $\mathbb{P}^3$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Enumerative consequences are discussed.

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Chapter 1

Introduction

An active problem over the last several decades in physics has been the formulation of a unified theory of gravity. It is now well known that Einstein's theory of general relativity and quantum mechanics are incompatible. General relativity is a theory of gravity on the large scale that does not agree with predictions at the quantum level. String theory is a formulation of high-energy physics that is a leading candidate for providing a unified explanation of gravity at the quantum and general levels. The fundamental idea behind string theory is to replace point particles, zero-dimensional objects, with one-dimensional strings. The paths traced out by such strings are surfaces, known as world sheets. For physical reasons, these take the form of pseudoholomorphic curves. It is thus natural in string theory to study maps from such world sheets, complex curves, into space-time. Further, space-time in string theory consists of the four dimensions in classical physics, and six extra dimensions. This is modeled as modeled (at least locally) as $\mathbb{R}^{3,1} \times X$, where X is a Calabi–Yau manifold of six real dimensions. Thus it is of physical interest to study maps from complex curves into a Calabi–Yau manifold X. This observation had a groundbreaking impact in enumerative geometry.

1.1 Enumerative Geometry: The Motivating Problems

Enumerative geometry is a very old subject, that has evolved extensively over the last two centuries. It was first active in the nineteenth century. Among the most famous problems in the subject was Hilbert's fifteenth problem, which concerned Schubert calculus and enumerative geometry. Although the former was understood through the topology and intersec-

tion theory of the Grassmannian, the latter remained unclear for several decades. Despite success in the early days of enumerative geometry, many of the fundamental problems eluded mathematicians for large portions of the twentieth century, until the subject was revolutionised in the 1990s by developments in high energy physics and string theory.

The basic question of enumerative geometry is stated as: *How many geometric structures of a given type satisfy a collection of geometric conditions?* In a modern guise, the geometric structures of a type are homology classes, and the geometric conditions are tangency conditions encoded by cohomology.

Definition 1. *A quintic threefold is a hypersurface $X \subset \mathbb{P}^4$ of degree five.*

A classical enumerative problem is the enumeration of rational curves (genus zero) of degree d in X, for X general. If we choose a parametrization $f : \mathbb{P}^1 \to X$ we can describe a rational curve as

$$f(x_0, x_1) = (f_0, f_1, f_2, f_3, f_4)$$

for homogeneous polynomials f_i of degree d. We may describe X as the zero locus of a homogeneous degree five polynomial F in variables $x_0, \ldots, x_4$. The condition that the image of f be contained in X is simply given by

$$F(f_0, \ldots, f_4) \equiv 0.$$

A simple dimension counting argument shows that the "expected" number of solutions for f is finite. In other words, the solution space has zero dimension. This leads us to the famous conjecture of Clemens.

Conjecture 2 (Clemens' Conjecture)**.** *Given any positive number d, the number of degree d rational curves in X, a general quintic threefold, is finite.*

Amazingly, this conjecture remains unresolved. The Clemens conjecture is known for $d \leq 9$, see Katz (2006). In low degree, the numbers of rational curves have been calculated. It was known in the nineteenth century that there are 2875 lines in a general quintic threefold. The number of degree two rational curves on a general quintic was worked out in 1985 to be 609,250 in Katz (1986). The $d = 3$ number was worked out in 1991 to be 317,206,375. In the same year, a group of physicists announced the number of rational curves for all degrees. These predictions were made using the now famous *Mirror Theorem* and the technique of *Gromov–Witten theory*.

1.2 Gromov–Witten Theory

The quintic threefold has the property that the first Chern class of its canonical bundle has vanishing first Chern class, and thus is a Calabi–Yau manifold. Such manifolds were of interest to physicists. The methods developed by string theory form what are now the well established mathematical subjects of Gromov–Witten theory and mirror symmetry. Gromov–Witten invariants historically arose as certain correlation functions associated to topological gravity on Calabi–Yau threefolds. Given a smooth algebraic variety X, and a smooth curve C, of homology class β, the Gromov–Witten invariant of β in X of genus g, denoted $\langle\ \rangle^X_{g,\beta}$, is a rational number that is an invariant given the input data. These numbers contain virtual enumerative information regarding the curves in the space. In particular, these numbers are invariant under (symplectic) deformations of X. The Gromov–Witten invariants for lines on the quintic threefold is given by

$$\langle\ \rangle^Q_{0,[\mathrm{line}]} = 2875,$$

while the invariant for degree 2 curves is given by

$$\langle\ \rangle^Q_{0,2[\mathrm{line}]} = 609250 + \frac{2875}{8}.$$

The fractional number in the sum is an artifact of the property that Gromov–Witten invariants are sensitive to double covers of lines being counted as conics. In this sense, the Gromov–Witten invariants contain quantum enumerative information.

1.2.1 The Gromov–Witten Approach to Enumerative Geometry

The approach of Gromov–Witten theory is elegant. We will form a type of parameter space $\overline{M}_{g,n}(X,\beta)$, which is a moduli stack of all isomorphism classes of (stable) maps $[f : C \to X]$ such that $f_\star[C] = \beta$. That is, we form a space whose points represent maps from a domain curve, to a target. Enumerative geometry computations on X amount to understanding intersections on this moduli stack. Though understanding the geometry of this stack is often a formidable task, the technique has been extremely lucrative in solving enumerative problems. It should however be noted that Gromov–Witten invariants are not always enumerative. That is, it is often the case in higher genus domain curves or targets that are not projective spaces, that enumerative geometric information cannot be extracted from

the invariants. The invariants do still provide meaningful and geometrically interesting properties concerning the target variety.

1.2.2 Toric Variety Targets

In many cases, properties of the target variety X can be used to ease the computations of the Gromov–Witten invariants. One shining example of such success has been toric geometry. Toric varieties are normal varieties that allow the embedding of an algebraic torus, $(\mathbb{C}^\times)^n$. This multiplicative group acts on the whole variety via a morphism, and often simplifies the study of the variety. The geometry of toric varieties are determined by certain combinatorial objects known as fans and polytopes. These can make the computations in Gromov–Witten theory far more tractable. Details and an introduction to toric geometry can be found in Chapter 2. Indeed, there are now several techniques that effectively compute the Gromov–Witten theory of toric threefolds. These include techniques such as fixed point localization, the topological vertex and most recently the remodeling conjecture. Each of these use the combinatorial data of the variety in a different way. The motivation of this project is to gain a deeper understanding of the role of combinatorial symmetries in developing computational techniques in toric Gromov–Witten theory. Our particular point of entry will be the study of the Gromov–Witten theory of toric varieties under blowups (see Chapter 2). The toric varieties of greatest relevance to our problem and results are projective spaces and products of projective spaces, particularly $\mathbb{P}^3$ and $(\mathbb{P}^1)^{\times 3}$. These two varieties are related via blowups, and in this paper we prove equivalence between a large family of their invariants, using the technique of toric symmetry.

1.3 Main Results

The main results of this thesis hinge on proving the equivalence of the all-genus virtual dimension zero nonexceptional GW theories of four spaces, illustrated in the following diagram, involving the blowups of $\mathbb{P}^3$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In the diagram below, the relationship between the invariants of X and $\hat{X}$ was established by Bryan and Karp (2005). The rest of the diagram was completed in this thesis.

In more detail, let $X = \mathbb{P}^3(6)$ be the blowup of $\mathbb{P}^3$ at six points, $p_1, \ldots, p_6$. Let $\tilde{X} = (\mathbb{P}^1)^{\times 3}(4)$ be the blowup of $(\mathbb{P}^1)^{\times 3}$ at $\tilde{p}_1, \ldots, \tilde{p}_4$. Let $\tilde{h}_{12}$, $\tilde{h}_{23}$, and $\tilde{h}_{13}$ be the three homology classes of lines pulled back from $(\mathbb{P}^1)^{\times 3}$, and h be

$$\begin{tikzcd} GW(\hat{X}) \arrow[r, dash, "\text{Isomorphism}"] \arrow[d, dash, "\text{Blowup}"'] & GW(\hat{\tilde{X}}) \arrow[d, dash, "\text{Blowup}"] \\ GW(X) \arrow[r, dash, "\text{Crepant}", "\text{Transformation}"'] & GW(\tilde{X}) \end{tikzcd}$$

the pullback of the line class in $\mathbb{P}^3$. The classes e_i and $\tilde{e}_i$ are the appropriate classes of lines in the exceptional divisors above the blowups of points.

Theorem 3. *Let X and $\tilde{X}$ be as above. If $\beta = dh - \sum_{i=1}^{6} a_i e_i \in A_1(X)$ with $\{a_5, a_6\} \neq \{0\}$, then for $\tilde{\beta} = \sum_{1 \leq i < j \leq 3} \tilde{d}_{ij} \tilde{h}_{ij} - \sum_{i=1}^{4} \tilde{a}_i \tilde{e}_i$, we have equality of all-genus invariants*

$$\langle\ \rangle^{X}_{g,\beta} = \langle\ \rangle^{\tilde{X}}_{g,\tilde{\beta}}$$

where the coefficients of β and $\tilde{\beta}$ are related as

$$\begin{aligned} \tilde{d}_{12} &= d - a_2 - a_3 \\ \tilde{d}_{13} &= d - a_1 - a_3 \\ \tilde{d}_{23} &= d - a_2 - a_3 \\ \tilde{a}_1 &= a_4 \\ \tilde{a}_2 &= d - a_1 - a_2 - a_3 \\ \tilde{a}_3 &= a_5 \\ \tilde{a}_4 &= a_6. \end{aligned}$$

Additionally, the invariants on $\tilde{X}$ above satisfy a symmetry given by the following theorem.

Theorem 4. *Let $\tilde{X}$ be as above. Then if $\tilde{\beta} = \sum_{1 \leq i < j \leq 3} \tilde{d}_{ij} \tilde{h}_{ij} - \sum_{i=1}^{4} \tilde{a}_i \tilde{e}_i$, and $\{a_3, a_4\} \neq \{0\} \in H_1(\tilde{X})$, we have*

$$\langle\ \rangle^{\tilde{X}}_{g,\tilde{\beta}} = \langle\ \rangle^{\tilde{X}}_{g,\tilde{\beta}'},$$

where $\tilde{\beta}' = \sum_{1 \leq i < j \leq 3} \tilde{d}'_{ij} \tilde{h}_{ij} - \sum_{i=1}^{4} \tilde{a}'_i \tilde{e}_i$ has coefficients given by

$$\begin{aligned} \tilde{d}'_{12} &= \tilde{d}_{12} + \tilde{d}_{23} - \tilde{a}_1 - \tilde{a}_2 \\ \tilde{d}'_{23} &= \tilde{d}_{23} \\ \tilde{d}'_{13} &= \tilde{d}_{13} + \tilde{d}_{23} - \tilde{a}_1 - \tilde{a}_2 \\ \tilde{a}'_1 &= \tilde{d}_{23} - \tilde{a}_2 \\ \tilde{a}'_2 &= \tilde{d}_{23} - \tilde{a}_1 \\ \tilde{a}'_3 &= \tilde{a}_4 \\ \tilde{a}'_4 &= \tilde{a}_3. \end{aligned}$$

The equality in Theorem 3 is determined by the push forward of the blowup-blowdown birational map, described in the figure above. In low degree, these theorems allow us to detect relationships between the stationary invariants of $\mathbb{P}^3$ and $(\mathbb{P}^1)^{\times 3}$.

Corollary 5. *The all–genus stationary Gromov–Witten theory of* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ *is equivalent to that of* $\mathbb{P}^3$ *in degree less than five.*

For further details on conventions, definitions and some background, the reader is encouraged to see later chapters, particularly Chapters 2–6.

1.4 Relation to the Crepant Transformation Conjecture

In general Gromov–Witten invariants are not functorial under birational maps. That is, given a birational map $\varphi : Y_1 \to Y_2$,

$$\langle \, \rangle^{Y_1}_{g,\beta} \neq \langle \, \rangle^{Y_2}_{g,\varphi_\star\beta}.$$

The general conditions under which the invariants can be pulled back are still unknown and this remains an open problem in Gromov–Witten theory. A leading proposal to answer this question is the Crepant transformation conjecture (CTC). The Gromov–Witten generating function frequently reveals properties that are not apparent at the level of invariants. CTC is stated in the language of generating functions. A map $\phi : Y_1 \to Y_2$ is said to be Crepant if

$$K_{Y_1} = \phi^\star K_{Y_2}.$$

That is, the canonical class of Y_1 is the pullback of the canonical class of Y_2. In this case it is conjectured that under an appropriate change of variables and an analytic continuation of the power series, the Gromov–Witten generating function of Y_1 is equal to that of Y_2.

As has been stated above, the main results are one example of a Crepant transformation. However, the results are stated in the language of invariants. Equality of invariants clearly implies the equality of the appropriate sectors of the generating function, and our arguments do not require analytic continuation or change of variables. The results of this thesis prove the equality of *nonexceptional* sectors of the Gromov–Witten invariants of the spaces described. Further, the main results of this thesis, together with previous results from Bryan and Karp (2005) involve proving equality between sectors of invariants under blowups, which are not Crepant transformations.

1.5 The Structure of this Document

In this document, we will overview the basic constructions in toric algebraic geometry, emphasizing the tools that will be required for the proofs of the main theorems. We will also explain the rudiments of Gromov–Witten theory, though we will not fully treat the more technical aspects such as the construction of the virtual class. In Chapters 2 and 3, we will introduce the constructions that drive this project, focussing on toric blowups of smooth projective toric varieties, and the concept of a toric symmetry. In Chapter 3 we will also review the results and techniques used by the author and collaborators (2011). In Chapter 4 we introduce the idea of Gromov–Witten theory and the moduli space of stable maps, as well as make concrete, the connection between toric symmetry and the Gromov–Witten theoretic motivations. With this machinery, in Chapter 5 we explicitly construct the toric blowup and analyse the toric symmetry associated to the common blowup mentioned above. This will lead us to the well-studied permutohedron, and the classical Cremona transform on $\mathbb{P}^n$. Finally we will prove the main theorems. The results follow from descent of nonexceptional invariants via the blowup map, in conjunction with the set up in Chapter 5.

Chapter 2

Toric Algebraic Geometry

In this chapter we introduce some basic notions in algebraic geometry and in particular toric geometry. We will define and construct fans and polytopes, and the toric varieties associated to them. We will define the intersection (cohomology) ring and explore the the combinatorial techniques used to compute it. We will also introduce the notion of blowup, and discuss the cohomology of toric blowups. For an introduction to affine and projective varieties, the Zariski topology and birational geometry, see the wonderful textbook *Undergraduate Algebraic Geometry* by Reid (1988). For a treatment of the scheme theoretic aspects of the subject, which are not covered here, see Eisenbud and Harris (2000). For a deeper treatment of toric geometry including intersection theory on toric varieties, the reader is encouraged to see the beautiful text on toric varieties by Fulton (1993).

2.1 Varieties, Projective Spaces and the Zariski Topology

Classically, the central objects of study in algebraic geometry are varieties. Intuitively they are geometric objects that are locally cut out by polynomials. Varieties are familiar objects from elementary geometry: lines, conic curves and surfaces defined by polynomials, are all examples of varieties. We will be chiefly concerned with projective varieties in this document, but we begin with the more simple notion of affine spaces and affine varieties.

Definition 6. *Let k be an algebraically closed field. Then $\mathbb{A}^n_k$, known as* affine n-space over k, *is the set of n-tuples of elements of k. A point in this space is of the form $P = (a_1, \ldots, a_n)$ with $a_i \in k$.*

In algebraic geometry, we usually do not work in the standard Euclidean topology, but rather a topology defined by vanishing sets of polynomials. This is known as the *Zariski topology* which we now describe. Let $k[x_1, \ldots, x_n]$ be the polynomial ring in n-variables over k, and let I be an ideal of polynomials in $k[x_1, \ldots, x_n]$. We define $V(I)$ to be the set of all points in $\mathbb{A}^n_k$ where all polynomials in I vanish under evaluation. That is,

$$V(I) = \{P \in \mathbb{A}^n_k : f(P) = 0 \ \forall f \in I\}.$$

Under the Zariski topology, we declare that the open sets are generated by the complements of sets of the form $V(I)$ for ideals $I \subset k[x_1, \ldots, x_n]$. It is left to the reader to check that these indeed form a topology.

An *algebraic set* is a set of the form $V(I)$ for some ideal $I \subset k[x_1, \ldots, x_n]$. An algebraic set $X \subset \mathbb{A}^n_k$ is said to be *irreducible* if there does not exist a decomposition

$$X = X_1 \cup X_2, \text{ with } X_1, X_2 \subsetneq X,$$

of X as a union of two strict algebraic subsets. For example the algebraic subset $V(xy) \subset \mathbb{A}^n_k$ is the locus consisting of the two coordinate axes, and hence is irreducible.

We define an affine variety as follows.

Definition 7. *An* affine variety *is an irreducible closed subset (under the Zariski topology) of $\mathbb{A}^n_k$.*

The reader is warned here that whether there exists a distinction between a variety and an algebraic set is a matter of convention and varies in the literature.

Although affine varieties are a source of many familiar examples, they are not without shortcomings. In particular, affine space is not compact and there are many problems with the theory of intersections in affine space. Consider the following example.

Example 8. *Nearly all the curves studied in introductory calculus courses are examples of varieties (over the complex numbers). The vanishing locus of the equation $y - mx$ gives a line in $\mathbb{A}^2$, while the equation $x^2 + y^2 - 1$ gives us a circle. The other conic sections are affine varieties as well. The ellipsoids and spheres are affine subvarieties of $\mathbb{A}^3$.*

Example 9. *Let us ask the enumerative question: "In how many points do two lines in $\mathbb{A}^2_k$ intersect?" This question has many answers. The answer could be infinity: if the two lines are identical. However this is largely a problem of convention, as we are really asking how many points do two distinct lines intersect*

at. The answer could be zero, if the lines are parallel. The general answer, and the answer that we would like to have consistently, is one. However, we cannot content ourselves with an answer of "usually one" for such an enumerative question. Intuitively, parallel lines in affine space can be seen as the limit of a family of intersecting lines. Since the space is not compact, this intersection property is "lost" in the limit of the family. This "point at infinity" is not a part of affine space.

One set of nonaffine varieties which resolve problems such as the above are projective varieties. Before we can define projective varieties, we will first define projective space.

Definition 10. *We define* projective space $k\mathbb{P}^n$ *as the set of all* $n+1$*-tuples* $(a_0, \ldots, a_n)$ *not all zero, under the equivalence* $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ *for all* $\lambda \in k^\times$. *A representative coordinate* $(a_0, \ldots, a_n)$ *gives a point of projective space.*

Projective space has many desirable properties. If $k = \mathbb{C}$, then $\mathbb{CP}^n$ is a compact, complex analytic manifold under the analytic topology. We will often use the complex analytic structure of projective space knowing that it is compatible with the Zariski topology.

To define projective varieties however, we cannot simply look at the vanishing sets of any polynomial. In fact this is not even well-defined. Notice that a polynomial f does not not necessarily have a well-defined value at a point in projective space. Indeed not even its vanishing set is well-defined. For instance, consider the polynomial $f(x_0, x_1) = x_0 + 1$. Observe that $f(\lambda a_0, \lambda a_1) = \lambda a_0 + 1$ changes with λ and could be zero for some λ and nonzero for others. To fix this problem, we instead use the graded ring of all homogenous polynomials in $n+1$ variables $x_0, \ldots, x_n$. These are polynomials where each term has the same total degree. As in the affine case, any homogeneous polynomial ideal defines the closed sets of the topology. This allows us to talk about a projective algebraic variety (embedded in projective space).

Definition 11. *Let* S *be a collection of homogeneous polynomials in* $n+1$ *variables. Then a* projective algebraic set *is the set* $Z(S) = \{(x_0 : \ldots : x_n) \in \mathbb{P}^n : f(x_0, \ldots, x_n) = 0, \forall f \in S\}$. *If this set is irreducible, then* $Z(S)$ *is a projective variety.*

Note that in our treatment of varieties, we assume an embedding, either into affine space (for an affine variety) or into projective space (for a projective variety). In the modern language of algebraic geometry, abstract varieties, free of an embedding, are defined using the language of schemes in similar spirit to the definition of abstract smooth manifolds.

2.1.1 Many Ways to View Projective Space

Projective space can be viewed in many different ways. We have seen the construction above via a quotient of affine space. That is, by taking $\mathbb{A}^{n+1}$ and removing the fixed points of $\mathbb{C}^\times$ action, then taking the quotient by the resulting fixed-point free action as in Definition 10. This procedure is quite general, and can in fact be applied to obtain homogeneous coordinates on a large class of *toric varieties*. For information on this construction see the textbook by Hori and colleagues (2003). Another way to view projective space is as the set of all lines through the origin in the space $\mathbb{A}^{n+1}$. Notice that the identification of $(x_0, \ldots, x_n)$ with the point $(\lambda x_0, \ldots, \lambda x_n)$ is precisely the identification of points on a line. Finally, we may view projective space as affine space compactified with a *hyperplane at infinity.* Consider $\mathbb{RP}^1$ with coordinates $(x_0 : x_1)$. We know that we cannot have both coordinates equal to zero, and thus either $x_0 \neq 0$ and $x_1 \neq 0$. In the Zariski topology these are both open sets, say U_0 and U_1. If $x_0 \neq 0$, we can map $(x_0 : x_1) \sim (1 : \frac{x_1}{x_0}) \mapsto \frac{x_1}{x_0}$, which is just a copy of $\mathbb{R}^1$. This leaves a single point in $\mathbb{RP}^1 \setminus U_0$. Thus $\mathbb{RP}^1$ is homoemorphic to a one-point compactification of $\mathbb{R}^1$, which is a circle.

Similarly, consider $\mathbb{C}^2 \setminus \{0\}$ under the projective equivalence relation. Every point in $\mathbb{CP}^1$ except one can be identified bijectively with a point of $\mathbb{C}$. Analogously with $\mathbb{RP}^1$, we add a point at infinity, yielding the familiar one-point compactification of $\mathbb{C}$ yielding the Riemann sphere S^2. In general, consider $\mathbb{P}^n$ with coordinates $(x_0 : \ldots : x_n)$. Let U be the set of points where $x_0 \neq 0$. Then $\mathbb{P}^n \setminus U$ is in natural one-to-one correspondence with $\mathbb{P}^{m-1}$ which we view as the hyperplane at infinity.

2.1.2 Projective Space in Local Coordinates

For computations in projective space, we often use techniques from the theory of manifolds. In fact, $\mathbb{P}^n$ is a complex manifold in the analytic topology. The charts on $\mathbb{P}^n$ are given by the complements of the vanishing sets of the coordinate functions. That is, on $\mathbb{P}^n$ we have distinguished open subsets, $U_i = \{(x_0 : \ldots : x_n) | x_i \neq 0\}$. Using projective scaling equivalence, we can associate every point in U_i uniquely with a point in $\mathbb{A}^n$. This association defines a map

$$\begin{array}{rcl} U_i \subset \mathbb{P}^n & \to & \mathbb{A}^n \\ (x_0 : \ldots : x_n) & \mapsto & (\dfrac{x_0}{x_i}, \ldots, \dfrac{x_n}{x_i}). \end{array}$$

It is elementary to observe that the sets $\{U_i\}$ cover $\mathbb{P}^n$. In many circumstances, we can do computations on $\mathbb{P}^n$ locally in $\mathbb{C}^n$ by consider such affine patches. A simple computation of the transition maps for these charts proves that $\mathbb{P}^n$ is a complex analytic manifold.

2.2 Toric Varieties

$\mathbb{P}^n$ is an example of an important class of varieties known as toric varieties, which we will define shortly. Toric varieties have an analogous role in algebraic geometry to that of CW-complexes in topology. They are specialized, but form a diverse class of examples on which more general theory can be tested. The motivation to study toric varieties comes from the fact that there is an action of the algebraic group $(\mathbb{C}^\times)^n$, an algebraic torus. Thus, studying properties of the variety that are equivariant with respect to the action of this group often reduces to studying certain invariant subsets of the action.

Definition 12. *A* toric variety *is a complex algebraic variety X, with a dense (Zariski) open subset isomorphic (as a linear algebraic group) to the algebraic torus $(\mathbb{C}^\times)^n$. Further we require that the group action of the torus on itself extends naturally to an action on X.*

To gain a clearer understanding of toric varieties, we consider the following examples.

Example 13. *Consider $\mathbb{CP}^2$ with homogeneous coordinates given by $(x_0 : x_1 : x_2)$. The dense open subset*

$$T = \{(1 : t_1 : t_2) \in \mathbb{CP}^2 : t_i \neq 0\}$$

is clearly isomorphic to $(\mathbb{C}^\times)^2$ and acts on the variety by coordinate wise multiplication. In similar fashion, $\mathbb{CP}^n$ contains a torus isomorphic to $(\mathbb{C}^\times)^n$ given by

$$T = \{(1 : t_1 : \ldots : t_n) \in \mathbb{CP}^n : t_i \neq 0\}.$$

2.2.1 Cones and Fans of Toric Varieties

As we have noted before, many computations involving toric varieties can be simplified to analyses of the combinatorics of objects known as fans or polytopes. For all varieties that we will work with, the fan and the polytope contain equivalent information, as the varieties are both normal and projective. We will first introduce the fan, with the next two definitions.

Let N be a lattice ($N \cong \mathbb{Z}^r$), and let $N_{\mathbb{R}} = N \otimes \mathbb{R}$. We first define a strongly convex rational polyhedral cone, which we will simply refer to as a cone.

Definition 14. *A* (strongly convex rational polyhedral) cone $\sigma \subset N_{\mathbb{R}}$ *is a set*

$$\sigma = \{a_1 v_1 + \cdots + a_k v_k : a_i \geq 0\}$$

generated by a finite set of vectors $v_1, \ldots, v_k$ in N such that $\sigma \cap (-\sigma) = \{0\}$.

A one-dimensional cone is often associated to the element of the lattice which spans it, and we will often confuse v_ρ with $\langle v_\rho \rangle$. This element v_ρ is known as a primitive generator. A fan is a way of putting cones together in a meaningful way, much like the facets, edges and vertices of a polyhedron are glued together to make a polyhedron.

2.2.2 Fans and Toric Varieties via Gluing

Just as projective space can be thought of as gluing affine open patches, as described above, more general toric varieties can be constructed by gluing *affine toric varieties*. Affine toric varieties are determined by a single cone in the lattice. In the natural way, a cone corresponds to an additive semigroup S. By taking the elements of S to be exponents of some formal variables, we can generate a semigroup algebra $\mathbb{C}[S]$. This semigroup algebra is a quotient of a polynomial ring, and canonically determines an affine variety with an embedded torus. To glue affine toric varieties is equivalent to understanding the gluing of the cones that determine them. This is done via a *fan*.

Definition 15. *A collection Σ of cones in $N_{\mathbb{R}}$ is called a* fan *if*

1. *Each face of a cone in Σ is also a cone, and*
2. *The intersection of two cones in Σ is a face of each.*

The set of one-dimensional cones of Σ is known as its one-skeleton, denoted $\Sigma^{(1)}$.

Although we will not go into the construction, from the fan of a (nice) toric variety we can completely recover the variety via its homogeneous coordinates. This is known as the *Cox construction.* The reader is encouraged to see Chapter 7 of the textbook by Hori and colleagues (2003) for details.

We will conclude this section by stating the properties of fans and their toric varieties that are of greatest relevance to us.

1. The cones of the fan correspond to torus invariant subvarieties of the toric variety. This correspondence is via *orbit closure* under the torus action.

2. Further, the dimension of the cone is equal to the codimension of the invariant subvariety. Thus the orbit closure of a maximal cone is a torus fixed point, the orbit closure of a dimension one cone is a divisor and so on.

3. If σ_1 is a face of σ_2, the the orbit closure of σ_1 contains the orbit closure of σ_2. In other words taking orbit closures is an *inclusion reversing bijection.*

2.2.3 A Note on Polytopes

Although we have introduced the fan as our primary combinatorial objects to study toric geometry, lattice polytopes form a crucial tool in understanding (projective) toric varieties. The polytope and fan are dual to each other and determine each other in the projective case. Given a polytope, the fan whose rays are normal to the facets of the polytope recovers the fan of the toric variety. Although we will not use the explicit construction to recover the toric variety from the polytope, it is often a useful tool to understand the geometry of toric varieties. The following is a useful dictionary between the polytope and the fan.

1. Given a lattice polytope Δ_X of dimension n, the dimension k faces of the polytope are in bijection with the codimension k cones of the fan over the faces of Δ_X.

2. The dimension k faces of the polytope hence correspond to torus invariant k-planes in X.

3. Consider two T-invariant subvarieties corresponding to cones σ_1 and σ_2, respectively. Let δ_1 and δ_2 be the faces of the polytopes corresponding to these subvarieties. If they intersect, the intersection of these subvarieties corresponds to the cone $\sigma_1 \cup \sigma_2$, and the face $\delta_1 \cap \delta_2$.

4. Let $Y_1 \subset Y_2$ be two T-invariant subvarieties corresponding to cones σ_1 and σ_2, respectively. Let δ_1 and δ_2 be the faces of the polytopes corresponding to these subvarieties. Then $\sigma_2 \subset \sigma_1$ and $\delta_1 \subset \delta_2$.

For instance, consider the tetrahedron, the polytope of $\mathbb{P}^3$. From this polytope, we can deduce that there are four torus invariant planes in $\mathbb{P}^3$,

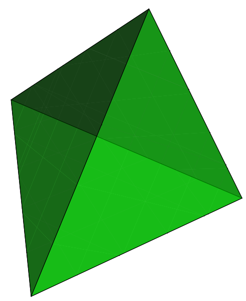


Figure 2.1 The polytope of $\mathbb{P}^3$.

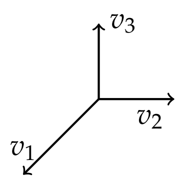


Figure 2.2 The fan of the toric variety $\mathbb{P}^2$.

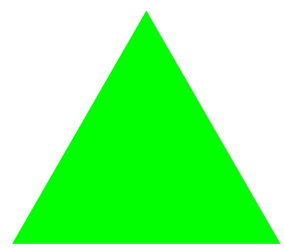


Figure 2.3 The polytope of the toric variety $\mathbb{P}^2$.

each containing three torus invariant lines. Further there are four torus invariant points. We can also deduce that any pair of torus invariant planes in $\mathbb{P}^3$ intersect at a torus invariant line. See Example 18 for the fan of $\mathbb{P}^3$.

We now explore these definitions with some examples of toric varieties and their fans. The next four examples will be fundamental to the work done in Chapter 3. Henceforth, all projective spaces will be assumed to be complex.

Example 16 ($\mathbb{P}^2$)**.** *As we have seen $\mathbb{P}^2$ is a toric variety. The fan Σ of $\mathbb{P}^2$ is the positive span of the vectors $\{v_1 = (-1,-1), v_2 = (1,0), v_3 = (0,1)\}$. There are a total of seven cones in Σ. The cone $\{0\}$ of dimension 0. The cones spanned by each v_i, $\langle v_i \rangle$, and the two-dimensional cones spanned by $\langle v_i, v_j \rangle_{i \neq j}$. The orbit closures of the two-dimensional cones are the points $(1:0:0)$, $(0:1:0)$, and $(0:0:1)$, while the one-dimensional cones correspond to the lines $(x:y:0)$, $(0:y:z)$, and $(x:0:z)$.*

Example 17 ($\mathbb{P}^1 \times \mathbb{P}^1$)**.** *Another example to consider is $\mathbb{P}^1 \times \mathbb{P}^1$. The fan Σ of*

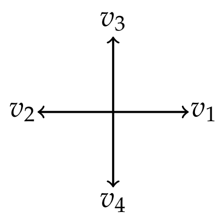


Figure 2.4 The fan of the toric variety $\mathbb{P}^1 \times \mathbb{P}^1$.

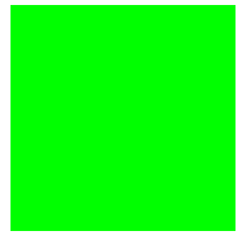


Figure 2.5 The polytope of the toric variety $\mathbb{P}^1 \times \mathbb{P}^1$

$\mathbb{P}^1 \times \mathbb{P}^1$ *is spanned by the four edges*

$$v_1 = (1,0), \quad v_2 = (-1,0), \quad v_3 = (0,1), \quad v_4 = (0,-1).$$

There are four two-dimensional cones: $\langle v_1, v_3 \rangle$, $\langle v_1, v_4 \rangle$, $\langle v_2, v_3 \rangle$, *and* $\langle v_2, v_4 \rangle$. *These correspond to the four torus fixed points in* $\mathbb{P}^1 \times \mathbb{P}^1$.

One may notice that the above toric varieties are determined by their one-skeleton, but we warn the reader that this is not generally the case, especially in higher dimensional examples. However, for *complete toric varieties*, all cones can be found by intersecting the maximal cones. Thus, every submaximal cone is a face of a maximal cone. We will use this fact to describe some basic examples.

Example 18 ($\mathbb{P}^3$)**.** *The fan* $\Sigma_{\mathbb{P}^3} \subset \mathbb{Z}^3$ *of* $\mathbb{P}^3$ *has one-skeleton with primitive generators*

$$\begin{aligned} v_1 &= (-1,-1,-1), & v_2 &= (1,0,0), \\ v_3 &= (0,1,0), & v_4 &= (0,0,1), \end{aligned}$$

and maximal cones given by

$$\begin{aligned} &\langle v_1, v_2, v_3 \rangle, \quad \langle v_1, v_2, v_4 \rangle, \\ &\langle v_1, v_3, v_4 \rangle, \quad \langle v_2, v_3, v_4 \rangle. \end{aligned}$$

Lower dimensional cones are found by intersecting higher dimensional ones.

Example 19 ($\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$)**.** *The fan* $\Sigma_{(\mathbb{P}^1)^{\times 3}} \subset \mathbb{Z}^3$ *of* $(\mathbb{P}^1)^{\times 3}$, *has one-skeleton generators*

$$\begin{aligned} u_1 &= (1,0,0), & u_3 &= (0,1,0), & u_5 &= (0,0,1), \\ u_2 &= (-1,0,0), & u_4 &= (0,-1,0), & u_6 &= (0,0,-1), \end{aligned}$$

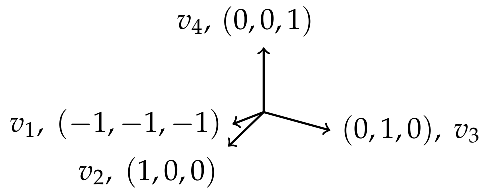


Figure 2.6 The fan of the toric variety $\mathbb{P}^3$.

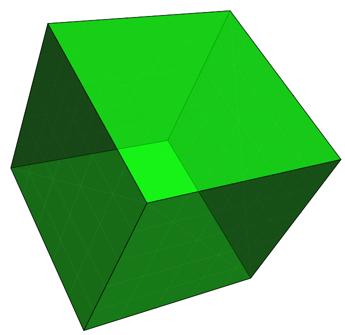


Figure 2.7 The polytope of $(\mathbb{P}^1)^{\times 3}$.

and maximal cones given by

$$\begin{aligned}
&\langle u_1, u_3, u_5\rangle, \quad \langle u_1, u_2, u_4\rangle, \quad \langle u_1, u_2, u_3\rangle, \quad \langle u_1, u_2, u_4\rangle,\\
&\langle u_2, u_4, u_6\rangle, \quad \langle u_2, u_3, u_4\rangle, \quad \langle u_1, u_2, u_3\rangle, \quad \langle u_1, u_2, u_4\rangle.
\end{aligned}$$

The polytope of $(\mathbb{P}^1)^{\times 3}$ *is the cube, shown in Figure 2.7.*

2.3 Blowups and Toric Blowups

The blowup, which we now introduce, is a geometric transformation that will form a fundamental tool for the rest of this thesis. Blowups can be used to resolve singularities, or create new varieties of interest. Further, blowups form an important class of birational transformations. The behaviour of properties of varieties under blowup is of basic interest in algebraic geometry.

Intuitively, a rational map is a partial function between varieties. Formally, it can be defined as follows.

Definition 20. *A* rational map $f : V \to W$ *between two varieties is an equivalence class of pairs* (f_U, U)*, in which* U *is an open set of* V *and* f_U *is a morphism*

to W. Two pairs (f_U, U) and $(f_{U'}, U')$ are equivalent if f_U and $f_{U'}$ coincide on $U \cap U'$.

It should be noted that this definition relies crucially on the Zariski topology, where open sets are quite large. In fact, if two morphisms coincide on an open set, they are equal. A birational map is roughly an invertible rational map.

Definition 21. *A rational map $f : V \to W$ is said to be* birational *if there exists a rational map $g : W \to V$ which is its inverse on the domain of definition.*

Two varieties are birational if and only if their function fields coincide. Birational equivalence is a weaker notion of equivalence than isomorphism. As we will see, any space is birational to a blowup of it, and these will form a large class of examples of birational transformations. A simpler example is given below.

Example 22. *$\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are birationally equivalent. Via the Segre embedding, $\mathbb{P}^1 \times \mathbb{P}^1$ is realised as a hypersurface in $\mathbb{P}^3$, as the variety X of points $(x : y : z : w)$ such that $yz - xw = 0$. The varieties X and $\mathbb{P}^2$ are clearly not isomorphic. Any two lines in $\mathbb{P}^2$ intersect, however the lines $w = x = 0$ and $y = z = 0$ cannot intersect in $\mathbb{P}^3$, and hence not in the Segre variety. A simple computation shows that the coordinate ring $\mathbb{C}(X) = \mathbb{C}[x, y, z]/(xy - z) \cong k[z, y]$. The function field of X is thus isomorphic to $\mathbb{C}(x, y)$ which is of course the function field of $\mathbb{P}^2$.*

In general, given a variety X, we may want to blowup a subvariety $Y \hookrightarrow X$ in X. However, rather than doing this for a general subvariety, we will explain the blowup of a point in $\mathbb{A}^n$ for intuition, and then move on to the toric case, where this process can be understood combinatorially. Note that the blowup of a point is completely described by the process below. To blowup a point in an arbitrary toric variety X, choose an open set U in X containing p such that U is isomorphic of $\mathbb{A}^n$. This can always be done, since every point is contained in some affine patch. Choose coordinates such that p is the origin, and use the construction below.

Example 23 (Blowup of $\mathbb{A}^n$ at the Origin)**.** *The blowup of $\mathbb{A}^n$ at the origin is a subset $X \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ together with a surjective projection map $\pi : X \to \mathbb{A}^n$. X is given by equations $\{x_i y_j = x_j y_i : i, j = 1, 2, \ldots, n\}$, where the coordinates on $\mathbb{A}^n$ and $\mathbb{P}^{n-1}$ are x_i and y_i, respectively. π is the obvious projection map, onto the first factor.*

The following are some general properties of blowups of general subvarieties.

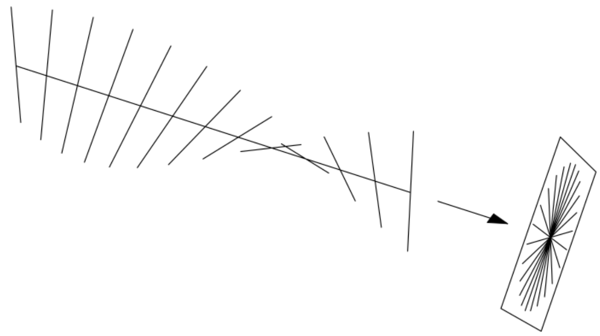


Figure 2.8 Blowup of the origin in $\mathbb{A}^2$.

1. Blowups are generally denoted together with the surjective projection map, $\pi : \hat{X} \to X$, where X is the base space, and $\hat{X}$ is the blowup space.

2. The subvariety being blown up is known as the *center of the blowup*, the *locus of the blowup* or the *exceptional locus.*

3. The blowup can be geometrically seen as inserting a divisor, an object of codimension 1, in place of the locus of the blowup.

4. Away from the exceptional locus π is an isomorphism, while the inverse image under π of the exceptional locus is a divisor. In the example above, we can see that if we are geometrically far from the point $(0, \ldots, 0)$, the coordinates y_i are completely determined (up to projective scaling) by the x_i.

2.3.1 Toric Blowups

The blowup of torus fixed $Z \hookrightarrow X$ in a toric variety X can be described combinatorially, by subdivision of the fan. We describe this now.

Definition 24. *A fan* Σ' subdivides *a fan* Σ *if*

1. $\Sigma(1) \subset \Sigma'(1)$ *and*

2. *Each cone of* Σ' *is contained in some cone of* Σ.

If $\Sigma' \neq \Sigma$ *then* $X_{\Sigma'}$ *is a blowup of* X_Σ. *Now suppose we have a subvariety* Z *which is the orbit closure of the cone* $\sigma_Z = \langle v_1, \ldots, v_s \rangle$. *To blowup* Z *we subdivide the*

cone σ_Z by introducing a new edge $v_Z = v_1 + \cdots + v_s$ and subdivide σ_Z in the natural way. Now combining these new cones with the cones of Σ except σ_Z but including all of its proper faces, we get a new fan Σ' which is the blowup of Σ at Z.

Subdivisions of fans correspond to blowups. From the dual point of view, blowups can be realized as truncations of polytopes. We will now compute a few relevant and important examples to better describe this process.

Example 25 (Blowup of $\mathbb{P}^2$ at a Point)**.** *Consider $\mathbb{P}^2$ and its fan as described in Example 16. We will blowup $\mathbb{P}^2$ at the point given by the orbit closure of the cone $\langle v_1, v_2\rangle$. To do this, we introduce a new element in the one-skeleton, given by $v_{12} = v_1 + v_2 = (0, -1)$. We then subdivide the cone as*

$$\langle v_1, v_2\rangle \to \langle v_1, v_{12}\rangle, \ \ \langle v_2, v_{12}\rangle.$$

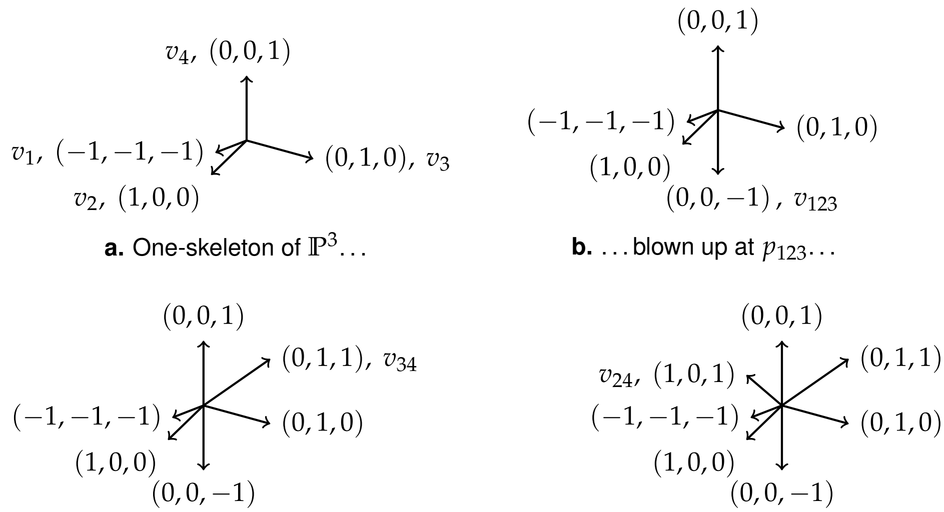
We remark here that if we blowup $\mathbb{P}^2$ as above at its three torus fixed points, we get a fan with a reflection through the origin symmetry. This will be of interest in Chapter 3, when we study the Permutohedron and the Cremona symmetry, of which this is an example. In dimensions higher than two, our study will require us to blowup lines in a threefold. We do an example of this now.

Example 26 (Blowup of a Line in $\mathbb{P}^3$)**.** *Consider the fan of $\mathbb{P}^3$ as described in Example 18. We will blowup the line ℓ_{12} corresponding to the orbit closure of $\sigma_{1,2} = \langle v_1, v_2\rangle$. To do this we introduce a new element of the one-skeleton $v_{12} = v_1 + v_2 = (0, -1, -1)$. We then subdivide the cone of the line and any cones containing it. This corresponds to blowing up any torus fixed points on the line, of which there are two in this example.*

$$\begin{aligned}
\langle v_1, v_2\rangle &\to \langle v_1, v_{12}\rangle, \ \ \langle v_2, v_{12}\rangle, \\
\langle v_1, v_2, v_3\rangle &\to \langle v_1, v_{12}, v_3\rangle, \ \ \langle v_2, v_{12}, v_3\rangle, \\
\langle v_1, v_2, v_4\rangle &\to \langle v_1, v_{12}, v_4\rangle, \ \ \langle v_2, v_{12}, v_4\rangle.
\end{aligned}$$

With all other cones unchanged, this subdivided fan is the blowup $\mathbb{P}^3(\ell_{12})$.

Example 27 (Fan of a Blowup of $\mathbb{P}^3$)**.** *The Figure 2.9 shows the successive blowups, through subdivision of cones, of $\mathbb{P}^3(p_{123}, \ell_{34}, \ell_{24})$. Only the one-skeleton is depicted.*



a. One-skeleton of $\mathbb{P}^3$...

b. ...blown up at p_{123}...

c. ...then at ℓ_{34}...

d. ...then at ℓ_{24}.

Figure 2.9 Constructing the fan for the iterated blowup.

2.4 Chow Ring and Cohomology

The cohomology of a variety X encodes information about intersections of subvarieties in X. Abstractly, cohomology is simply a contravariant functor from the category of algebraic varieties (or schemes) to the category of graded rings. Intuitively, in dimension k, the Chow group $A^k(X)$ is generated by all the subvarieties in X of dimension k with some form of equivalence between these subvarieties. In the case of cohomology, this equivalence is a homological equivalence induced by coboundary maps. In the case of the Chow ring, the equivalence is much weaker, and is known as rational equivalence. The equivalence is rational, because any two points on a rational curve (a copy of $\mathbb{P}^1$) are equivalent, but not on a general curve, which is the case for homological equivalence. We will not go further into either form of equivalence in detail, as it requires a greater level of algebra-geometric and topological sophistication than is within the scope of this manuscript. The expert reader should be able to read this treatment with the subtleties involved, however for the general reader we will focus on intuition and computation.

The equivalence relation on elements in the Chow ring allows us to algebraically formalize the notion of a "class of subvarieties." For instance,

the class of all hyperplanes in $\mathbb{P}^3$, or the class of all conics in $\mathbb{P}^3$. The ring structure of the Chow ring is an intersection product. That is, if $[Z], [Y]$ are two classes of subvarieties in Chow, the product $[Z] \cdot [Y]$ can roughly be thought of as the class of the intersection $[Z \cap Y]$. If Y and Z are sufficiently general, this is precisely the product. The condition of generality is consistent with the idea of transversal intersection of smooth submanifolds in de Rham cohomology. For nonsingular toric varieties that have the structure of a smooth manifold in the analytic topology, the Chow ring coincides with the de Rham cohomology of the variety. In this document, both Chow and cohomology will be taken with integer coefficients. With the product structure, we interpret these as $\mathbb{Z}$-algebras, and are usually written as quotients of the polynomial algebra over $\mathbb{Z}$. We will follow the notation that $A^\star(X)$ is the Chow ring of X and $H^\star(X;\mathbb{Z})$ is the cohomology of X with integer coefficients. The dual notion to the Chow ring, which we will call the intersection ring, $A_\star(X)$ coincides with homology $H_\star(X;\mathbb{Z})$ for all the cases we consider.

Example 28 (The Chow Ring of $\mathbb{P}^n$)**.** *$A^\star(\mathbb{P}^n)$ is generated by the class of a hyperplane in $\mathbb{P}^n$. For the sophisticated reader, this is the first Chern class of the dual of the tautological bundle on $\mathbb{P}^n$. However, in Chow, we can think of this as the class of any hyperplane, say $x_0 = 0$. The intersection product on can be reasoned as follows. Two hyperplanes intersect at a general codimension 2 linear subvariety. In fact the codimension two Chow group is generated by this class H^2. Similarly H^k generated the codimension k group. n general hyperplanes intersect at a point in $\mathbb{P}^n$. The class of a point $H^n = [pt]$ generates $A^n(\mathbb{P}^n)$. Finally, since $n+1$ hyperplanes in general do not intersect, $H^{n+1} = 0$. In summary, we give the structure of the cohomology as*

$$A^\star(\mathbb{P}^n) = \mathbb{Z}[H]/\langle H^{n+1}\rangle.$$

The Chow ring in the toric case can be computed using combinatorial techniques in a very straightforward manner from the fan. The process is described by the following theorem:

Theorem 29 (Fulton (1993))**.** *For a nonsingular projective variety X, $A^*X = \mathbb{Z}[D_1, \ldots, D_d]/I$, where I is the ideal generated by all*

1. *$D_{i_1} \times \cdots \times D_{i_k}$ for $v_{i_1}, \ldots, v_{i_k}$ not in a cone of Σ.*
2. *$\sum_{i=1}^{d} \langle u, v_i \rangle D_i$ for u in M.*

In fact in type 1 it suffices to include only the sets of v_i without repeats, and in type 2 one needs only those u from a basis of M.

Let us now use this to compute the Chow ring of $\mathbb{P}^3$.

Example 30 (Chow Ring of $\mathbb{P}^3$)**.** *Consider the fan as in Example 18 of $\mathbb{P}^3$. Observe that the relations of type 2 above give us*

$$D_1 = D_2, \;\; D_1 = D_3, \;\; D_1 = D_4.$$

Notice that the only set of generators that do not span a cone is the set $\{v_1, v_2, v_3, v_4\}$, and thus,

$$D_1 D_2 D_3 D_4 = 0.$$

Thus setting $D_i = H$, we recover the calculation of Example 28.

Example 31 (Chow Ring of $(\mathbb{P}^1)^{\times 3}$)**.** *Consider the fan as in Example 19 of $(\mathbb{P}^1)^{\times 3}$. Observe that the relations of type 2 give us*

$$D_1 = D_2, \;\; D_3 = D_4, \;\; D_5 = D_6.$$

Recall that v and $-v$ cannot be contained in the same cone, and we get

$$D_1^2 = D_3^2 = D_5^2 = 0.$$

Setting $D_1 = H_1$, $D_3 = H_2$, and $D_5 = H_3$, we see that $A^1((\mathbb{P}^1)^{\times 3}) = \mathbb{Z}[H_1, H_2, H_3]$.

The above example can also be interpreted geometrically with the help of our intuition regrading geometry of $\mathbb{R}^3$. There are three classes of planes in $(\mathbb{P}^1)^{\times 3}$. These can be seen as parallel classes of planes. In general two planes from any single class are parallel and hence do not intersect, and thus $H_i^2 = 0$. Any two such planes intersect at a line. These classes of lines $\{h_{ij}\}$ for $\{i, j\} \subset \{1, 2, 3\}$ span $A^2((\mathbb{P}^1)^{\times 3})$.

Example 32 (Chow Ring of $\mathbb{P}^2(1)$)**.** *Consider the fan as in Example 25 of $\mathbb{P}^2(1)$. Observe that the relations of type 2 now give us*

$$-D_1 + D_2 + D_{23} = 0, \;\; -D_1 + D_{23} + D_3 = 0.$$

Notice that the cone $\langle v_2, v_3 \rangle$ has now been subdivided, and hence is no longer a cone. Thus,

$$D_2 D_3 = 0, \;\; D_i D_{23} = 0.$$

We see that by setting $D_1 = H$ and $D_{23} = E$, $A^1(\mathbb{P}^2(1)) = \mathbb{Z}[H, E]$. Further we have that $H^2 + E^2 = 0$. We have $H^2 = pt$ and $E^2 = -pt$.

Chapter 3

Toric Symmetry

In this chapter we introduce and discuss the use of toric symmetry as a computational technique in Gromov–Witten theory. Toric symmetries are special types of automorphisms of toric varieties that arise from the fan or polytope. The first systematic use of toric symmetry was carried out the author and colleagues (2011), generalising ideas from a previously studied symmetry known as Cremona symmetry. To illustrate the techniques and results that are relevant to the general question of finding toric symmetries, we will review the key results of this paper for the case of $\mathbb{P}^3$ and review some of the difficulties in extending this technique to $(\mathbb{P}^1)^{\times 3}$. We begin with an introduction to toric symmetries and analyse when a toric symmetry is nontrivial.

3.1 (Nontrivial) Toric Symmetries

Definition 33 (Cox and Katz (1999))**.** *Let X be a toric variety, with fan $\Sigma_X \subset \mathbb{Z}^n$. A* toric symmetry *is an automorphism of $\mathbb{Z}^n$, in other words, an element $\tau \in GL(\mathbb{Z}^n)$ such that for any cone $\sigma \in \Sigma_X$, $\tau(\sigma)$ is a cone of Σ_X. That is, τ is a lattice isomorphism that preserves the cones in the fan.*

Note that the automorphism group of a toric variety is larger in general than its group of toric symmetries. In particular, there are automorphisms that extend from automorphisms of the torus $(\mathbb{C}^{\times})^n$ itself.

Recall from Chapter 2, that for a smooth projective toric variety X, the intersection ring is generated as an algebra by classes corresponding to the one-dimensional cones of Σ_X. Now given the action of a toric symmetry τ on Σ_X, there is a natural action of τ on the one-skeleton $\Sigma_X^{(1)}$. Since the

elements of $\Sigma_X^{(1)}$ generate the homology, by linearly extending the action of τ and defining it to commute with the product structure in homology, we get a an automorphism

$$\sigma_\star : A_\star(X) \to A_\star(X).$$

This ring automorphism is the pushforward of σ to the homology. When we speak of cohomology, the analogous map is $\sigma^\star$, the pullback. The differences for us are minor, but we simply note that homology is covariant, and cohomology is contravariant, and thus reverses arrows.

Now recall from Chapter 2, that the process of blowing up a torus fixed subvariety introduces new elements into the one-skeleton of Σ_X. These divisors, the exceptional divisors above the blowup locus, have a different intersection theory than the hyperplane divisor classes pulled back from the base space. Consider the example of $\mathbb{P}^1 \times \mathbb{P}^1$. The module structure of the homology of this space is described by

$$A_\star(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}^2 \oplus \mathbb{Z}.$$

In particular, in algebraic degree 1, the homology is generated by classes H_1 and H_2 corresponding to the two parallel classes of lines in $\mathbb{P}^1 \times \mathbb{P}^1$. Now if we blowup $\mathbb{P}^1 \times \mathbb{P}^1$ at a single point, a new codimension one class is introduced, the class E. Thus,

$$A_\star(\mathbb{P}^1 \times \mathbb{P}^1(1)) = \mathbb{Z} \oplus \mathbb{Z}^3 \oplus \mathbb{Z}.$$

However the generator of the new class in $A_1(X)$ does not behave like the generic hyperplane class. $H_i \cdot H_j = [pt]$ when $i \neq j$ and 0 otherwise. However $E \cdot E = -[pt]$. That is E has self intersection -1.

We are interested not simply in finding the toric symmetries of a toric variety, but understanding those symmetries that give us a nontrivial relationship amongst the homology classes. Thus, a toric symmetry σ of $\mathbb{P}^1 \times \mathbb{P}^1(1)$ that sends the class H_1 to H_2 is deemed uninteresting, since the map is essentially the identity up to relabeling. However, a map that sends either of the class H_i to E would be deemed a nontrivial toric symmetry, because it exchanges an exceptional divisor with a nonexceptional one. This brings us to our definition of nontrivial toric symmetry. Since we will be working almost exclusively with toric threefolds, we will restrict our definition for brevity and notational convenience.

Definition 34. *Let X be a toric threefold. Let $X(p_1, \ldots, p_r, \ell_1, \ldots, \ell_s)$ be the blowup of this threefold at points $p_1, \ldots, p_r$ and lines $\ell_1, \ldots, \ell_s$. Let $\mathfrak{H}$ be the set*

of classes pulled back from X to the blowup, and $\mathfrak{E}$ and $\mathfrak{F}$ be the set divisor classes above the blowups of points and lines, respectively. Then a toric symmetry τ of $X(p_1, \ldots, p_r, \ell_1, \ldots, \ell_s)$ is said to be trivial *if $\tau_\star$ stabilizes each of these sets, $\mathfrak{H}$, $\mathfrak{E}$, and $\mathfrak{F}$. If τ is not trivial, it is said to be* nontrivial*.*

3.2 A Computational Approach to Toric Symmetry: $\mathbb{CP}^3$

The nature of Definition 34 lends itself to computational and algorithmic characterization. For a toric variety X with polytope Δ_X, the group of toric symmetries is clearly finite since the group of automorphisms of any lattice polytope is finite. Thus, if $\mathrm{Aut}(\Delta_X)$ can be found explicitly, the action of an element $\tau \in \mathrm{Aut}(\Delta_X)$ on the fan can be computed quite easily. Further, the computation of the pullback of τ on the cohomology of X extends linearly from the action of τ on the faces of the polytope Δ_X. This is due to the fact that the faces of Δ_X correspond to torus fixed subvarieties, whose classes in turn generate the cohomology of X. Thus, using Definition 34, we can find all elements $\tau \in \mathrm{Aut}(\Delta_X)$ which pullback nontrivially to the cohomology (and hence the Gromov–Witten theory) of X.

This approach was actualized by the author and collaborators (Karp et al., 2011). Using a combinatorial characterization of Definition 34, we exhausted all the nontrivial toric symmetries of $\mathbb{P}^3$ and its iterated toric blowups. Note that $\mathbb{P}^3$ has four torus fixed points and six torus fixed lines. Our results and the computational algorithm are summarized by Theorem 35.

Theorem 35. *There exist precisely four classes of toric blowups of $\mathbb{P}^3$ which have nontrivial toric symmetry. These four classes, labelled A, B, C, and D, are described in Theorems 36, 37, 38, and 39, respectively.*

Moreover, a space of Class A, B, or D admits a unique toric symmetry (up to relabeling), whereas there are precisely four distinct nontrivial symmetries for Class C varieties.

We now construct these four families and describe their toric symmetries. In what follows X will be an iterated blowup of $\mathbb{P}^3$ at a specified configuration of points and lines. Throughout this work, we say the variety X is a *toric blowup* of $\mathbb{P}^3$ if X is an iterated blowup of $\mathbb{P}^3$ only along torus invariant subvarieties of $\mathbb{P}^3$ (or their proper transforms). In particular, we are not interested in spaces obtained by blowups with centers in the exceptional locus, as their geometry is far from $\mathbb{P}^3$.

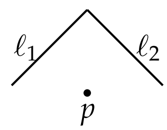


Figure 3.1 The ordered Class A blowup center.

We denote by H the pullback to X of the hyperplane class in $\mathbb{P}^3$. We denote by E_α the exceptional divisor above a point p_α and by $F_{\alpha'}$ the exceptional divisor above a line $\ell_{\alpha'}$ for appropriate indices α, α'. Further, we will let h and e_α denote the classes of a line in the divisors H and E_α, respectively, and $f_{\alpha'}$ denotes the fiber class in $F_{\alpha'}$. The homology groups $H_4(X;\mathbb{Z})$ and $H_2(X;\mathbb{Z})$ are spanned by divisor and curve classes, respectively:

$$H_4(X;\mathbb{Z}) = \langle H, E_\alpha, F_{\alpha'} \rangle\,, \quad H_2(X;\mathbb{Z}) = \langle h, e_\alpha, f_{\alpha'} \rangle\,.$$

Theorem 36. *Let X be the blowup of $\mathbb{P}^3$ along a point p and two intersecting distinct lines ℓ_1 and ℓ_2, such that $p \neq \ell_1 \cap \ell_2$. We call such a space a* Class A *blowup; see Figure 3.1. Then, given $\beta = dh - a_1 e - a_2 f_1 - a_3 f_2 \in H_2(X;\mathbb{Z})$, there exists a unique nontrivial toric symmetry τ_A on X, and its action on homology is given by*

$$(\tau_A)_*\beta = \beta',$$

where $\beta' = d'h - a_1' e - a_2' f_1 - a_3' f_2$ has coefficients given by

$$\begin{aligned} d' &= 2d - a_1 - a_2 - a_3 \\ a_1' &= d - a_2 - a_3 \\ a_2' &= d - a_1 - a_3 \\ a_3' &= d - a_1 - a_2. \end{aligned}$$

Theorem 37. *Let X be the sequential blowup of $\mathbb{P}^3$ at distinct points p_1 and p_2 and three pairwise intersecting lines ℓ_1, ℓ_2, and ℓ_3 such that $p_1 \in \ell_1$, $p_2 \in \ell_3$, and $p_1, p_2 \notin \ell_2$. Then X is called a* Class B *blowup; see Figure 3.2. Then, given $\beta = dh - a_1 e_1 - a_2 e_2 - a_3 f_1 - a_4 f_2 - a_5 f_3$, there exists a unique toric symmetry τ_B on X and its action on homology is given by*

$$(\tau_B)_*\beta = \beta',$$

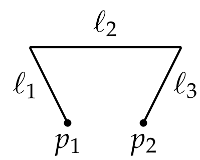


Figure 3.2 The ordered Class B blowup center.

where $\beta' = d'h - a_1'e_1 - a_2'e_2 - a_3'f_1 - a_4'f_2 - a_5'f_3$ *has coefficients given by*

$$\begin{aligned} d' &= 2d - a_1 - a_2 - a_3 - a_4 \\ a_1' &= a_5 \\ a_2' &= d - a_1 - a_3 - a_4 \\ a_3' &= d - a_2 - a_4 - a_5 \\ a_4' &= d - a_1 - a_2 - a_3 \\ a_5' &= a_1. \end{aligned}$$

Theorem 38. *Let X be the sequential blowup of $\mathbb{P}^3$ at distinct points p_1 and p_2 and three pairwise intersecting lines ℓ_1, ℓ_2, and ℓ_3 such that $p_1 \in \ell_2$, $p_2 \in \ell_3$, and $p_1, p_2 \notin \ell_1$. We term such spaces* Class C *blowups; see Figure 3.3. Then, there exist precisely four nontrivial toric symmetries $\tau_C, \sigma_C, \sigma_C^2$, and $\sigma_C\tau_C$ of X. With $\beta = dh - a_1e_1 - a_2e_2 - a_3f_1 - a_4f_2 - a_5f_3$, their action on cohomology is given by*

$$(\tau_C)_*\beta = \beta' \qquad\qquad (\sigma_C)_*\beta = \beta'',$$

where $\beta' = d'h - a_1'e_1 - a_2'e_2 - a_3'f_1 - a_4'f_2 - a_5'f_3$ *has coefficients given by*

$$\begin{aligned} d' &= 2d - a_1 - a_2 - a_3 - a_4 \\ a_1' &= a_5 \\ a_2' &= d - a_2 - a_3 - a_4 \\ a_3' &= d - a_1 - a_2 - a_4 \\ a_4' &= a_2 \\ a_5' &= d - a_1 - a_3 - a_5, \end{aligned}$$

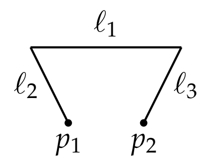


Figure 3.3 The ordered Class C blowup center.

and $\beta'' = d''h - a_1''e_1 - a_2''e_2 - a_3''f_1 - a_4''f_2 - a_5''f_3$ *has coefficients given by*

$$\begin{aligned} d'' &= 2d - a_1 - a_2 - a_3 - a_5 \\ a_1'' &= a_4 \\ a_2'' &= d - a_1 - a_3 - a_5 \\ a_3'' &= d - a_1 - a_2 - a_5 \\ a_4'' &= a_1 \\ a_5'' &= d - a_2 - a_3 - a_4. \end{aligned}$$

Theorem 39 (Bryan and Karp (2005)). *Let X be the sequential blowup of $\mathbb{P}^3$ at four distinct points $p_1, \dots, p_4$ and the six distinct lines ℓ_{ij} between them. Let β be given by*

$$\beta = dh - \sum_{i=1}^{4} a_i e_i - \sum_{1 \le i < j \le 4} b_{ij} f_{ij} \in H_2(X; \mathbb{Z}).$$

There exists a unique toric symmetry τ_D of X, and its action on homology is given by $(\tau_D)_*\beta = \beta'$, where $\beta' = d'h - \sum_i a_i' e_i - \sum_{ij} b_{ij}' f_{ij}$ has coefficients given by

$$\begin{aligned} d' &= 3d - 2\sum_{i=1}^{4} a_i \\ a_i' &= d - a_j - a_k - a_l - b_{ij} - b_{ik} - b_{il} \\ b_{ij}' &= b_{kl}, \end{aligned}$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. We refer to X as a Class D *blowup of $\mathbb{P}^3$; see Figure 3.4.*

Proof of Theorem 35

Let X be a toric blowup of $\mathbb{P}^3$. Let Σ_X be the fan of X. Notice that the primitive generators of Σ_X include the standard basis of $\mathbb{Z}^3$. Also observe that the elements of the one-skeleton are sums of $v_1, \dots, v_4$, as constructed

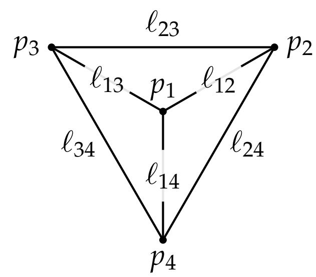


Figure 3.4 The ordered Class D blowup center.

in Chapter 2. Since τ acts on $\Sigma_X^{(1)}$, the standard basis elements of $\mathbb{Z}^3$ must be mapped by τ to elements whose entries are in the set $\{-1, 0, 1\}$. Therefore the automorphism group of the fan is finite. Moreover, this analysis yields a computational method to determine this group of lattice isomorphisms. We know that the action of τ on $\Sigma_X^{(1)}$ yields a map τ^* on $A^*(X)$.

The sets $\{D_1, \ldots, D_4\}$, $\{D_{ijk}\}$, and $\{D_{rs}\}$ correspond to H's, E_α's and $F_{\alpha'}$'s, respectively. Thus, nontrivial toric symmetries are characterized by those which exchange elements of these three sets amongst each other.

Since the number of toric blowups is finite, this characterization for maps which pushforward nontrivially to $A_\star(X)$ allows us to computationally find all nontrivial toric symmetries.

The result of this search identifies precisely those symmetries of classes A, B, C, and D. This computation also shows that there are no further nontrivial maps that are the pushforwards of toric symmetries, the result of Theorem 35. Pseudocode for the computational technique is shown in Figure 3.5. The four toric symmetries found in these results can be computed in a straightforward manner once they are identified by the algorithm.

3.2.1 Extension to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Recall that the fan of $(\mathbb{P}^1)^{\times 3}$ has one-skeleton generators

$$\begin{aligned} u_1 &= (1,0,0), & u_3 &= (0,1,0), & u_5 &= (0,0,1), \\ u_2 &= (-1,0,0), & u_4 &= (0,-1,0), & u_6 &= (0,0,-1), \end{aligned}$$

and maximal cones given by

$$\begin{array}{cccc} \langle u_1, u_3, u_5 \rangle, & \langle u_1, u_2, u_4 \rangle, & \langle u_1, u_2, u_3 \rangle, & \langle u_1, u_2, u_4 \rangle, \\ \langle u_2, u_4, u_6 \rangle, & \langle u_2, u_3, u_4 \rangle, & \langle u_1, u_2, u_3 \rangle, & \langle u_1, u_2, u_4 \rangle. \end{array}$$

Its polytope is the 3-cube. As we can then see, this variety has eight torus invariant points, and 12 torus invariant lines. The corresponding numbers for $\mathbb{P}^3$ are four and six. Thus, there are 333,327,704,320 possible iterated blowup configurations of this space, as compared to 31,312. This is the single biggest hurdle to completing an exhaustive analysis of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and renders this approach computationally intractable. In Chapter 7 we discuss the beginnings of a taxonomical study of the toric symmetry of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

3.2.2 Toric Symmetry of $\mathbb{P}^2$

The same algorithm used in the Section 3.2 can be used in lower dimension. The polytope of $\mathbb{P}^2$, as discussed previously, is a triangle. An exhaustive study of the toric symmetry of $\mathbb{P}^2$ and its blowups yields the following theorem.

Theorem 40. *The Cremona transformation induced by reflection through the origin on the two-dimensional permutohedral variety is (up to composition with isomorphisms) the only nontrivial toric symmetry of a blowup of* $\mathbb{P}^2$.

The Cremona transform in $\mathbb{P}^2$ was studied by Göttsche and Pandharipande (1998). This map sends degree one curves in $\mathbb{P}^2$ to conics, and can be used to provide an elegant Gromov–Witten theoretic proof of the existence of precisely one conic through five generic points in the projective plane.

3.3 The Computational Setup

In Section 3.2, we computationally analysed the nontrivial toric symmetries of blowups of $\mathbb{CP}^3$. This computation was carried out using SAGE, an open source computer algebra system. SAGE now has an extensive toric geometry package that has been developed, but our computational setup was created before the implementation of this package. In the following sections we will describe the important aspects of the setup.

3.3.1 Representing Toric Varieties

Given a toric variety X we represent it in SAGE via its fan Σ_X. The toric varieties relevant to this project are $\mathbb{CP}^3$ and its blowups, and as a result are smooth, complete, normal, and projective. Since the fans are complete, we need to store only the primitive generators of the one-skeleton and the

highest dimensional cones. Lower dimensional cones can then be generated by intersecting the higher dimensional cones. Cones themselves were stored as triples $\sigma_{ijk} = (v_i, v_j, v_k)$, where v_i, v_j, and v_k generate the cone σ_{ijk}. Thus a toric variety is represented as a pair (P, G), where P was a list of the generators of the one-skeleton, and G was a list of triples, each representing a three-dimensional cone.

3.3.2 Computing Toric Blowups

The *blowup* function takes in a toric variety, along with a locus of torus fixed points and lines. These points and lines are each given by a cone whose dimension is equal to the codimension of the variety. Recall that the cone is stored in terms of its generators. The function works differently for points and lines.

Points Given a point p corresponding to the cone $\sigma = (v_i, v_j, v_k)$, the function first removes σ from Σ_X and introduces a new element $v_{ijk} = v_i + v_j + v_k$ into the one-skeleton. It then introduces top dimensional cones from the subdivision of σ by the introduction of v_{ijk}.

Lines In this case there is an additional step. Since any two-dimensional cone is a common face of two three-dimensional cones, these higher dimensional cones are also subdivided by the same algorithm.

3.3.3 Finding Aut(Σ_X)

Given a matrix $M \in GL(3, \mathbb{Z})$, M acts on $\mathbb{Z}^3$ and hence the cones of Σ_X. Such a matrix M is an automorphism of Σ_X if it permutes the cones of Σ_X. Thus, given a matrix M, we act on the representation of the toric variety, on each of the primitive and maximal cones. If M permutes these cones, we flag M as a toric symmetry. Using the observation described in Section 35, we reduce the automorphism group of Σ_X to a subgroup of $GL(3, \mathbb{Z}_3)$. This is a finite group in the SAGE libraries, and we can simply scan through the elements of $GL(3, \mathbb{Z}_3)$, flagging and collecting each toric symmetry

3.3.4 Characterizing Nontrivial Toric Symmetries

Given a toric symmetry M in matrix form, we use a dictionary between the primitive generators of the one-skeleton and their indices to identify an element of the symmetric group that acts on the indices of $\{v_i\}$. That is we have the correspondence $v_1 \leftrightarrow (-1, -1, -1)$, $v_2 \leftrightarrow (1, 0, 0)$, and so on.

Given:

- The set of possible torus fixed subvarieties to blow up: $\{p_{123},\ p_{124},\ p_{134},\ p_{234},\ \ell_{12},\ \ell_{13},\ \ell_{14},\ \ell_{23},\ \ell_{24},\ \ell_{34}\}$.
- The primitive generators associated with each index: $v_1 = (-1,-1,-1),\ v_2 = (1,0,0),\ v_3 = (0,1,0),\ v_4 = (0,0,1)$.

Algorithm:

- Collect the loci for all possible toric blowups of $\mathbb{P}^3$:
  - ○ Collect all ordered collections of torus invariant subvarieties—points first (if any), followed by lines (if any). This leaves 31,312 collections of interest.
  - ○ If any two collections are equivalent up to relabeling of the fan of $\mathbb{P}^3$, remove one; their blowups are isomorphic. This leaves 1319 distinct toric blowups.
- For each toric blowup, find all nontrivial toric symmetries:
  - ○ Generate the blowup space sequentially by subdividing the fan, as in Chapter 2, for each object in the blowup configuration.
  - ○ For each element M in $GL(\mathbb{F}_3)$, check that M maps the set of primitives to itself. If it does, collect M as a potential symmetry, and record the permutation g_M of the primitive generators.
  - ○ Check if g_M maps the maximal cones to themselves. If it does, record M as a toric symmetry.
  - ○ Check if g_M is a nontrivial symmetry using Definition 34.

Figure 3.5 The computational technique by which we exhausted all possible toric symmetries of sequential toric blowups of $\mathbb{P}^3$.

Once we have a permutation representation τ_M of M, we simply check to see if there is an exchange between each set of one-skeleton elements that correspond to the various types of divisors described in Definition 34.

Chapter 4

Gromov–Witten Theory

In this chapter we will review the rudiments of Gromov–Witten theory. We will not treat many of the technical details concerning the construction of these invariants and the virtual class of moduli space, but we will supply sufficient details and motivation to familiarize the reader with the subject. The axiomatic description in the following section should be sufficient for the reader to be able to understand the results section of this thesis.

4.1 The Idea Behind Gromov–Witten Theory

In ideal cases, Gromov–Witten invariants count the number of curves in a smooth variety X of given genus g and of a given curve class β, with specified tangency conditions. Tangency conditions are prescribed by the data of cohomology classes γ_i dual to the subvarieties tangent to the curve. All of this data is usually denoted

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{g,\beta}^X.$$

Axiomatically, we simply describe the following data to be a rational number that gives a virtual or quantum count of the curves satisfying the input data.

The fundamental idea behind Gromov–Witten theory is to probe the geometry of the target X, using maps from an abstract complex algebraic curve, whose fundamental class pushes forward to the target class β. To do this, we form a moduli space of all stable maps f from smooth genus g curves C representing curve class β. If there are tangency conditions, the curve C has marked points. That is, we consider the space of isomorphism

classes of maps from a curve with n-marked points $(C, p_1, \ldots, p_n)$ to X, where n is the desired number of tangency conditions,

$$f : C \to X, \;\; f_\star[C] = \beta, \;\; f(p_i) \in \gamma_i.$$

Two such maps are isomorphic if there is a reparametrization of the domain, compatible with the marked points, that takes one map to another. In other words, points of the moduli space are isomorphism classes of the maps described above.

This move to consider maps from abstract curves with marked points, rather than embedded curves with tangencies, was made by Kontsevich, and is consistent with the notion of nonlinear sigma models. The origins of these ideas are intricately tied with notions of two-dimensional quantum field theory coupled to gravity. We denote this moduli space $M_{g,n}(X, \beta)$. In general, there are many obstructions to forming a well behaved moduli space. Intuitively this is due to the fact that curves may have nontrivial automorphism groups, and thus points in the moduli space have nontrivial automorphisms. In general, algebraic stacks provide the right framework to deal with these difficulties, although the geometry of stacks is formidable and will largely remain untreated in this document. The condition that the automorphism group of any point is finite is the key to the definition of stability.

Finally, we will usually want compactify the moduli space to form the space $\overline{M}_{g,n}(X, \beta)$. To do this we will allow curves that have (at worst) nodal singularities, since these can arise as limits of smooth curves. The compactification using this approach space is due to Kontsevich and was central to the development of the theory. This moduli space is often called the Kontsevich moduli space of stable maps. Among the many sources for the beautiful theory are the two books: Hori et al. (2003) and Cox and Katz (1999).

4.2 Defining Gromov–Witten Invariants

We now begin our exploration stability and stable maps, a key ingredients of Gromov–Witten theory. We explicitly describe stability conditions on rational maps from trees to projective space targets, before dealing with the general case.

4.2.1 What is a Stable Map?

The points of the moduli space $\overline{M}_{g,n}(X,\beta)$ can often have nontrivial automorphism groups, as described above. Stability, in this context, is the condition that this automorphism group is finite. This condition is necessary for the moduli space to have desirable properties. The Gromov–Witten theoretic moduli spaces are smooth Deligne–Mumford stacks.

The notion of stability can be intuitively understood by considering $\mathbb{P}^1$. Recall that $\mathbb{P}^1$ is topologically the Riemann sphere. Consider as a thought experiment, the automorphism group of a sphere fixing different numbers of points. It is easy to see that if there are no fixed points, the sphere has infinitely many automorphisms. Also notice that fixing one, or two points on the sphere, there are still infinitely many automorphisms. However, if we fix three points, the sphere has trivial automorphism group. The spheres with zero, one, and two fixed points are unstable, while the sphere with three fixed points is stable. The general conditions described in Definition 49 impose a similar stability conditions on the domain curve C, to ensure that the map to X is stable.

4.2.2 Stability: Genus-0 Maps from Trees of $\mathbb{P}^1$'s to $\mathbb{P}^n$

Before considering the general case of a genus g curve and arbitrary target, we will first introduce the notion of a genus zero stable map to $\mathbb{P}^n$ through the following examples and definitions. Rather than developing the theory of algebraic curves, we streamline the treatment as is done by Katz (2006).

A *tree of* $\mathbb{P}^1$*'s*, defined in Definition 41, is roughly an algebraic curve obtained from finite collections of $\mathbb{P}^1$'s, by gluing them together in a manner that doesn't introduce cycles. Construct a curve C from $C_1, \ldots, C_n$, each being isomorphic to $\mathbb{P}^1$ and a collection of points $\{p_j, q_j\}$ where $p_j \in C_{k(j)}$ and $q_j \in C'_{l(j)}$ for some indexes $k(j) \neq l(j)$. The curve C is obtained by identifying p_j and q_j. Each C_i is called a component. The points where components are glued together are called nodes. Now, the no-cycles condition is as follows. Form the *dual graph* G_C *of* C by representing each C_i by a vertex, and introduce an edge between vertices C_i and C_j if they share a node. C has no cycles if and only if G_C has no cycles as a graph.

Definition 41. *A* tree of $\mathbb{P}^1$'s *is a curve C which is the union of copies of $\mathbb{P}^1$'s glued along pairs of points, such that G_C is a graph theoretic tree.*

Some concrete examples of trees can be obtained as follows.

Example 42. *Consider a line pair $C \subset \mathbb{P}^2$ defined by the vanishing of the homogeneous polynomial $p(x_0, x_1, x_2) = x_1x_2$. It is clear that this is the union of the two lines $x_1 = 0$ and $x_2 = 0$. A tree is obtained by gluing at $(1:0:0)$.*

Example 43. *The union of three lines in $\mathbb{P}^2$ defined by the vanishing of the polynomial $p(x_0, x_1, x_2) = x_0x_1x_2$ is not a tree due to the presence of cycles. For instance, there is a cycle going from $(1:0:0)$ to $(0:1:0)$, to $(0:0:1)$, and back to $(1:0:0)$.*

We now introduce, in this specific case, the concept of a morphism from a tree $C = \cup C_i$ to $\mathbb{P}^n$.

Definition 44. *Let $C = \cup C_i$ be a tree of $\mathbb{P}^1$'s with parametrizations $\phi_i : \mathbb{P}^1 \to C_i$. A* morphism *$f : C \to \mathbb{P}^n$ is a mapping such that each*

$$f_i = f \circ \phi_i : \mathbb{P}^1 \to \mathbb{P}^n$$

is a parametrized rational curve or a constant map. The degree of f is the sum of those of f_i.

Example 45. *Let $C = \mathbb{P}^1$ with parametrization the identity. Let f be the embedding of a line into $\mathbb{P}^n$:*

$$f(x_0, x_1) = (x_0 : x_1 : 0, \ldots : 0).$$

f is a morphism of degree one. The map $g : C \to \mathbb{P}^n$ defined by

$$g(x_0, x_1) = (x_0^2 : x_1^2 : 0 : \ldots : 0)$$

is a morphism of degree two.

We now have the ingredients necessary to define a stable map.

Definition 46. A genus zero stable map to $\mathbb{P}^n$ *is a morphism*

$$f : C \to \mathbb{P}^n$$

from a tree C such that if f is constant when restricted to a component C_i then C_i is required to contain at least three nodes of C.

To form our moduli space, we need isomorphism classes of stable maps. An isomorphism of stable maps is defined below.

Definition 47. *Let $f : C \to \mathbb{P}^n$ and $f' : C' \to \mathbb{P}^n$ be two genus zero stable maps with components parametrized by $\phi_i : \mathbb{P}^1 \to C_i$ and $\phi_i' : \mathbb{P}^1 \to C_i'$. An* isomorphism from f to f' *is a map $g : C \to C$ of domain curves, such that the following conditions hold:*

- $f' \circ g = f$.
- *For each* C_i *we have* $f(C_i) = C'_j$ *for some* j *and this correspondence between* i *and* j *is unique.*
- ${\phi'}_j^{-1} \circ g \circ \phi_i : \mathbb{P}^1 \to \mathbb{P}^1$ *is a degree one parametrized rational curve whenever* $g(C_i) \subset C'_j$.

With this machinery, we can now define $\overline{M}_{0,0}(\mathbb{P}^n, \beta)$. Note that $\beta \in H_2(\mathbb{P}^n) = \mathbb{Z}$.

Definition 48. The moduli space of stable maps $\overline{M}_{0,0}(\mathbb{P}^n, d)$ *is the set of all isomorphism classes of degree* d *genus zero stable maps to* $\mathbb{P}^n$

4.2.3 Stability: General Case

We now turn our attention to the general case. The stability conditions on maps from an arbitrary genus curve C to a target X are analogous to the stability conditions discussed previously. The points of $\overline{M}_{g,n}(X, \beta)$ are triples $(C, \{p_i\}, f)$ where C is a genus g complex curve with n distinct nonsingular marked points $p_1, \ldots, p_n$ and f a map $C \to X$ such that $f_\star[C] = \beta$. A stable map is then defined as follows.

Definition 49. *An* n-pointed stable map *consists of a connected domain curve with marked points* $(C, \{p_i\})$ *and a morphism* $f : C \to X$ *satisfying the following properties,*

1. *The only singularities of* C *are ordinary double points.*
2. $p_1, \ldots, p_n$ *are distinct ordered nonsingular points of* C.
3. *If* C_i *is a component of* C*, such that* $C_i \cong \mathbb{P}^1$*, then* C_i *contains at least three special points (marked or nodal).*
4. *If* C *has arithmetic genus one and* $n = 0$ *(i.e.,* C *is elliptic) then* f *is nonconstant.*

Given the first two conditions, the final two conditions are equivalent to ensuring that the data $(C, \{p_i\}, f)$ has a finite automorphism group. For the informed reader familiar with $\overline{M}_{g,n}$ the moduli space of stable curves, notice the difference between this notion of stability and the Deligne–Mumford stability of curves: for a stable map, only components which contract to a point need to be stable in the sense of stability of curves.

4.2.4 Fundamental Classes and The Virtual Class

Intuitively, for an smooth orientable manifold M, the fundamental class corresponds to the homology class of "the whole manifold." That is, it is generally the class that generates $H^{2n}(X)$ for a $2n$ real dimensional manifold. Pairing cohomology classes with this fundamental class can be seen as an abstract form of integration on this space. In fact if the cohomology theory is the de Rham cohomology, this is the standard Riemann integral. However, if the space is not smooth, or not orientable, the fundamental class cannot be defined. That is there is no class that corresponds to the "whole space." In general the space $\overline{M}_{g,n}(X,\beta)$ is badly behaved—extremely singular, not connected, and even nonequidimensional. Thus there is usually no fundamental class that can be defined except for special cases such as projective spaces in genus zero. The moduli space usually has components of higher than expected dimension. However via deformation theory, the moduli space can be shown to have a perfect obstruction theory. This obstruction theory can then be used to construct a *virtual fundamental class* which behaves much like a fundamental class, and has the expected dimension. Gromov–Witten invariants are defined by integration against this virtual fundamental class. We cannot go into the construction in this manuscript, but it is important to note the existence of such a class. In particular, the virtual class has degree

$$\operatorname{vdim} \overline{M}_{g,n}(X,\beta) = (\dim X - 3)(1-g) - K_X \cdot \beta + n.$$

Here K_X is the canonical class of the target space X. There are only a few cases where the moduli space admits a fundamental class. In such cases, the moduli space is said to be homogeneous. For example, for target projective spaces, with no insertions in genus zero, the moduli space turns out to be a smooth complex orbifold, which admits a fundamental class. However, even blowing up at a single point in $\mathbb{P}^n$, the moduli spaces are no longer orbifold, and a virtual class argument is necessary.

4.2.5 Evaluation Morphisms

Notice that the classes γ_i are classes on the target space X. Thus, we need a way to pullback classes from the target space, to the moduli space. The way this is done is by pulling back via evaluation maps from $\overline{M}_{g,n}(X,\beta)$ to X. Observe that our maps $f : C \to X$ can be evaluated on the marked

points. Thus, for every map $f \in \overline{M}_{g,n}(X,\beta)$, we have

$$\begin{aligned} \mathrm{ev}_i : \overline{M}_{g,n}(X,\beta) &\to X \\ f &\mapsto f(p_i). \end{aligned}$$

The evaluation morphisms give us a way to a way to pull cohomology classes back from X to the moduli stack. Finally, we have a definition for a (primary) Gromov–Witten invariant:

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{g,\beta}^X = \int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} \bigwedge_{i=1}^{n} \mathrm{ev}_i^\star(\gamma_i).$$

The class $\bigwedge_{i=1}^n \mathrm{ev}_i^\star(\gamma_i)$ is sometimes referred to as the Gromov–Witten class. Notice that the Gromov–Witten invariant is zero if the sum of the degrees of the sum of the degrees of the inserted cohomology classes is not equal to the degree of the virtual class. In particular this means that if we consider classes without insertions, known as *virtual dimension zero* or *Calabi–Yau* classes,

$$\langle\ \rangle_{g,\beta}^X = \int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} 1.$$

We can deduce from the dimension formula that $-K_X \cdot \beta = 0$.

4.3 Gromov–Witten Theory of Blowups at Points

In general, if $\pi : \hat{X} \to X$ is a blowup of X, there need not be any meaningful relationship between the invariants of X and $\hat{X}$. In particular, observe that the canonical class of $\hat{X}$ is not the pullback of the canonical class of X,

$$\pi^\star K_X \neq K_{\hat{X}}.$$

In fact, via adjunction, we know that if we blowup a dimension k submanifold $Z \subset X$, then

$$K_{\hat{X}} = \pi^\star K_X + (n-k-1)E,$$

where E is the class of the projectivization of the normal bundle of Z in X, $\mathbb{P}(N_{Z/X})$. The dimension of the virtual class is dependent on the canonical class, which explicitly tells us that certain invariants are zero, when the degree of the integrand is not equal to the virtual dimension. The main result of this thesis follows from a proof that if we can "stay away" from the locus of the blowup, then these issues are overcome.

The blowup of the target variety at points turns out to be a more interesting case than the general blowup. In such a case, statements can be made about certain invariants of classes that pass through the exceptional locus, namely the points. This problem was first studied by Gathmann for projective space. A more general case was proved by Bryan and Leung (2000), by explicitly proving statements about the virtual class of the moduli stacks. Their results allows us to trade point insertions in the base space, with exceptional classes in the homology class in the blowup. More formally,

Lemma 50 (Bryan and Leung (2000))**.** *Let Y be a smooth algebraic variety and $p : \hat{Y} \to Y$ the blowup of Y at a point. Let $\beta \in A_1(Y)$ and $\hat{\beta} = p^!(\beta)$. Then we have the equality of invariants,*

$$\langle pt \rangle^{Y}_{g,\beta} = \langle \, \rangle^{\hat{Y}}_{g,\hat{\beta}-\hat{e}}.$$

Here $\hat{e}$ is the class of a line above the exceptional locus, and $p^!(\beta) = [p^\star[\beta]^{\mathrm{PD}}]^{\mathrm{PD}}$.

Here PD denotes the Poincare duality.

Invariants where the insertions are only classes of points, where the total degree of the insertions is zero, are sometimes known as *stationary invariants.* Let X be the blowup of $\mathbb{P}^3$ at two points p_1 and p_2. Then from Lemma 50, we get the following equality of invariants

$$\langle pt, pt \rangle^{\mathbb{P}^3}_{0,h} = \langle \, \rangle^{X}_{0,h-e_1-e_2} = 1.$$

In fact the invariant on the left corresponds to the class of a line through two points in $\mathbb{P}^3$, of which there is precisely one. Using this result of Bryan and Leung, we may use Theorem 3 to make statements about the stationary invariants on the base spaces and recover enumerative results. In Section 5.4, we will explicitly use this technique.

4.4 A Note on Donaldson–Thomas Theory

Another modern approach to studying enumerative geometry, is through Donaldson–Thomas theory (DT) and the closely related Pandharipande–Thomas theory of stable pairs (PT). Though we will not present proofs of our main results in these contexts, there are powerful duality theorems between the Gromov–Witten and Donaldson–Thomas theories. This includes a proof of duality in the case of toric-threefolds. Although our results are not proved in the Donaldson–Thomas setting, there is reason to suspect

that they are true, including an overarching Gromov–Witten Donaldson–Thomas duality, conjectured by Maulik, Okounkov, Nekrasov and Pandharipande (2006b). One natural extension of this work would be to prove the descent, correspondence and symmetry theorems for the Donaldson–Thomas invariants. One should note that the toric GW/DT duality does not immediately force these results to extend to the DT setting. The duality between Gromov–Witten and Donaldson–Thomas theories is at the level of generating functions, not invariants. For the purposes of this larger context, we now present a terse overview of DT invariants. The basic approach of DT theory is to replace the study of maps to a target variety X, with the study of sheaves on X. Donaldson–Thomas invariants also have a presence in physics. Sheaves are considered as models for D-branes in the topological B-model of string theory.

4.4.1 Defining Donaldson–Thomas Invariants

Donaldson–Thomas invariants also virtually count curves in a smooth projective threefold X of class $\beta \in H_2(X;\mathbb{Z})$ intersecting Poincaré duals of cohomology classes $\gamma_1, \ldots, \gamma_r \in H^\star(X)$. For an ideal sheaf[1] $\mathcal{I}$, there exists an injection into its double dual

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{I}^{\vee\vee}.$$

But

$$\mathcal{I}^{\vee\vee} \cong \mathcal{O}_X,$$

so $\mathcal{I}$ determines a subscheme Y given by

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Since $\mathcal{I}$ has trivial determinant, Y has components of dimension zero and one. The weighted one dimensional components of Y determine a homology class

$$[Y] \in H_2(X;\mathbb{Z}).$$

The moduli space of ideal sheaves $\mathcal{I}$ with holomorphic Euler characteristic $\chi(\mathcal{O}_Y) = n$ and class $[Y] = \beta \in H_2(X;\mathbb{Z})$ is denoted $I_n(X,\beta)$. Similar to GW invariants, DT invariants are defined by integrating against the virtual class $[I_n(X,\beta)]^{\text{vid}}$ of dimension

$$\dim[I_n(X,\beta)]^{\text{vir}} = \int_\beta c_1(T_X).$$

[1] According to the convention in Maulik et al. (2011), an ideal sheaf is a rank 1 torsion free sheaf with trivial determinant.

The construction of this virtual class and other foundational aspects of DT theory may be found in the paper by Maulik and colleagues (2006a) and Thomas (2000).

In order to integrate against the virtual class, we need to pull back the classes γ_i from X to $I_n(X,\beta)$. This is done using the universal ideal sheaf and the associated universal subscheme.

By results from the paper by Maulik and colleagues (2011: Section 1.2), there exists a universal ideal sheaf

$$\mathfrak{I} \longrightarrow I_n(X,\beta) \times X$$

with well-defined Chern classes[2]. Let π_i denote the respective projection maps. The DT invariants are defined by push-pulling Chern classes via π_i. For each $\gamma \in H^\star(X)$, define the operator $c_2(\gamma)$ by, for any $\xi \in H_\star(I_n(X,\beta))$,

$$c_2(\gamma)(\xi) = \pi_{1\star}\left(c_2(\mathfrak{I}) \cdot \pi_2^\star(\gamma) \cap \pi_1^\star(\xi)\right). \tag{4.1}$$

For details of this construction, including the pullback of the homology class ξ in Equation 4.1, see the paper by Maulik and colleagues (2011: Section 1.2).

The class (n,β) DT invariant of X with insertions $\gamma_1,\dots\gamma_r$ is defined by

$$DT^X_{n,\beta}(\gamma_1,\dots,\gamma_r) = \langle \gamma_1,\dots,\gamma_r\rangle^X_{n,\beta} = \int_{[I_n(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{r} c_2(\gamma_i).$$

[2]The second Chern class $c_2(\mathfrak{I})$ is interpreted as the universal subscheme as in Maulik et al. (2011).

Chapter 5

Cremona Symmetry and the Permutohedron

One of the classical examples of a birational transformation is the Cremona birational map on $\mathbb{P}^n$. This map can be resolved via a sequence of toric blowups of $\mathbb{P}^n$. The polytope of this toric blowup of $\mathbb{P}^n$ is a polytope known as the permutohedron, an object of independent interest in combinatorics. The permutohedron is part of a larger class of polytopes known as *generalised associahedra*. For more regarding these combinatorial applications, as well application as to real moduli spaces, see for instance Devadoss (2009). The results of this thesis are intimately related to the three-dimensional permutohedron and analogues to the Cremona transformation. In this chapter we create the framework for the main theorems of this thesis by discussing the toric geometry relevant to the Cremona transformation and the permutohedron. We will then discuss analogous birational transformations on $(\mathbb{P}^1)^{\times 3}$, as well as enumerative applications of the results.

5.1 Toric Blowups and the Permutohedron

In this section we will construct the permutohedron and its associated toric variety. For further treatment regarding the combinatorics and topological applications of the permutohedron and related polytopes, see the papers by Carr and Devadoss (2006), Devadoss (2009), and Postnikov (2005). For more regarding toric blowups, their polytopes and symmetries, see the paper by the author and colleagues (Karp et al., 2011).

Let X be a toric variety with fan Σ_X. We will denote torus fixed subvarieties in multi-index notation corresponding to generators of their cones.

$$\begin{tikzcd}
X_{\Pi_2} \arrow[r, "\psi"] \arrow[d, "\pi_1"] & X_{\Pi_2} \arrow[d, "\pi_2"] \\
\mathbb{P}^2 \arrow[r, dashed, "\varphi"] & (\mathbb{P}^1)^{\times 2}
\end{tikzcd}$$

Figure 5.1 The variety X_{Π_2} as a blowup.

For instance, $p_{i_1 \dots i_k}$ will denote the torus fixed point which is the orbit closure of the cone $\sigma = \langle v_{i_1}, \dots, v_{i_k} \rangle$, for $v_i \in \Sigma_X^{(1)}$. Similarly $\ell_{i_1 \dots i_r}$ will denote the line which is the orbit closure of $\sigma = \langle v_{i_1}, \dots, v_{i_r} \rangle$, and so on. Further, $X(Z_1, \dots, Z_s)$ will denote the iterated blowup of X at the subvarieties $Z_1, \dots, Z_s$. By abuse of notation, we will denote $X(k)$ as the blowup of X at k points where it causes no ambiguity.

5.1.1 The Permutohedron in Dimension Two

Recall that the fan $\Sigma_{\mathbb{P}^2} \subset \mathbb{Z}^2$ of $\mathbb{P}^2$ has a one-skeleton whose primitive generators are

$$v_1 = (-1, -1), \;\; v_2 = (1, 0), \;\; v_3 = (0, 1),$$

and maximal cones given by

$$\langle v_1, v_2 \rangle, \;\; \langle v_2, v_3 \rangle, \;\; \langle v_1, v_3 \rangle.$$

Also recall that the fan $\Sigma_{(\mathbb{P}^1)^{\times 2}} \subset \mathbb{Z}^2$ of $(\mathbb{P}^1)^{\times 2}$, has one-skeleton generators

$$u_1 = (1, 0), \;\; u_2 = (-1, 0), \;\; u_3 = (0, 1), \;\; u_4 = (0, -1).$$

with maximal cones given by

$$\langle u_1, u_3 \rangle, \;\; \langle u_1, u_4 \rangle, \;\; \langle u_2, u_3 \rangle, \;\; \langle u_2, u_4 \rangle.$$

In dimension two, the permutohedron Π_2 (a hexagon) can be realized as the dual polytope to two toric varieties. First, of $\mathbb{P}^2(p_{12}, p_{23}, p_{13})$, the blowup of $\mathbb{P}^2$ at its three torus fixed points. And second, of the toric variety $(\mathbb{P}^1)^{\times 2}(p_{13}, p_{24})$. Thus the variety associated to Π_2 is a common blowup for $\mathbb{P}^2$ and $(\mathbb{P}^1)^{\times 2}$, providing a birational map between these varieties via blowup–blowdown. By functoriality of Gromov–Witten invariants, this gives us a way to relate the invariants on $\mathbb{P}^2$ blown up at points to that

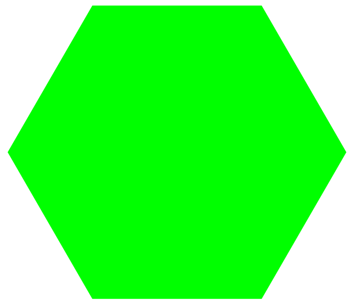


Figure 5.2 The two-dimensional permutohedron.

of $(\mathbb{P}^1)^{\times 2}$ blown up at points. Via Lemma 50 we then have results concerning the stationary invariants on $\mathbb{P}^2$ and $(\mathbb{P}^1)^{\times 2}$. The polytope of the two-dimensional permutohedral variety is depicted in Figure 5.2

Our goal now is to use this combinatorial observation about the permutohedron in higher dimensions.

5.1.2 The Permutohedron in Dimension Three

Recall that the fan $\Sigma_{\mathbb{P}^3} \subset \mathbb{Z}^3$ of $\mathbb{P}^3$ has one-skeleton with primitive generators

$$\begin{aligned} v_1 &= (-1,-1,-1), & v_2 &= (1,0,0), \\ v_3 &= (0,1,0), & v_4 &= (0,0,1), \end{aligned}$$

and maximal cones given by

$$\begin{aligned} &\langle v_1, v_2, v_3 \rangle, \quad \langle v_1, v_2, v_4 \rangle, \\ &\langle v_1, v_3, v_4 \rangle, \quad \langle v_2, v_3, v_4 \rangle. \end{aligned}$$

Also note that the fan $\Sigma_{(\mathbb{P}^1)^{\times 3}} \subset \mathbb{Z}^3$ of $(\mathbb{P}^1)^{\times 3}$, has one-skeleton generators

$$\begin{aligned} u_1 &= (1,0,0), & u_3 &= (0,1,0), & u_5 &= (0,0,1), \\ u_2 &= (-1,0,0), & u_4 &= (0,-1,0), & u_6 &= (0,0,-1), \end{aligned}$$

and maximal cones given by

$$\begin{aligned} &\langle u_1, u_3, u_5 \rangle, \quad \langle u_1, u_2, u_4 \rangle, \quad \langle u_1, u_2, u_3 \rangle, \quad \langle u_1, u_2, u_4 \rangle, \\ &\langle u_2, u_4, u_6 \rangle, \quad \langle u_2, u_3, u_4 \rangle, \quad \langle u_1, u_2, u_3 \rangle, \quad \langle u_1, u_2, u_4 \rangle. \end{aligned}$$

Analogously to Π_2, the three-dimensional permutohedron Π_3 can be realized as the dual polytope of the blowup of $\mathbb{P}^3$ at its four torus fixed points

$$\begin{tikzcd} X_{\Pi_3} \arrow[r, "\sigma"] \arrow[d, "\pi_1"] & X_{\Pi_3} \arrow[d, "\pi_2"] \\ \mathbb{P}^3(4) \arrow[r, dashed, "\tau"] & (\mathbb{P}^1)^{\times 3}(2) \end{tikzcd}$$

Figure 5.3 The variety X_{Π_3} as a blowup.

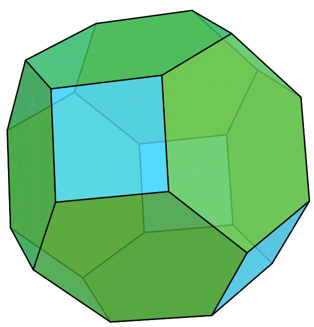


Figure 5.4 The polytope of the three-dimensional permutohedral variety.

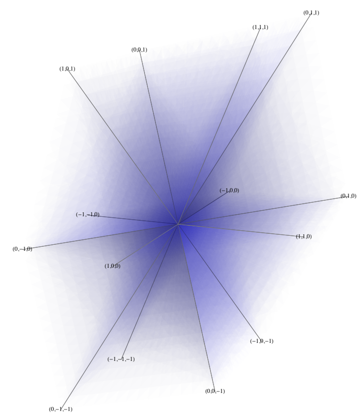


Figure 5.5 The fan of the three-dimensional permutohedral variety.

and the six torus invariant lines between them,

$$X_{\Pi_3} = \mathbb{P}^3(p_{123}, p_{124}, p_{134}, p_{234}, \ell_{12}, \ell_{13}, \ell_{14}, \ell_{23}, \ell_{24}, \ell_{34}).$$

It can also be realized as the dual polytope of a blowup of $(\mathbb{P}^1)^{\times 3}$. In particular,

$$X_{\Pi_3} \cong (\mathbb{P}^1)^{\times 3}(p_{135}, p_{246}, \ell_{13}, \ell_{15}, \ell_{35}, \ell_{24}, \ell_{26}, \ell_{46}).$$

The above blowup can be viewed as the blowup of two antipodal vertices on the 3-cube and the six invariant lines intersecting these points. This common blowup gives us a birational map between the spaces $\mathbb{P}^3$ and $(\mathbb{P}^1)^{\times 3}$. Appealing to the aforementioned relationship between the Gromov–Witten theories of varieties, and their blowups at points, we will consider these blowups of $\mathbb{P}^3(4)$ and $(\mathbb{P}^1)^{\times 3}(2)$, respectively. The situation is depicted in Figure 5.3.

Remark 51. This construction can be generalised to higher dimensions. The permutohedron Π_n is the dual polytope corresponding to the blowup of

$\mathbb{P}^n$ at all its torus invariant subvarieties up to dimension $n-2$. Note that $\Delta_{(\mathbb{P}^1)^{\times n}}$, the dual polytope of $(\mathbb{P}^1)^{\times n}$ is the n-cube. Then Π_n is the dual polytope of the variety corresponding to the blowup of $(\mathbb{P}^1)^{\times n}$ at the points corresponding to antipodal vertices on $\Delta_{(\mathbb{P}^1)^{\times n}}$, and all the torus invariant subvarieties intersecting these points, up to dimension $n-2$.

5.1.3 Chow Ring of X_{Π_3}

We now turn our attention to a description of the cohomology and Chow ring of these toric varieties. Recall that if X is a smooth projective toric variety, the $H^\star(X) = A^\star(X)$. In particular $A^\star(X)$ is generated by the divisor classes coming from the orbit closures of the elements in $\Sigma_X^{(1)}$. We will use D_α for the divisor class corresponding to v_α or u_α. Finally, we will label a new element of the one-skeleton, introduced to subdivide the cone $\sigma = \langle v_i, \ldots, v_j \rangle$, by $v_{i \cdots j}$. For a deeper treatment of the Chow ring and intersection theory of toric varieties, see Fulton (1993).

Notation

Note that throughout the rest of this paper, the undecorated classes will be classes on $\mathbb{P}^3(k)$, tilde classes, such as $\tilde{H}_i$ or $\tilde{e}_{ijk}$ will be classes on $(\mathbb{P}^1)^{\times 3}(k)$. Classes pulled back via the blowup to the variety X_{Π_3} will be decorated with a hat.

As a Toric Blowup of $\mathbb{P}^3$

The Chow ring of $\mathbb{P}^3$ is generated by the first Chern class of hyperplane bundle on $\mathbb{P}^3$. Let $\hat{H}$ be the pullback of this class to X_{Π_3} and $\hat{H} \cdot \hat{H} = \hat{h}$ the class of a general line in $A_1(X)$. Let $\hat{E}_\alpha$ be the class of the exceptional divisor above the blowup of p_α, and $\hat{e}_\alpha$ be the line class in the exceptional divisor. Let $\hat{F}_{\alpha'}$ the class of the exceptional divisor above the blowup of the line $\ell_{\alpha'}$. Note that that this divisor is abstractly isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, so we let $\hat{f}_\alpha$ and $\hat{s}_\alpha$ be the section and fiber class, respectively. Observe that

$$A_2(X_{\Pi_3}) = \langle \hat{H}, \hat{E}_\alpha, \hat{F}_{\alpha'} \rangle, \;\; A_1(X_{\Pi_3}) = \langle \hat{h}, \hat{e}_\alpha, \hat{f}_{\alpha'} \rangle.$$

The divisor classes corresponding to $\Sigma^{(1)}_{X_{\Pi_3}}$, are written in terms of this basis as

$$\begin{aligned} D_i &= \hat{H} - \sum_{i\in\alpha} \hat{E}_\alpha - \sum_{j\in\alpha'} \hat{F}_{\alpha'} \\ D_{ij} &= \hat{F}_{ij} \\ D_{ijk} &= \hat{E}_{ijk}. \end{aligned}$$

As a Toric Blowup of $(\mathbb{P}^1)^{\times 3}$

Let $\hat{\hat{H}}_1$, $\hat{\hat{H}}_2$, and $\hat{\hat{H}}_3$ be the three hyperplane classes pulled back from the Künneth decomposition of the homology of $(\mathbb{P}^1)^{\times 3}$. That is, $H_1 = pt \otimes [\mathbb{P}^1] \otimes \mathbb{P}^1$, $H_2 = [\mathbb{P}^1] \otimes pt \otimes [\mathbb{P}^1]$, and $H_3 = [\mathbb{P}^1] \otimes [\mathbb{P}^1] \otimes pt$. Then $\hat{\hat{H}}_i$ is the pullback of H_i through the blowup map. We let $\hat{h}_{ij}$ be the line class $\hat{\hat{H}}_i \cdot \hat{\hat{H}}_j$ and $\hat{\hat{E}}_\alpha$, $\hat{\hat{e}}_\alpha$, $\hat{\hat{F}}_{\alpha'}$, $\hat{\hat{f}}_{\alpha'}$, and $\hat{\hat{s}}_{\alpha'}$ be the aforementioned divisor and curve classes. These classes generate the Chow groups in the appropriate degree. The divisor classes corresponding to $\Sigma^{(1)}_{X_{\Pi_3}}$ are given by

$$\begin{array}{ll} D_1 = \hat{\hat{H}}_1 - \hat{\hat{E}}_{135} - \hat{\hat{F}}_{13} - \hat{\hat{F}}_{15}, & D_2 = \hat{\hat{H}}_3 - \hat{\hat{E}}_{246} - \hat{\hat{F}}_{24} - \hat{\hat{F}}_{26} \\ D_3 = \hat{\hat{H}}_2 - \hat{\hat{E}}_{135} - \hat{\hat{F}}_{13} - \hat{\hat{F}}_{35}, & D_4 = \hat{\hat{H}}_2 - \hat{\hat{E}}_{246} - \hat{\hat{F}}_{24} - \hat{\hat{F}}_{46} \\ D_5 = \hat{\hat{H}}_3 - \hat{\hat{E}}_{135} - \hat{\hat{F}}_{13} - \hat{\hat{F}}_{25}, & D_6 = \hat{\hat{H}}_3 - \hat{\hat{E}}_{246} - \hat{\hat{F}}_{26} - \hat{\hat{F}}_{46} \\ \qquad D_{ijk} = \hat{\hat{E}}_{ijk}, & \qquad D_{ij} = \hat{\hat{F}}_{ij}. \end{array}$$

Notice that in Figure 5.3, σ is an isomorphism induced by a relabeling of the fan $\Sigma_{X_{\Pi_3}}$. In particular, inspecting $\sigma_\star$ on $A_1(X_{\Pi_3})$, we see that

$$\begin{aligned} \sigma_\star \hat{h} &= \hat{\hat{h}}_{12} + \hat{\hat{h}}_{13} + \hat{\hat{h}}_{23} - \hat{\hat{e}}_{246} \\ \sigma_\star \hat{e}_{123} &= \hat{\hat{h}}_{13} + \hat{\hat{h}}_{23} - \hat{\hat{e}}_{246} \\ \sigma_\star \hat{e}_{124} &= \hat{\hat{h}}_{12} + \hat{\hat{h}}_{23} - \hat{\hat{e}}_{246} \\ \sigma_\star \hat{e}_{134} &= \hat{\hat{h}}_{12} + \hat{\hat{h}}_{13} - \hat{\hat{e}}_{246} \\ \sigma_\star \hat{e}_{234} &= \hat{\hat{e}}_{135} \\ \sigma_\star \hat{f}_{12} &= \hat{\hat{s}}_{46} = \hat{\hat{h}}_{23} - \hat{\hat{e}}_{246} + \hat{\hat{f}}_{46} \\ \sigma_\star \hat{f}_{13} &= \hat{\hat{s}}_{26} = \hat{\hat{h}}_{13} - \hat{\hat{e}}_{246} + \hat{\hat{f}}_{26} \\ \sigma_\star \hat{f}_{14} &= \hat{\hat{s}}_{24} = \hat{\hat{h}}_{12} - \hat{\hat{e}}_{246} + \hat{\hat{f}}_{24} \\ \sigma_\star \hat{f}_{34} &= \hat{\hat{f}}_{35} \end{aligned}$$

$$\begin{aligned} \sigma_\star \hat{f}_{24} &= \hat{\hat{f}}_{15} \\ \sigma_\star \hat{f}_{23} &= \hat{\hat{f}}_{13}. \end{aligned}$$

5.2 Cremona Symmetry

The classical Cremona transformation is the rational map

$$\xi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$$

defined by

$$(x_0 : \ldots : x_n) \mapsto (x_1 \ldots x_n : \ldots : \prod_{j \neq i} x_j : \ldots : x_0 \ldots x_{n-1}).$$

Note that the map is undefined on the union of the torus invariant subvarieties of codimension at least two, and is resolved by the maximal blowup of $\mathbb{P}^n$, namely the blowup $\pi : X_{\Pi_n} \to \mathbb{P}^n$. In the language of toric geometry, the resolved Cremona involution on X_{Π_n} is a toric symmetry, namely it is induced by the reflection through the origin symmetry on $\Sigma_{X_{\Pi_n}}$. Note that the resolved Cremona map, $\hat{\xi}$ pushes forward nontrivially to the Chow ring of X_{Π_n}. For a more detailed treatment on toric symmetries the Cremona symmetry in $\mathbb{P}^3$ see Bryan and Karp (2005). A result from a previous paper by the author (Karp et al., 2011) shows that all the automorphisms $\Sigma_{X_{\Pi_3}}$ are either identity on cohomology, or are equal to $\hat{\xi}_\star$ (perhaps up to relabeling). The Cremona symmetry can be stated as follows:

Lemma 52 (Bryan and Karp (2005))**.** *Let X_{Π_3} be the permutohedral variety as a blowup of $\mathbb{P}^3$. Let β be given by*

$$\beta = d\hat{h} - \sum_{i=1}^{4} a_i \hat{e}_i - \sum_{1 \leq i < j \leq 6} b_{ij} \hat{f}_{ij} \in H_2(X; \mathbb{Z}).$$

There exists a toric symmetry $\hat{\xi}$ that resolves ξ, such that $\hat{\xi}_\star \beta = \beta'$, where $\beta' = d'\hat{h} - \sum_i a'_i \hat{e}_i - \sum_{ij} b'_{ij} \hat{f}_{ij}$ has coefficients given by

$$\begin{aligned} d' &= 3d - 2\sum_{i=1}^{4} a_i \\ a'_i &= d - a_j - a_k - a_l - b_{ij} - b_{ik} - b_{il} \\ b'_{ij} &= b_{kl}, \end{aligned}$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

5.2.1 Descent of Cremona Symmetry

Notice in Lemma 52, that the classes $\hat{f}_i$ form an orbit under $\hat{\hat{\xi}}_\star$. In particular if a curve class β has $b_\alpha = 0$ for all α, then $\hat{\hat{\xi}}_\star\beta$ also has $b'_\alpha = 0$ for all α. The corresponding equality of Gromov–Witten invariants does not in general descend to $\mathbb{P}^3(4)$. However Bryan and Karp (2005) prove that if we blow up at additional points p_5, p_6, to $\hat{X} = X_{\Pi_3}(2) \to \mathbb{P}^3(6)$, such that $\hat{\hat{\xi}}_\star\hat{e}_5 = \hat{e}_6$, then for all classes $\beta = dh - \sum_{1\le i\le 6} a_i e_i$ with a_5 or a_6 nonzero, there is an equality of invariants

$$\langle\,\rangle^{\hat{X}}_{g,\beta} = \langle\,\rangle^{\mathbb{P}^3(6)}_{g,\pi_\star\beta}.$$

With the above result, it immediately follows that the invariants of $\pi_\star\beta$ are preserved under pushforward by the birational map ξ. In Chapter 6, we will recall the proof of descent by Bryan and Karp.

5.2.2 Reinterpreting the Cremona Transform on $\mathbb{P}^3$

Let $\hat{X} \to \mathbb{P}^3(4)$ be the permutohedral variety. The resolved Cremona transformation on $\hat{X}$ is induced by reflecting the permutohedron through the origin. On $\hat{X}$, this map exchanges the classes E_i above the blowups of points, with the classes corresponding to the faces of the tetrahedron.

Considering $\hat{\hat{X}}$ as a blowup over $(\mathbb{P}^1)^{\times 3}(2)$, the reflection through the origin symmetry simply exchanges the classes $\tilde{E}_1$ and $\tilde{E}_2$ of divisors above the blowups of the points. In particular, the reflection through the origin symmetry is a trivial symmetry when $\hat{\hat{X}}$ is viewed as a blowup over $(\mathbb{P}^1)^{\times 3}(2)$. The blowup over $\mathbb{P}^3$ sends a curve class of degree 1 to a class of degree 3. Thus, a trivial toric symmetry of $\hat{\hat{X}}$ composed with the induced isomorphism to $\hat{X}$ induces a nontrivial symmetry on $\hat{X}$. This illustrates that the base space of the blowup is crucial in our definition of *nontrivial* toric symmetries.

5.3 An Analogue of Cremona Symmetry

We now show that the blowup $X_{\Pi_3} \to (\mathbb{P}^1)^{\times 3}$, also has a nontrivial toric symmetry analogous to Cremona involution. For this next lemma we will view X_{Π_3} as a blowup of $(\mathbb{P}^1)^{\times 3}$, and will use the homology discussed in Section 5.1.3. We will however label the classes in single index notation, e_1, e_2, and $f_1, \ldots, f_6$ following the previously discussed order for blowup.

Consider the rational map

$$\zeta : (\mathbb{P}^1)^{\times 3} \dashrightarrow (\mathbb{P}^1)^{\times 3}$$

defined by

$$((x_0 : x_1), (y_0 : y_1), (z_0, z_1)) \mapsto ((x_1 y_0 z_0 : x_0 y_1 z_1), (y_0 : y_1), (z_0, z_1)).$$

Lemma 53. *Let* $\beta = \sum_{1\le i\le j\le 3} d_{ij}\hat{\hat{h}}_{ij} - a_1\hat{\hat{e}}_1 - a_2\hat{\hat{e}}_2 - \sum_{i=1}^{6} b_i\hat{\hat{f}}_i \in A_\star(X_{\Pi_3})$. X_{Π_3} *admits a nontrivial toric symmetry* $\hat{\zeta}$*, which resolves* ζ*, whose action on homology is given by*

$$\zeta_\star\beta = \beta'$$

where $\beta' = \sum_{1\le i\le j\le 3} d'_{ij}\hat{\hat{h}}_{ij} - a'_1\hat{\hat{e}}_1 - a'_2\hat{\hat{e}}_2 - \sum_{i=1}^{6} b'_i\hat{\hat{f}}_i$ *has coefficients given by*

$$\begin{array}{rcl}
d'_{12} &=& d_{12} + d_{23} - a_1 - a_2 - b_2 - b_5 \\
d'_{23} &=& d_{23} \\
d'_{13} &=& d_{13} + d_{23} - a_1 - a_2 - b_1 - b_4 \\
a'_1 &=& d_{23} - a_2 - b_4 - b_5 \\
a'_2 &=& d_{23} - a_1 - b_2 - b_2 \\
b'_1 &=& b_5,\ \ b'_2 = b_4 \\
b'_3 &=& b_3,\ \ b'_4 = b_2 \\
b'_5 &=& b_1,\ \ b'_6 = b_6.
\end{array}$$

Proof. Observe that choosing $\hat{\zeta}$ to be the toric symmetry

$$\hat{\zeta} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

$\zeta_\star$ on $A_\star(X_{\Pi_3})$ has the desired action on homology, and the natural blowup–blowdown composition with $\hat{\zeta}$ gives the birational map ζ.

Just as for the Cremona symmetry, observe in Lemma 53, we see that the classes $\hat{f}_\alpha$ also form an orbit under $\zeta_\star$. Paired with the discussion in Section 5.1.3, observe that the proofs of Theorems 3 and 4, amount to proving the descent of nonexceptional invariants from X_{Π_3} to $(\mathbb{P}^1)^{\times 3}(4)$ when $\tilde{a}_3$ or $\tilde{a}_4$ is nonzero, where $\tilde{p}_3$ and $\tilde{p}_4$ are points that are not fixed by the torus action. In Chapter 6, we will prove descent in such cases. □

$$\begin{tikzcd}
X_{\Pi_3} \arrow[r, "\hat{\zeta}"] \arrow[d] & X_{\Pi_3} \arrow[d] \\
(\mathbb{P}^1)^{\times 3} \arrow[r, dashed, "\zeta"] & (\mathbb{P}^1)^{\times 3}
\end{tikzcd}$$

Figure 5.6 The rational map ζ and its resolution.

5.4 Results in Enumerative Geometry

In this section we illustrate the use of the Cremona symmetry on $\mathbb{P}^3$ and its analogue on $(\mathbb{P}^1)^{\times 3}$, to prove basic enumerative results on these spaces. We also will use the main result of this paper, Theorem 3, to recover classical enumerative consequences through the machinery of Gromov–Witten theory.

5.4.1 Lines in $(\mathbb{P}^1)^{\times 3}$

Observe that given a line class, say $\tilde{h}_{12}$ in $(\mathbb{P}^1)^{\times 3}$, there exists only one line of this class through a fixed point. That is

$$\langle \tilde{p}_1 \rangle_{0,\tilde{h}_{12}}^{(\mathbb{P}^1)^{\times 3}} = 1.$$

This result is not obvious, and we will use by using Theorem 3. Using the result of Lemma 50, we can write this stationary invariant as a virtual dimension zero invariant of a blowup at four points, $p_1, \ldots, p_4$,

$$\langle \tilde{p}_1 \rangle_{0,\tilde{h}_{12}}^{(\mathbb{P}^1)^{\times 3}} = \langle\, \rangle_{0,\tilde{h}_{12}-\tilde{e}_1}^{(\mathbb{P}^1)^{\times 3}(4)} = 1.$$

Now using the correspondence in Theorem 3, we know that

$$\langle\, \rangle_{0,\tilde{h}_{12}-\tilde{e}_1}^{(\mathbb{P}^1)^{\times 3}(4)} = \langle\, \rangle_{0,h-e_1-e_2}^{\mathbb{P}^3(6)}.$$

Again using Lemma 50, we write this as a stationary invariant, and see that

$$\langle p_1, p_2 \rangle_{0,h}^{\mathbb{P}^3} = 1.$$

The final inequality is evident from the fact that there is precisely one line through two points in $\mathbb{P}^3$.

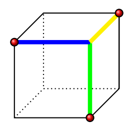


Figure 5.7 The curve of class $h_{12} + h_{13} + h_{23}$ through three given points.

5.4.2 The Rational Normal Curve in $\mathbb{P}^3$

Another classical result in $\mathbb{P}^3$ is that there exists precisely one cubic curve through six general points. This cubic is in fact the twisted cubic, and is an example of a rational normal curve. We will prove this result using the correspondence theorem.

Observe from the above result that there is precisely one curve of class $\tilde{h}_{12} + \tilde{h}_{13} + \tilde{h}_{23}$ through three given points. This can be combinatorially seen from the polytope of $(\mathbb{P}^1)^{\times 3}$.

In fact, this result can be derived by using the Cremona analogue symmetry on $(\mathbb{P}^1)^{\times 3}$. Observe that from Theorem 4, given the class $\tilde{h}_{12} - \tilde{e}3$, the symmetry gives us equality of the following invariants

$$\langle\,\rangle^{(\mathbb{P}^1)^{\times 3}}_{0,\tilde{h}_{12}-\tilde{e}_3} = 1 = \langle\,\rangle^{(\mathbb{P}^1)^{\times 3}}_{0,\tilde{h}_{12}+\tilde{h}_{13}+\tilde{h}_{23}-\tilde{e}_1-\tilde{e}_2-\tilde{e}_3}.$$

Using the correspondence in Theorem 3, we get

$$\langle\,\rangle^{(\mathbb{P}^1)^{\times 3}(4)}_{0,\tilde{h}_{12}+\tilde{h}_{13}+\tilde{h}_{23}-\tilde{e}_1-\tilde{e}_2-\tilde{e}_3} = \langle\,\rangle^{\mathbb{P}^3(6)}_{0,3h-e_1-\cdots-e_6}.$$

Rewriting this in terms of stationary invariants,

$$\langle pt^{\otimes 6}\rangle^{\mathbb{P}^3}_{0,3h} = 1.$$

This gives a Gromov–Witten theoretic proof of the existence of a unique cubic through six general points in $\mathbb{P}^3$. The "real cartoon" of this curve is depicted in Figure 5.8.

5.5 The Permutohedron in Higher Dimensions

In general the permutohedron can be constructed purely combinatorially. Consider the vector $(1, 2, \ldots, n)$. By permuting the coordinates in all possible ways, and taking the convex hull of the resulting collection of points, one arrives at the permutohedron. As we have discussed above however, the permutohedron also arises as the truncation of lattice polytopes corresponding to toric varieties. The constructions in the preceding sections extend to the higher dimensional cases naturally and are stated below.

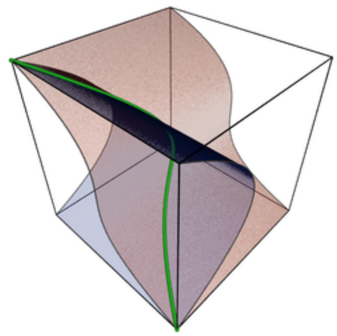


Figure 5.8 The unique cubic through six general points.

Theorem 54. *Let X be the sequential blowup of $\mathbb{P}^n$ at each of its torus fixed subvarieties up to dimension $n-2$. The polytope Δ_X of X is combinatorially equivalent to the permutohedron. The reflection through the origin toric symmetry corresponds to the resolved Cremona transform.*

A similar construction exists for $(\mathbb{P}^1)^{\times n}$.

Theorem 55. *Let $\tilde{Y}$ be the blowup of $(\mathbb{P}^1)^{\times n}$ at p_1 and p_2, the orbit closures of antipodal vertices of the polytope $\Delta_{(\mathbb{P}^1)^{\times n}}$. Let Y be the blowup of $\tilde{Y}$ at all torus fixed subvarieties containing p_1 and p_2. The polytope Δ_Y of Y is combinatorially equivalent to the permutohedron.*

This reproves a combinatorial result due to Carr and Devadoss (2006).

Corollary 56. *The permutohedron can be obtained via truncations of the cube.*

$\tilde{Y}$ above also has a nontrivial toric symmetry analogous to the map discussed previously. This map is given by the following lattice isomorphism of $\mathbb{Z}^n$:

$$\sigma = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 \end{pmatrix}.$$

Chapter 6

Nef Divisors and Descent via Blowup

We now turn our attention to the proofs of the main results of this thesis. The results are proved through a study of divisors on $\mathbb{P}^3$, the permutohedral variety and $(\mathbb{P}^1)^{\times 3}$. These divisors allow us to investigate intersection properties of the images of stable maps. These properties have implications to the moduli space and its virtual class, and force the moduli spaces of the base space and its blowup to be isomorphic for nonexceptional invariants. We will introduce the notions of effective curves and nef divisors, and review some of their basic properties. We will then formulate our problem for the descent of toric symmetry. First we review the proof by Bryan–Karp of the descent of the Cremona symmetry in $\mathbb{P}^3$, and then via methods of birational geometry, provide a proof of descent of the Cremona symmetry in the case of $(\mathbb{P}^1)^{\times 3}$ and thus proofs of our main theorems.

6.1 Numerically Effective Divisors and Effective Curves Classes

Simply speaking, a divisor is an object of codimension 1. In more detail, let X be a smooth projective variety. Then a divisor D is given by a formal sum of hypersurfaces

$$D = \sum_{V_i} a_i V_i,$$

where a_i is zero for almost all i, and the sum is taken over all hypersurfaces V_i. D is thus an element of the free abelian group generated by the divisors.

Note that in the general case, what we are describing are Weil divisors, as opposed to their counterparts, Cartier divisors, but for smooth projective varieties, the notions coincide. Since each hypersurface V_i defines an element of the Chow ring $A^1(X)$, D also defines a class in the Chow ring, $[D]$. A divisor is said to be effective if $a_i \geq 0$ for all i.

A divisor D is said to be *numerically effective*, or *nef*, if for any curve $C \subset X$, the product $D \cdot C \geq 0$. The product here is interpreted as the intersection product in the Chow ring.

Given a class $\beta \in A^{n-1}(X)$, β is called effective if there exists a dimension 1 subvariety $C \subset X$ whose class is β. For instance, in $\mathbb{P}^3$, there exists no curve of class dh for $d < 0$. Effective curve classes are not in general well behaved under blowup. For instance, let $X = \mathbb{P}^3(p_{123}, p_{134})$ be the blowup of $\mathbb{P}^3$ at the two specified points. Let ℓ_{13} be the proper transform of the line between p_{123} and p_{134}. Now let $Y = X(\ell_{13})$. The class $h - e_{123} - e_{134}$ is clearly effective in X, but its pullback is not effective in Y, since there is no longer a curve of that class—the single curve of that class has been blown up.

6.2 Effective Curve Classes and nef Divisors on X_{Π_3}

On $\mathbb{P}^3$, the first Chern class of the hyperplane bundle, which we have previously referred to as H, is a numerically effective divisor. Similarly, the classes $\tilde{H}_i$ are all nef on $(\mathbb{P}^1)^{\times 3}$. However, when we blowup subvarieties, the inserted divisors are not nef. In particular, on $\mathbb{P}^3(1)$, if E is the class of the exceptional divisor above the blowup, then notice that

$$E \cdot e = -pt.$$

Since every line in the exceptional divisor above the blowup has class e, e is an effective class. Thus E is not nef. It is also interesting to note that $-E$ is also not nef, since by blowing up at another point p' in addition to p,

$$-E \cdot (h - e - e') = -1,$$

and clearly the proper transform of the line through p and p' has class $h - e - e'$. For the proof of descent we will need an understanding of the nef divisors on X_{Π_3}. In particular, let $\{j, k\}$ be a two element subset of $\{1, 2, 3, 4\}$. With the notation from Chapter 5, we let

$$\hat{D}_{jk} = 2\hat{H} - (\hat{E}_1 + \cdots + \hat{E}_6) - F_{jk} - F_{j'k'}.$$

Above, $\{j,k,j',k'\} = \{1,2,3,4\}$. We now state a result due to Bryan and Karp (2005) and present their proof of this result.

Lemma 57. *D_{jk} is numerically effective.*

Proof. We proceed as in Bryan and Karp (2005). Let $\hat{D}'$ and $\hat{D}''$ be the proper transforms of the planes through the points $\{p_j, p_j, p_5\}$ and $\{p_{j'}, p_{k'}, p_6\}$. Then

$$\begin{aligned} \hat{D}' &= \hat{H} - \hat{E}_j - \hat{E}_k - \hat{E}_5 - \hat{F}_{jk} \\ \hat{D}'' &= \hat{H} - \hat{E}_{j'} - \hat{E}_{k'} - \hat{E}_6 - \hat{F}_{j'k'}. \end{aligned}$$

To see that $\hat{D}_{jk}$ is nef, it suffices to check that $\hat{D}_{jk} \cdot C \geq 0$ for any $C \subset \hat{D}'$. Notice that $\hat{D}'$ is isomorphic to $\mathbb{P}^2(3)$, and the classes have the following identification under the standard basis for the Chow ring of $\mathbb{P}^2(3)$.

$$h' = \hat{h} - \hat{f}_{jk}, \;\; e'_j = \hat{e}_j - f_{jk}, \;\; e'_k = \hat{e}_k - f_{jk}, \;\; e'_5 = e_5.$$

We know that effective curves in $\mathbb{P}^2(3)$ have the form

$$\beta = dh' - a_j e'_j - a_k e'_k - a_5 e'_5,$$

where d and a_i are all positive. Note that since $h' - e'_5$ is a nef divisor in $\hat{D}'$, we must have that

$$d \geq a_5.$$

The first Chern class of the normal bundle of D' in X_{Π_3} is

$$(\hat{H} - \hat{E}_j - \hat{E}_k - \hat{E}_5 - \hat{F}_{jk})^2 = \hat{e}_5 = e'_5.$$

Using this fact, we see that

$$D_{jk} \cdot C = -a_5 + d \geq 0,$$

and thus D_{jk} is numerically effective. □

6.3 Proof of Main Results

Let $\tilde{\pi} : \hat{X} = X_{\Pi_3}(2) \to \tilde{X} = (\mathbb{P}^1)^{\times 3}(4)$ as discussed previously. That is, we blowup two additional points that are not fixed by the torus action, but otherwise follow the constructions of Section 5.1.2 We will now prove the descent of nonexceptional invariants for (virtual dimension zero) classes

of the form $\hat{\beta} = \sum_{1\leq i<j\leq 3} \hat{d}_{ij}\hat{\hat{h}}_{ij} - \sum_{i=1}^{4} \hat{a}_i\hat{\hat{e}}_i$ on $\hat{X}$ via $\tilde{\pi}_\star$. We require that $\{\hat{a}_3, \hat{a}_4\} \neq \{0\}$. We will argue that any stable map in the isomorphism class $[\hat{f}] \in \overline{M}_g(\hat{X}, \hat{\beta})$ has an image disjoint from $F = \cup\hat{\hat{F}}_{jk}$ where the union is taken over all the exceptional divisors above line blowups. We will similarly show that any stable map $[f] \in \overline{M}_g(\tilde{X}, \beta)$ has an image disjoint from $\ell = \cup\ell_{jk}$. It then follows that the map on moduli stacks induced by $\tilde{\pi}$ is an isomorphism. Note that by abuse of notation we will use capital letters to denote both subvarieties and their classes.

Let $[f : C \to \tilde{X}] \in \overline{M}_g(X, \beta)$. Suppose that $f_\star C \cap \ell_{rs} \neq \emptyset$ where ℓ_{rs} is one of the six lines in the locus described in Section 5.1.2. Without loss of generality, since $\tilde{a}_3 \neq 0$, $\mathrm{Im}(f) \not\subseteq \ell_{rs}$. As a result we may write the class of the image as

$$f_\star C = C' + b\ell_{rs}, \ \ (b \geq 0).$$

Here C' meets ℓ_{rs} at finitely many points for topological reasons. Let $\hat{C}'$ be the proper transform via $\tilde{\pi}$ of C'. Since $C' \cap \ell_{rs} \neq \emptyset$, $\hat{C}' \cdot \hat{F}_{rs} = m > 0$. Thus, we may write

$$\hat{C}' = \hat{\beta} - b(\hat{\hat{h}}_{ij} - \hat{\hat{e}}_\alpha) - m\hat{\hat{f}}_{rs}.$$

Here $\alpha \in \{1, 2\}$, or in other words, e_α is the exceptional lines above one of the torus fixed points, and $\{i, j\}$ is such that $[\ell_{rs}] = \tilde{h}_{ij}$. Now push forward this class $\hat{C}'$ via the inverse of the map σ described in Section 5.1.3. Observe then that we get a curve in X_{Π_3}, whose class can be written as

$$\sigma_!\hat{C}' = \sigma_\star^{-1}\hat{C}' = d\hat{h} - \sum_{i=1}^{6} a_i\hat{e}_i - b(\hat{h} - \hat{e}_\gamma - \hat{e}_\delta) - m\hat{f}_{pq},$$

where $\{\gamma, \delta\} \subset \{1, 2, 3, 4\}$. In particular, using the map in Section 5.1.3, we see that $dh - \sum_{i=1}^{6} a_i e_i$ must have virtual dimension zero since $\tilde{\beta}$ and $\hat{\beta}$ have virtual dimension zero, in other words, $2d = \sum_{i=1}^{6} a_i$. Further, $\sigma_\star\hat{f}_{pq} = \hat{\hat{f}}_{rs}$. Now consider the divisor

$$\hat{D}_{pq} - 2\hat{H} - (\hat{E}_1 + \cdots + \hat{E}_6) - \hat{F}_{pq} - \hat{F}_{p'q'},$$

where $\{p, q, p', q'\} = \{1, 2, 3, 4\}$. From Bryan–Karp (2005) we know that this divisor is nef. However, clearly $\hat{D}_{pq} \cdot \sigma_\star^{-1}\hat{C} = mF_{pq} \cdot f_{pq} = -m < 0$, which is a contradiction. Thus, $f_\star C \cap \ell_{rs} = \emptyset$.

We argue in similar vein for the $\overline{M}_g(\hat{X}, \hat{\beta})$. Let $[\hat{f} : C \to \hat{X}]$. Suppose $\mathrm{Im}(\hat{f}) \cap \hat{\hat{F}}_{rs} \neq \emptyset$. Since $\hat{\beta} \cdot \hat{\hat{F}}_{rs} = 0$, $f_\star C$ must have a component C'' completely contained in $\hat{\hat{F}}_{rs}$, where we have

$$f_\star C = C' + C'',$$

where C' is nonempty since $\hat{\beta} \cdot \hat{\hat{E}}_4 \neq 0$. Since $C'' \subset \hat{\hat{F}}_{rs}$ is an effective class in $\hat{\hat{F}}_{rs} \cong \mathbb{P}^1 \times \mathbb{P}^1$, it must be of the form $C'' = a\hat{f}_{rs} + b\hat{s}_{rs}$ for $a, b \geq 0$, and $a + b > 0$. Writing $\hat{D}_{pq}$ in the basis induced by σ^{-1} and intersecting, we see that $\hat{D}_{pq} \cdot C' = -a - b$ contradicting the fact that D_{pq} is nef. Thus, $\mathrm{Im}(\hat{f}) \cap \hat{\hat{F}}_{rs} = \emptyset$, and the result follows.

Chapter 7

Taxonomy of Blowups of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

The approach to studying toric symmetry discussed in Section 3.2 can be adapted, in theory, to any complete toric variety. However, from a viewpoint of complexity, the algorithm described previously is not sufficiently fast to compute all the toric symmetries for many spaces. The naive approach to compute blowups used in the case of $\mathbb{CP}^3$ would have to compute the fans of 333,327,704,320 different blowups and perform an analysis of $GL(\mathbb{Z}^3)$ on each of those fans to find all nontrivial toric symmetries. This is in contrast to 31,312 spaces for $\mathbb{P}^3$. Even with considerable improvements made to the algorithm used in $\mathbb{P}^3$ by the author and colleagues (Karp et al., 2011) in this thesis, this approach to the problem is computationally not tractable. However, considering a fixed ordering of the 12 lines in $(\mathbb{P}^1)^{\times 3}$, blowups of this space have been completely studied where the blowup locus contains fewer than seven torus fixed lines. This chapter contains the main results of a taxonomical study, under a fixed ordering of the T-fixed lines, of the manifestations of nontrivial toric symmetry in blowups of $(\mathbb{P}^1)^{\times 3}$.

7.1 Results of Taxonomical Study

There are eighteen blowups discovered in the analysis described above that admit nontrivial toric symmetry. The following are a subset of this collection, where the symmetries and their action on the Chow ring are described. They are enumerated as in Section 7.3. Note that one of these

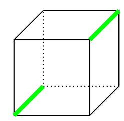


Figure 7.1 The toric blowup locus for Space 1.

eighteen spaces is the blowup of $(\mathbb{P}^1)^{\times 3}$ described in Chapter 5, the permutohedral variety.

Space 1 Consider the space constructed by the following blowup.

$$X_1 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1(\ell_{13}, \ell_{24}).$$

This may be viewed as $(\mathbb{P}^1 \times \mathbb{P}^1(p_{13}, l_{24})) \times \mathbb{P}^1$. The map $(183)(274)(56)$ on the one-skeleton induces a nontrivial action on the curve classes. Note that $A_1(X_1) = \mathbb{Z}[h_{ij}, f_1, f_2]$. The map on $A_1(X_1)$ is then given by

$$\begin{array}{rcl} h_{12} & \mapsto & h_{12} \\ h_{23} & \mapsto & h_{23} \\ h_{13} & \mapsto & h_{13} + h_{23} - f_1 - f_2 \\ f_i & \mapsto & h_{23} - f_i. \end{array}$$

Observe that two lines classes are fixed, and the third is mapped to a class of tridegree $(0, 1, 1)$. This will be a theme we will see throughout this analysis.

Space 2 We now consider the space

$$X_2 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1(\ell_{13}, \ell_{35}, \ell_{24}).$$

Notice that we have added ℓ_{35} to the locus in X_1. Now, the map $(19)(27)$ induces a nontrivial action on curve classes. In this case

$$\begin{array}{rcl} h_{12} & \mapsto & h_{12} \\ h_{13} & \mapsto & h_{13} \\ h_{23} & \mapsto & h_{13} + h_{23} - f_1 - f_2 \\ f_1 & \mapsto & h_{13} - f_3 \\ f_2 & \mapsto & f_2 \\ f_3 & \mapsto & h_{13} - f_1. \end{array}$$

Note that at the level of curve classes, this map is very similar to that of X_1.

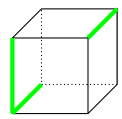


Figure 7.2 The toric blowup locus for Space 2.

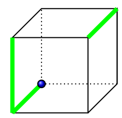


Figure 7.3 The toric blowup locus for Space 3.

Space 3 Now consider the space constructed via the following blowup:

$$X_3 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1(p_{136}, \ell_{13}, \ell_{35}, \ell_{24}).$$

With the usual basis for the Chow ring, we have the map

$$\begin{array}{rcl} h_{12} & \mapsto & h_{12} \\ h_{23} & \mapsto & h_{23} \\ h_{13} & \mapsto & h_{13} + h_{23} - f_1 - f_3 \\ e & \mapsto & h_{23} + f_1 + f_2 \\ f_1 & \mapsto & h_{23} - f_1 \\ f_2 & \mapsto & e - f_2 \\ f_3 & \mapsto & h_{23} - f_3. \end{array}$$

Space 11 The space constructed by the following blowup has the property that intersecting lines are only blown up after the point in their intersection. As a result the order of blowup is irrelevant for this space, as is the case for the permutohedral spaces. The space below exhibits a similar symmetry to the Cremona symmetry on X_{Π_3} discussed previously.

$$X_{11} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1(p_{135}, p_{246}, \ell_{13}, \ell_{35}, \ell_{15}, \ell_{26}, \ell_{24}).$$

The map $(18)(27)(9,12)(11,13)$ yields a nontrivial map on curve classes given as follows. This map can be shown to descend nontrivially to

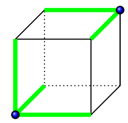


Figure 7.4 The toric blowup locus for Space 11.

$(\mathbb{P}^1)^{\times 3}$, and is equal to the descent of the map discussed in Chapter 5.

$$\begin{array}{rcl} h_{12} & \mapsto & h_{12} \\ h_{23} & \mapsto & h_{23} \\ h_{13} & \mapsto & h_{13} + h_{23} + h_{12} - e_1 - e_2 \\ e_1 & \mapsto & h_{13} + h_{12} - e_2 \\ e_2 & \mapsto & h_{13} + h_{12} - e_1 \\ f_1 & \mapsto & s_4 = h_{13} - e_2 + f_4 \\ f_2 & \mapsto & f_2 \\ f_3 & \mapsto & s_5 = h_{12} - e_2 + f_5 \\ f_4 & \mapsto & s_1 = h_{12} - e_1 + f_1 \\ f_5 & \mapsto & s_3 = h_{13} - e_1 + f_3. \end{array}$$

7.2 Ascent of Toric Symmetry from $(\mathbb{P}^1)^{\times 2}$ to $(\mathbb{P}^1)^{\times 3}$

The symmetry described above on X_1 is induced by the lattice automorphism

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Notice that this matrix fixes the z-axis. By composing σ with the projection

$$\begin{array}{rcl} \pi : \mathbb{Z}^3 & \to & \mathbb{Z}^2 \\ (x, y, z) & \mapsto & (x, y), \end{array}$$

we get an automorphism $\tau : \mathbb{Z}^2 \to \mathbb{Z}^2$ of the polytope of $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at two torus fixed points, shown in Figure 7.5. This polytope $\tilde{\Delta}$ is the two-dimensional permutohedron. As a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$. Let h_1 and h_2 be the line classes in $X_{\tilde{\Delta}}$ and let e_1 and e_2 be the classes of lines in the exceptional

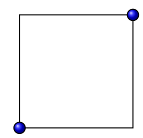


Figure 7.5 The polytope of $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at two torus fixed points.

divisors above the points. The action of $\tau_\star$ on $A_\star(X_{\tilde{\Delta}})$ is given as follows:

$$\begin{array}{rcl} h_i & \mapsto & h_i \\ e_1 & \mapsto & h_1 - e_1 \\ e_2 & \mapsto & h_2 - e_2. \end{array}$$

Compare this to the classes h_{ij} and f_i in X_1 above. This nontrivial toric symmetry on $\mathbb{P}^1 \times \mathbb{P}^1(2)$ is thus being lifted to $\sigma_\star$ on $A_\star(X_1)$.

In the analysis of toric symmetries of blowups of $\mathbb{P}^3$ and $(\mathbb{P}^1)^{\times 3}$, we observe that a necessary condition for a nontrivial toric symmetry of $\hat{X}$ a blowup at points and lines to descend to X, the blowup at just points, is that lines are separated. That is, if ℓ_1 and ℓ_2 are part of the blowup locus and $\ell_2 \cap \ell_2 = p$, then p also belongs to the blowup locus. This is the case for the permutohedral blowups, and X_{11} described above. In fact, this is a necessary condition for the classes pulled back from X to form an orbit under the toric symmetry. An interesting future direction concerning these results would be to study ascent and descent of the various blowup spaces described above. That is, if the toric symmetries are descending from a further blowup of these spaces, or are being lifted.

7.3 Collection of Nontrivial Toric Symmetries

The following is the output of the SAGE program that was written to perform the analysis of toric symmetry in $(\mathbb{P}^1)^{\times 3}$.

```
Note: The following is SAGE output.
All matrices are over GF(3) = {0,1,2}.
We identify 2 with -1.

Space 1 : ['L13', 'L24']
There are 16 interesting automorphisms of this fan.

1 (interesting!):
```

```
(1,8,3)(2,7,4)(5,6)
[2 1 0]
[2 0 0]
[0 0 2]

Space 2 : ['L13', 'L35', 'L24']
There are 1 interesting automorphisms of this fan.

1 (interesting!):
(1,9)(2,7)
[2 0 0]
[2 1 0]
[0 0 1]

Space 2

['L13', 'L24', 'L35']
There are 1 interesting automorphisms of this fan.

1 (interesting!):
(1,8)(2,7)
[2 0 0]
[2 1 0]
[0 0 1]


Space 3 : ['p136', 'L13', 'L35', 'L24']
There are 1 interesting automorphisms of this fan.

1 (interesting!):
(1,2)(3,8)(4,10)(5,6)(7,9)
[2 1 0]
[0 1 0]
[0 0 2]

Space 4 : ['p245', 'L13', 'L35', 'L24']
There are 1 interesting automorphisms of this fan.

1 (interesting!):
(3,10)(4,8)(7,9)
```

```
[1 2 0]
[0 2 0]
[0 0 1]

Space 5 : ['p246', 'L13', 'L35', 'L24']
There are 1 interesting automorphisms of this fan.

1 (interesting!):
(3,10)(4,8)(5,6)(7,9)
[1 2 0]
[0 2 0]
[0 0 2]

Space 6 : ['p245', 'p246', 'L13', 'L15', 'L16', 'L45', 'L24']
There are 1 interesting automorphisms of this fan.

1 (interesting!):
(1,13)(2,9)(7,10)(8,11)
[2 0 0]
[2 1 0]
[0 0 1]

Space 7 : ['L13', 'L15', 'L16', 'L24', 'L26']
There are 1 interesting automorphisms of this fan.

1 (interesting!):
(3,10)(4,7)
[1 2 0]
[0 2 0]
[0 0 1]

Space 8 : ['p245', 'L13', 'L15', 'L16', 'L24', 'L36']
There are 1 interesting automorphisms of this fan.

1 (interesting!):
(3,11)(4,8)(5,6)(7,12)(9,10)
[1 2 0]
[0 2 0]
[0 0 2]
```

```
Space 9 : ['p246', 'L13', 'L15', 'L16', 'L24', 'L36']
There are 1 interesting automorphisms of this fan.

1 (interesting!):
(3,11)(4,8)(7,12)
[1 2 0]
[0 2 0]
[0 0 1]

Space 10 : ['p245', 'p246', 'L13', 'L15', 'L16', 'L24', 'L36']
There are 1 interesting automorphisms of this fan.

1 (interesting!):
(1,12)(2,9)(7,10)(8,11)
[2 0 0]
[2 1 0]
[0 0 1]

Space 11 : ['p135', 'p246', 'L13', 'L35', 'L15', 'L26', 'L24']
There are 2 interesting automorphisms of this fan.
Lines are separated!

1 (interesting!):
(1,8)(2,7)(9,12)(11,13)
[2 0 0]
[2 1 0]
[2 0 1]

2 (interesting!):
(1,8)(2,7)(3,5)(4,6)(9,13)(11,12)
[2 0 0]
[2 0 1]
[2 1 0]

Space 12 : ['p245', 'L13', 'L15', 'L24', 'L26', 'L35', 'L16']
There are 1 interesting automorphisms of this fan.

1 (interesting!):
(3,10)(4,8)(7,12)
[1 2 0]
```

```
[0 2 0]
[0 0 1]

Space 13 : ['p245', 'L13', 'L15', 'L24', 'L26', 'L35', 'L25']
There are 1 interesting automorphisms of this fan.

1 (interesting!):
(3,10)(4,8)(7,12)
[1 2 0]
[0 2 0]
[0 0 1]

Space 14 : ['p136', 'p245', 'L13', 'L15', 'L24', 'L26', 'L35', 'L45']
There are 1 interesting automorphisms of this fan.

1 (interesting!):
(1,11)(2,9)(7,12)(8,10)
[2 0 0]
[2 1 0]
[0 0 1]

Space 15 : ['p135', 'p246', 'L13', 'L15', 'L24', 'L26', 'L35', 'L46']
There are 36 interesting automorphisms of this fan.
Isomorphic to Permutohedron

1 (interesting!):
(1,7,3,6)(2,8,4,5)(10,13,12,14)
[1 0 2]
[1 0 0]
[1 2 0]

2 (interesting!):
(1,6)(2,5)(3,7)(4,8)(10,12)
[0 1 2]
[0 1 0]
[2 1 0]

Space 16 : ['p135', 'p246', 'L13', 'L15', 'L24', 'L26', 'L35', 'L46',
 'L14']
There are 2 interesting automorphisms of this fan.
```

```
1 (interesting!):
(5,8)(6,7)(10,14)(12,13)
[1 0 2]
[0 1 2]
[0 0 2]

2 (interesting!):
(1,4)(2,3)(5,7)(6,8)(9,11)
[0 2 1]
[2 0 1]
[0 0 1]

Space 17 : ['p135', 'p246', 'L13', 'L15', 'L24', 'L26', 'L35',
 'L46', 'L14', 'L23']
There are 4 interesting automorphisms of this fan.
There are 8 automorphisms of this fan:

1 (interesting!):
(5,8)(6,7)(10,14)(12,13)
[1 0 2]
[0 1 2]
[0 0 2]

3 (interesting!):
(1,3)(2,4)(5,8)(6,7)(10,12)(13,14)(15,16)
[0 1 2]
[1 0 2]
[0 0 2]

5 (interesting!):
(1,4)(2,3)(5,7)(6,8)(9,11)
[0 2 1]
[2 0 1]
[0 0 1]

6 (interesting!):
(1,2)(3,4)(5,7)(6,8)(9,11)(10,13)(12,14)(15,16)
[2 0 1]
[0 2 1]
```

```
[0 0 1]

Space 18 : ['L13', 'L24', 'L15', 'L16']
There are 2 interesting automorphisms of this fan.

2 (interesting!):
(3,8)(4,7)
[1 2 0]
[0 2 0]
[0 0 1]

3 (interesting!):
(3,8)(4,7)(5,6)(9,10)
[1 2 0]
[0 2 0]
[0 0 2]
```

Chapter 8

Future Work

The main results of this paper lend themselves to various extensions and generalisations which we now explain.

8.1 Cremona Higher Dimensions

It is believed, though not proved, that the Cremona symmetry on virtual dimension zero invariants extends analogously to $\mathbb{P}^n$. Blowing up $\mathbb{P}^n$ at all the torus fixed subvarieties up to dimension $n-2$, we get a variety whose dual polytope is the permutohedron. Reflection through the origin of this polytope is an involution that resolves the Cremona transform on $\mathbb{P}^n$. This map is a nontrivial toric symmetry on X_{Π_n}, though descent is unknown.

It can also be shown that the blowup of $(\mathbb{P}^1)^{\times n}$ at the points corresponding to antipodal vertices on the n-cube, and in increasing dimension up to $n-2$, all subvarieties intersecting these points, we get a variety whose dual polytope is also the permutohedron. This gives us a general birational map between these two spaces. The analogue of Cremona, that gives rise to the symmetry of invariants on $(\mathbb{P}^1)^{\times 3}$ can be generalised to n-dimensions as well. Proof of descent through these blowup maps would prove analogous results to the main results of this paper, in n-dimensions.

The main difficulties are twofold. First, the standard technique, degeneration and deformation to the normal cone, discussed briefly in Appendix A, is much less tractable in higher dimensions. In fact, even in low dimensions, high genus degeneration computations have many difficulties, though mostly technical ones. Secondly, the use of nef divisors, the technique used to prove the main results in this paper requires the understanding of the intersection theory of the maximal blowup of $\mathbb{P}^n$. Though

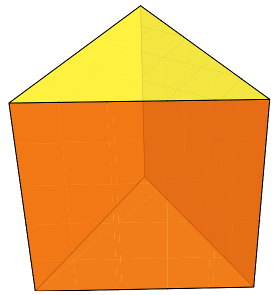


Figure 8.1 The polytope of $\mathbb{P}^2 \times \mathbb{P}^1$.

in theory this is easily described using toric techniques, choosing the right basis to describe the intersection theory is crucial in extending the argument.

It should be noted that both these difficulties are technical ones and can likely be overcome with an understanding of higher dimensional intersection theory and choosing the right basis for the Chow ring of the blowups.

8.2 Higher Virtual Dimension

The equality of the Gromov–Witten invariants of Calabi–Yau classes on $\mathbb{P}^3(6)$ and $(\mathbb{P}^1)^{\times 3}(4)$, and the corresponding equality of stationary invariants of $\mathbb{P}^3$ and $(\mathbb{P}^1)^{\times 3}$, can both possibly be extended to non–Calabi–Yau classes. However the technique of using nef divisors to ensure that the corresponding moduli spaces are isomorphic with isomorphic virtual classes does not extend naturally. The standard technique for this extension is degeneration and relative invariants.

8.3 Toric Symmetry in Other Spaces

Another first step in extending these results would be to study the Gromov–Witten theory of the space $\mathbb{P}^2 \times \mathbb{P}^1$. This is a toric variety whose polytope is shown in Figure 8.1.

Blowups of this space have known nontrivial toric symmetries ascending from the permutohedral blowup of $\mathbb{P}^2$, by blowing up three torus fixed lines of a given parallel class. However, the toric symmetries of blowups of this space at points are unknown. Further, $\mathbb{P}^2 \times \mathbb{P}^1$ is birational to both $\mathbb{P}^3$ and $(\mathbb{P}^1)^{\times 3}$. It would be of interest to choose a birational map between

these spaces and understand the behaviour of the Gromov–Witten invariants of the threefold projective spaces.

Appendix A

The Degeneration Formula

The main results of this thesis relate the Gromov–Witten theory of a blowup to that of the base space. The standard technique for such problems is to use the *degeneration formula* for relative Gromov–Witten invariants, due to Jun Li, to derive the conditions under which the invariants are unchanged under blowup. This appendix describes the essentials of this technique. An extension of the results of this thesis to the case of invariants with insertions could require the use of this technique. Note that we do not go deeply into the background of relative invariants here, and record the technique here only for its relevance as a technique in analysing the Gromov–Witten theories of blowups. For the fundamentals of relative invariants see the groundbreaking papers by Li (2001, 2002).

A.1 The Idea of Relative Invariants

Given X, a nonsingular projective threefold, and a nonsingular divisor $S \subset X$, one can define Gromov–Witten invariants of X *relative to* S. Given a curve class $\beta \in A_1(X)$ satisfying $S \cdot \beta \geq 0$, the idea is to use the intersection points of the curve with the divisor to have relative insertions. These insertions are cohomology classes in the divisor S. By allowing stable maps to have possibly disconnected domains, one can study a moduli space of maps from X relative to S of a curve class β. In this case the target X is allowed to be a degeneration of X along the divisor S. These relative moduli spaces admit a virtual fundamental class, and invariants are defined as usual by integration against this virtual class.

A.2 The Idea of Degeneration

We want to study the absolute invariants of a blowup of a subvariety $Z \hookrightarrow X$ using the technique of relative invariants. In the blowup space $\pi : \hat{X} \to X$, let F be the exceptional divisor above Z. To compare the invariants of X with those of $\hat{X}$ we use degeneration, and in our case, deformation to the normal cone. To do this we consider the variety $X \times \mathbb{C}$, and the blowup of $Z \times \{0\} \hookrightarrow X \times \mathbb{C}$. Note that $\mathbb{C}$ could be replaced by any nonsingular algebraic curve.

To understand what this blowup of $Z \times \{0\} \hookrightarrow X \times \mathbb{C}$ we can use the technique of deformation to the normal cone. This tells us that the section above a complex number t (in the second factor) is X for all $t \neq 0$. At $t = 0$, we have a normal crossing divisor consisting of two pieces intersecting at F. The first piece is $\hat{X}$, while the second piece is given by

$$P = \mathbb{P}_F(N_{Z/\hat{X}} \oplus \mathcal{O}),$$

the total space of the projective completion of the normal bundle of Z in $\hat{X}$.

A.3 The Formula

With the setup above, the degeneration formula can be expressed in terms of the Gromov–Witten generating functions as follows. Let $\lambda : \chi \to C$ be a nonsingular fourfold fibered over a nonsingular irreducible curve C (for our purposes we can let $C = \mathbb{P}^1$ or $\mathbb{C}$.) Let X be a nonsingular fiber of λ and $X_1 \cup_S X_2$ be a reducible special fiber, consisting of two nonsingular threefolds intersecting transversely along the the nonsingular surface (the relative divisor) S. The degeneration formula for the absolute invariants of X in terms of the relative invariants of X_1/S and X_2/S, is given as follows:

$$\begin{aligned} &Z'(X \mid \prod_{i=1}^{r} \tau_0(\gamma_{l_i})_\beta \\ &\qquad = \sum Z'(\frac{X_1}{S} | \prod_{i \in P_1} \tau_0(\gamma_{l_i}))_{\beta_1,\eta} \xi(\eta) u^{2l(\eta)} Z'(\frac{X_2}{S} | \prod_{i \in P_2} \tau_0(\gamma_{l_i})_{\beta_2,\eta^V}. \end{aligned}$$

In the above formula, η is a weighted cohomology partition, and $\xi(\eta) = \prod_i \eta_i |Aut(\eta_i)$. Z' is the Gromov–Witten partition function. The degeneration formula can also be expressed at the level of invariants, as is done below. Geometrically, it expresses the invariants of the ordinary fiber in terms

of a convolution of the invariants of the two pieces of the reducible special fiber, namely $\hat{X}$ and P. For simplicity we express the formula without any absolute insertions, although it is easily generalised.

$$\langle\ \rangle^X_{g,\beta} = \sum_{\hat{\beta}_1+\hat{\beta}_2=\hat{\beta}} \sum_{\varphi_i \in H_\star(F)} \langle\ |\varphi_i\rangle^{(\hat{X}/F)}_{g,\hat{\beta}_1} \langle\ |\varphi^i\rangle^{(\hat{P}/F)}_{g,\hat{\beta}_2}.$$

where φ_i and φ^i are a dual basis for $H_\star(F)$. When descent of invariants is expected, the large numbers of the summands have at least one invariant equal to zero for degree reasons. An easy way to identify some of these terms is via the degree axiom for Gromov–Witten invariants. This is stated below for primary fields only.

Axiom 58 (Degree Axiom)**.** *This axiom states that for homogenous classes γ_i, the GW invariant*

$$\langle \gamma_1, \dots, \gamma_n \rangle_{g,\beta}$$

is nonzero only if

$$\sum_{i=1}^{n} deg\gamma_i = 2((dimX - 3)(1-g) - K_X \cdot \beta + n).$$

Here degree refers to the cohomological degree (that is the real degree). This says that the algebraic (in Chow) degree of all the insertions has to match the virtual dimension of the moduli space $\overline{M}_{g,n}(X, \beta)$.

A.4 The Intersection Theory of $\mathbb{P}_F(N_{Z/\hat{X}} \oplus \mathcal{O})$

One obstacle to using degeneration to analyse the permutohedral blowup is understanding the intersection theory of P, the projective completion of the normal bundle of the exceptional locus, as described in the previous section. We will now describe the intersection theory on this bundle.

The intersection theory is computed using the Chow ring package for SAGE. The intersection ring on P is abstractly $\mathbb{Z} \oplus \mathbb{Z}^3 \oplus \mathbb{Z}^3 \oplus \mathbb{Z}$. That is, the ring is torsion free, and divisors and curve classes are both rank 3. The divisor group is generated by three classes. As seen on the polytope in Figure A.1, these classes are the orbit closures of the top face, and either pair of four adjacent planes of the pyramid. Let us call these classes D_H, D_1, and D_2, respectively. The class D_H is the first Chern class of the relative

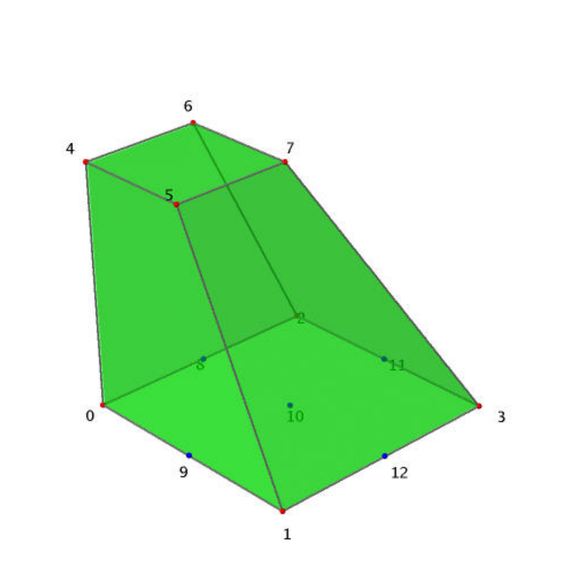


Figure A.1 The polytope of $P = \mathbb{P}_F(N_{F/\hat{X}} \oplus \mathcal{O})$.

$\mathcal{O}(1)$ on the total space of $\mathcal{O}(-1) \boxtimes \mathcal{O}(-1)$ over $\mathbb{P}^1 \times \mathbb{P}^1$. The canonical class of this space is largely what we are interested in and it is given by

$$-K_P = \sum_{i \text{ is a facet}} D_i = 2D_h + D_1 + D_2,$$

where the second equality comes from the Danilov relations. The curve class group, $A^1(P)$ is generated again by three classes, h, f, and s, which are the line classes on the base (bottom face). We are largely interested in the classes f and s as these classes must be identified with classes on the relative divisor F (which corresponds to the base of the polytope). The intersection pairings for the classes f and s are given by

$$\begin{aligned} D_H \cdot f &= D_2 \cdot f = 0, \ \ D_1 \cdot f = 1 \\ D_H \cdot s &= D_1 \cdot s = 0, \ \ D_2 \cdot s = 1. \end{aligned}$$

Using several methods (SAGE, Leray–Hirsh, and intersection theory from Σ_P), the pairing with the canonical class was computed to be

$$-K_P \cdot s = -K_P \cdot f = 1.$$

Finally, for a curve class $\alpha \in A_1(P)$ to have a nonzero relative GW invariant, it must pair nonnegatively with the class of the relative divisor. Hence, we

are interested in $\alpha \cdot F$. However, we find that

$$F \cdot s = F \cdot f = -1.$$

The remaining part of the analysis above is to derive the situations in which $\beta_2 \neq 0$ in the degeneration formula, the invariants of either β_1 or β_2 are zero, and thus deduce that the invariants of the blowup and its base space coincide. This technique was explored, but the method of proof using nef divisors was more lucrative for this problem. The method of nef divisors however cannot be used to analyse invariants with insertions and hence we must use degeneration to extend the results of this thesis.

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