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Nonlinear Wave Equations and Solitary Wave Solutions in Mathematical Physics

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Abstract

In this report, we study various nonlinear wave equations arising in mathematical physics and investigate the existence of solutions to these equations using variational methods. In particular, we look for particle-like traveling wave solutions known as solitary waves. This study is motivated by the prevalence of solitary waves in applications and the rich mathematical structure of the nonlinear wave equations from which they arise. We focus on a semilinear perturbation of Maxwell's equations and the nonlinear Klein–Gordon equation coupled with Maxwell's equations. Physical ramifications of these equations are also discussed.

Contents

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Chapter 1

Introduction

1.1 Nonlinear Wave Equations

Before discussing nonlinear wave equations, we first present the n-dimensional linear wave equation, which models the propogation of waves. It is among the most fundamental partial differential equations, as it describes the motion of various linear waves, such as light, water, and sound waves. In $\mathbb{R}^n$, we have the inhomogeneous wave equation

$$u_{tt}(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) = f(\mathbf{x}, t), \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$, $\Delta u = u_{xx} + u_{yy} + u_{zz}$, and $f(\mathbf{x}, t)$ is a continuously differentiable function. Note that when $f = 0$ and $n = 1$, then we obtain the one-dimensional homogeneous wave equation, which is the familiar model of a vibrating string. We can simplify notation by introducing the d'Alembert operator,

$$\Box = \frac{\partial^2}{\partial t^2} - \Delta, \tag{1.2}$$

yielding

$$\Box u(\mathbf{x}, t) = f(\mathbf{x}, t). \tag{1.3}$$

We will be considering semilinear wave equations, which add an extra nonlinear term g which depends on the function u but not on its derivatives. The general semilinear wave equation is then

$$\Box u(\mathbf{x}, t) + g(u) = f(\mathbf{x}, t). \tag{1.4}$$

1.2 Solitary Waves

Solitary waves are particle-like waves that arise from a balance between nonlinear and dispersive effects. A *soliton* is a solitary wave which maintains its shape when it moves at constant speed and conserves amplitude, shape, and velocity after a collision with another soliton. They have become increasingly popular due to their stability particle-like behavior. Solitons naturally arise in several areas of mathematical physics, such as in nonlinear optics, fluid mechanics, plasma physics, and quantum field theory. The classical example of an equation yielding solitary wave solutions is the Korteweg-de Vries equation, which is model of waves on shallow water surfaces:

$$u_t + u_{xxx} + 6uu_x = 0. \tag{1.5}$$

By assuming wave-like solutions traveling at a speed c, the equation can be simplified, yielding the solution

$$u(x,t) = \frac{1}{2}c\text{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct - a)\right), \tag{1.6}$$

where a is an arbitrary constant. This is a soliton which translates with constant velocity c.

Chapter 2

Background

2.1 Function Spaces

In this section, we review some of the relevant mathematics necessary for our investigations. Let Ω be a measurable subset of $\mathbb{R}^n$. For all intents and purposes, we can think of Ω as an open subset, and we will try to limit the amount of measure theory required. Now, we define several function spaces on Ω, each with their own special integrability, differentiability, and continuity constraints.

Definition 2.1. *$C^k(\Omega)$ denotes the space of k-times differentiable continuous functions $u : \Omega \to \mathbb{R}$.*

This space is endowed with the supremum norm:

$$||u|| = \sup_{x \in \Omega} |u(x)|. \tag{2.1}$$

Note that when $k = \infty$, this denotes the space of smooth functions defined on Ω. When we write C_0^k, the zero indicates that the functions have compact support. This means that $u = 0$ for all x in Ω, except on a compact subset $K \subset \Omega$. We also need an adequate space for integration, as we will often recast differential equations in terms of integral equations.

Definition 2.2. *$L^p(\Omega)$ is the space of measurable functions $f : \Omega \to \mathbb{R}$ such that*

$$\int_\Omega |f|^p dx < \infty. \tag{2.2}$$

It is actually a Banach space (a complete normed vector space), equipped with the norm

$$||f||_p = \left(\int_\Omega |f|^p \right)^{\frac{1}{p}}. \tag{2.3}$$

Here, complete means that every Cauchy sequence in L^p converges to an element of L^p. Note that when $p = 2$, there is an additional inner product structure. The inner product is given by

$$\langle f, g \rangle = \int_\Omega f(x)g(x)dx. \tag{2.4}$$

The space $L^2(\Omega)$ is an example of a Hilbert space; that is, a complete inner product space. The additional structure provided by the inner product makes this infinite-dimensional space easier to conceptualize and work with, as it generalizes the notion of the Euclidean space $\mathbb{R}^n$ to infinite dimensions and synthesizes ideas from linear algebra and analysis. We also need to impose conditions on derivatives to provide an appropriate functional setting for the partial differential equations we will study.

Definition 2.3. *The Sobolev space $W^{k,p}(\Omega)$ consists of all functions $u \in L^p(\Omega)$ such that each weak partial derivative of order $\leq k$ is also in L^p.*

Let us take a moment to discuss the notion of *weak* derivatives and solutions. Using multi-index notation for the partial derivatives of u, we say that a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ has order

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \tag{2.5}$$

and we employ the following notation for the partial derivative:

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \equiv \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u. \tag{2.6}$$

When $|alpha| = 1$, we will often use the more familiar symbol ∇ for the gradient operator; that is, $\nabla u = Du$ for some function u. Likewise, we use the traditional notation for the Laplacian, so that Δu denotes $\mathrm{div}(Du)$.

The idea behind weak derivatives is to extend the notion of differentiability to functions only assumed to be integrable.

Definition 2.4. *A weak α-th partial derivative of u is a locally integrable function v (i.e., $v \in L^1(K)$ for every compact set $K \subset \Omega$) which satisfies*

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega \phi v dx, \tag{2.7}$$

for all functions ϕ in $C_0^\infty(\Omega)$.

With the idea of weak derivatives in mind, we can also speak of weak solutions to differential equations. Essentially, the idea is to put a differential equation into integral form by multiplying the equation by a suitable test function ϕ (usually in C_0^∞) and integrating by parts. Functions which satisfy the resultant integral relation are deemed weak solutions. This concept allows for a wider variety of solutions, as weak solutions may not be sufficiently smooth to satisfy the original PDE. Thus, Sobolev spaces provide an excellent functional framework for investigating the existence of solutions. The norm associated with a Sobolev space is a natural extension of the L^p norm; for example, the space $W^{1,2}$ is endowed with the norm

$$||u||_{W^{1,2}} = \left(\int_\Omega |u|^2 + |\nabla u|^2 dx \right)^{\frac{1}{2}}. \tag{2.8}$$

When working with inner product spaces ($p = 2$), we make use of the shorthand notation H^k, which indicates that the Sobolev spaces are also Hilbert spaces. A final related function space is closely related to Sobolev spaces:

Definition 2.5. *$D^{1,2}(\Omega)$ denotes the completion of C_0^∞ with respect to the norm*

$$||u||_{D^{1,2}} = \left(\int_\Omega |\nabla u|^2 dx \right)^{1/2}. \tag{2.9}$$

This provides another norm to work with, which will be useful in our analysis.

2.2 Variational Principles

We will also examine the variational structure of systems of PDEs, so we briefly discuss concepts from the calculus of variations, in which one seeks minimizers, maximizers, saddle points, and other critical points of functionals. In a physical framework, ideas like Fermat's principle and the principle of least action correspond to finding extrema of these functionals. For this section, we use the notation for partial derivatives introduced in the previous section for convenience. First, we look at the one-dimensional (ODE) case. Consider functionals of the form

$$J(u) = \int_0^1 L(x, u, Du) dx, \tag{2.10}$$

where $u : [0,1] \to \mathbb{R}^m$ is a continuously differentiable function of one variable and D denotes the derivative with respect to x. Note that $J : X \to \mathbb{R}$ is defined on a Hilbert or Banach space so that we ensure the existence and integrability of derivatives. The domain of J is determined by the structure of the function L, which we call the *Lagrangian*. For example, with the Lagrangian

$$L = (Du)^2 + G(u), \tag{2.11}$$

we use the Hilbert space H^1, so that the integral exists. Let L_x denote the partial derivative of the Lagrangian L with respect to x, and similarly for L_u and L_{Du}. Note that if $u = (u_1, \ldots, u_m)$, then these partial derivatives denote the gradient; that is,

$$L_u = (L_{u_1}, \ldots, L_{u_m}), L_{Du} = (L_{Du_1}, \ldots, L_{Du_m}). \tag{2.12}$$

In order to find a critical point of the functional J, we find where all the directional derivatives vanish; that is,

$$\langle J'(u), v \rangle = \frac{d}{d\epsilon} J(u + \epsilon v)\Big|_{\epsilon=0} = 0, \tag{2.13}$$

for all smooth functions v with compact support. After integrating by parts and removing the integral, we eventually arrive at the *Euler–Lagrange equation*:

$$- DL_{Du} + L_u = 0. \tag{2.14}$$

If $u = u(u_1, \ldots, u_m)$, this becomes

$$- DL_{Du_i} + L_{u_1} = 0, i = 1, 2, \ldots, m. \tag{2.15}$$

This extends to multiple dimensions (PDEs). Suppose that

$$J(u) = \int_\Omega L(x, u, Du) dx, \tag{2.16}$$

where $\Omega \subset \mathbb{R}^n$, $u : \Omega \to \mathbb{R}$, and

$$Du = (D_1 u, D_2 u, \ldots, D_n u) \tag{2.17}$$

denotes the derivative of u, and each D_i is the partial derivative with respect to x_i. The corresponding Euler–Lagrange equation is

$$- \sum_{i=1}^{n} D_i L_{D_i u} + L_u = 0. \tag{2.18}$$

The calculus of variations is a powerful method for finding equations of motion in physics. As an example, let us show that we can derive Newton's second law using the principle of stationary action. This principle states that for fixed initial and final positions, the trajectory of a particle (i.e., the equation of motion) is a stationary point of the action, which is a functional whose integrand is the Lagrangian of the system. In a physical setting, the Lagrangian is the difference between kinetic and potential energy. Hence, for a particle at position $x(t)$ subject to a potential $V(x)$, the action is

$$S(x) = \int_a^b \left\{ \frac{1}{2} m\dot{x}^2 - V(x) \right\} dt. \tag{2.19}$$

We then differentiate to find a stationary point of the action

$$\frac{d}{d\epsilon} J(u + \epsilon v)\Big|_{\epsilon=0} = \int_a^b \{ m\dot{x} \cdot v - \nabla V(x) \cdot v \} \, dt. \tag{2.20}$$

Integrating by parts and using the fact that the boundary terms vanish, we obtain

$$\langle J'(u), v \rangle = -\int_a^b \{ m\ddot{x} + \nabla V(x) \} \cdot v dt = 0. \tag{2.21}$$

The integral vanishes for arbitrary $v(t)$, so we obtain

$$m\ddot{x} + \nabla V(x) = 0, \tag{2.22}$$

which is indeed Newton's second law.

2.3 Poincaré Group

One last area of interest before investigating some nonlinear wave equations is the concept of the Poincaré group. Many of the fundamental equations of physics are invariant under this group, including Maxwell's equations. Let us now state the definition of the Poincaré group and some of its basic properties. The Poincaré group is the group of isometries (distance-preserving maps) of Minkowski spacetime. In Euclidean space, isometries include reflections, translations, and rotations. Minkowski spacetime is similar to ordinary Euclidean space, except that it also includes a special timelike dimension, reflecting the geometry of special relativity. This is accomplished by substituting the standard inner product with the bilinear form

$$\langle u, v \rangle_M = -u_0 v_0 + \sum_{i=1}^{3} u_i v_i, \tag{2.23}$$

where u and v are vectors in $\mathbb{R}^4$. The Poincaré group is then the group of transformations which preserve the quadratic form (akin to a norm) induced by the bilinear form mentioned above (akin to an inner product).

Note that we can associate physical space with the three spacelike dimensions ($\mathbb{R}^3$), which gives the Poincaré group a nice substructure. In $\mathbb{R}^3$, we call the group of isometries the Euclidean group. This group is generated by space translations and rotations, and we can represent elements of the Euclidean group as

$$gx = Ox + v, \tag{2.24}$$

where O is a three-by-three orthogonal matrix (a rotation such that $O^T = O^{-1}$) and v is a vector in $\mathbb{R}^3$ (a translation). As an example of a rotation matrix, the following is a matrix R rotates a vector by a positive angle θ about the z axis:

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A pure rotation can be written as

$$r = Rr', \tag{2.25}$$

where r is an unprimed position vector in $\mathbb{R}^3$, r' is the corresponding primed vector, and R is a three-by-three rotation matrix satisfying the orthogonality condition as well as having its determinant equal to 1.

The transformations comprising the Poincaré group include space translations, space rotations, as well as time translations, Lorentz transformations, time inversions, and parity inversions (reflections). These transformations are all fairly straightforward, with the exception of Lorentz transformations.

A Lorentz transformation A satisfies the following conditions:

$$A^T \eta A = \eta, \tag{2.26}$$

and

$$|A| = 1, \tag{2.27}$$

where η is the four-by-four matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, notice that if we employ the parameterization

$$\gamma = \cosh \xi, \tag{2.28}$$

then we can use the identity $\cosh^2 \xi - \sinh^2 \xi = 1$ to show that

$$\gamma\beta = \sinh \xi, \tag{2.29}$$

where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. Since Lorentz transformations can be written in the form

$$ct = \gamma(ct' + \beta x'), \tag{2.30}$$
$$x = \gamma(x' + \beta ct'), \tag{2.31}$$
$$y = y', \tag{2.32}$$
$$z = z', \tag{2.33}$$

we can use the above parameterizations to write Lorentz transformations in the form

$$r = Ar', \tag{2.34}$$

where r is the unprimed four-vector, r' is the primed four-vector, and A is the following matrix:

$$\begin{pmatrix} \cosh \xi & \sinh \xi & 0 & 0 \\ \sinh \xi & \cosh \xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.35}$$

Notice how this resembles a rotation but instead involves hyperbolic functions and a sign flip, providing some insight into the hyperbolic geometry of Minkowski spacetime.

Invariance under these transformations tells us that the laws of physics are independent of orientation and preserve the principle of relativity (experiment in inertial frame gives same results as noninertial frame), as well as that experiments performed at different times or in different places give the same results. However, there are also useful physical theories that do not satisfy all of these principles; a key example is the (nonlinear) Schrödinger equation, which does not satisfy the principle of relativity. For this example, we replace the Lorentz transformations with Galilean transformations, giving Galilean invariance.

The simplest equation invariant for the Poincaré group is

$$\Box\psi = 0, \tag{2.36}$$

where $\Box$ is the wave operator defined in Chapter 1. Note that in this simple form, a factor of $\frac{1}{c^2}$ is left out from the temporal derivative; we have assumed units so that we can let $c = 1$. Now, let us prove that the wave operator is invariant under space translations, space rotations, time translations, and Lorentz transformations. First, suppose we have a space translation:

$$t' = t, \tag{2.37}$$
$$x' = x + x_0, \tag{2.38}$$
$$y' = y, \tag{2.39}$$
$$z' = z. \tag{2.40}$$

From the chain rule, we have

$$\frac{\partial \psi}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial \psi}{\partial x'} \tag{2.41}$$
$$= \frac{\partial \psi}{\partial x'}, \tag{2.42}$$

from which it is easy to see that the Laplace operator eliminates the constant term, leaving the equation invariant. The proof is analogous for space translations in x or y, as well as for time translations. In the case of time inversions or parity inversions, we simply notice that

$$\frac{\partial \psi}{\partial t} = -\frac{\partial \psi}{\partial t'}, \tag{2.43}$$

and similarly for $(x, y, z) \to (-x, -y, -z)$. Taking the second derivatives will get rid of all the negative signs, yielding the invariance under inversions. Now, let us consider space rotations. We will only consider counterclockwise rotations about the z axis, as the proofs of the remaining types of rotations are similar. We obtain the coordinate transformation

$$t' = t, \tag{2.44}$$
$$x' = x\cos\theta - y\sin\theta, \tag{2.45}$$
$$y' = x\sin\theta + y\cos\theta, \tag{2.46}$$
$$z' = z. \tag{2.47}$$

The chain rule gives

$$\frac{\partial \psi}{\partial x} = \cos\theta\frac{\partial \psi}{\partial x'} + \sin\theta\frac{\partial \psi}{\partial y'}, \tag{2.48}$$
$$\frac{\partial \psi}{\partial y} = \cos\theta\frac{\partial \psi}{\partial y'} - \sin\theta\frac{\partial \psi}{\partial x'}. \tag{2.49}$$

Taking the second derivatives eliminates all the mixed partials, and using the identity $\cos^2\theta + \sin^2\theta = 1$ yields the desired invariance. Finally, we consider a Lorentz transformation in the x direction:

$$t' = \gamma(t - v_1 x), \tag{2.50}$$
$$x' = \gamma(x - v_1 t), \tag{2.51}$$
$$y' = y, \tag{2.52}$$
$$z' = z. \tag{2.53}$$

Once again, we turn to the chain rule:

$$\frac{\partial\psi}{\partial t} = \gamma\frac{\partial\psi}{\partial t'} - \gamma v_1\frac{\partial\psi}{\partial x'}, \tag{2.54}$$
$$\frac{\partial\psi}{\partial x} = -\gamma v_1\frac{\partial\psi}{\partial t'} + \gamma\frac{\partial\psi}{\partial x'}. \tag{2.55}$$

We can then show the invariance by factoring out γ and v_1 and using the fact that $\gamma^2 = 1/(1 - v_1^2)$.

Chapter 3

Introducing Nonlinearities

3.1 A Simple Semilinear Wave Equation

Now, we investigate the behavior of the wave equation with a nonlinearity. We follow the procedure used in Benci and Fortunato (2007) to find the simplest semilinear equation yielding solitary waves. To this end, we consider the Lagrangian

$$\mathcal{L} = \left(\frac{\partial \psi}{\partial t}\right)^2 - |\nabla \psi|^2 - W(\psi), \tag{3.1}$$

where $W : \mathbb{C} \to \mathbb{R}$ satisfies

$$W(e^{i\theta}\psi) = W(\psi). \tag{3.2}$$

Note that this "Lagrangian" is actually a Lagrangian density function; we will refer to it as the Langrangian for simplicity. The added nonlinearity is rotationally symmetric on the complex plane, meaning that it is only a function of the norm of ψ. This is the simplest Lagrangian giving rise to nonlinear Euler–Lagrange equations—in particular, we obtain the *semilinear wave equation*

$$\Box \psi + W'(\psi) = 0. \tag{3.3}$$

This is deemed semilinear because the nonlinearity does not depend the derivatives of ψ. Note that if $W'(\psi)$ is linear (in particular, $W'(\psi) = \mu^2\psi$ with $\mu^2 > 0$), then the semilinear wave equation reduces to the linear Klein–Gordon equation:

$$(\Box + \mu^2)\psi = 0. \tag{3.4}$$

It can be thought of as a relativistic form of Schrödinger's equation, and is technically the equation of motion of a quantum field with spinless particles. For our purposes, we need only to think of this as a wave equation producing relativistic matter waves. This means that it is a free wave equation with an additional mass term accounting for matter. However, it does not admit solitary wave solutions—instead it produces wave packet solutions which initially behave as solitary waves but disperse over time.

The easiest way to obtain solitary wave solutions for the semilinear wave equation is to first solve the static case,

$$-\Delta u + W'(u) = 0, \tag{3.5}$$

and then apply a coordinate transformation to u to obtain a solution in terms of time and space. We can reasonably assume that

$$W \geq 0, \tag{3.6}$$

since this would then correspond to a solution with positive energy. Another simplification we can make is to substitute standing waves of the form

$$\psi_0(t, x) = u(x)e^{i\omega_0 t}, \quad u \geq 0 \tag{3.7}$$

into the semilinear wave equation. Making this substitution yields

$$\frac{\partial^2}{\partial t^2}\left(u(x)e^{-i\omega_0 t}\right) - \Delta\left(u(x)e^{-i\omega_0 t}\right) + W'\left(u(x)e^{-i\omega_0 t}\right) = 0. \tag{3.8}$$

Upon computing the derivatives and dividing by the positive factor $e^{-i\omega_0 t}$, we obtain

$$-\Delta u + W'(u) = \omega_0^2 u, \tag{3.9}$$

which we will call the *reduced static equation*. We know the Lagrangian is invariant under Lorentz transformations, so we can obtain another solution by performing a Lorentz transformation in the first spatial variable; that is,

$$t' = \gamma(t - vx_1), \tag{3.10}$$

$$x_1' = \gamma(x_1 - vt), \tag{3.11}$$

$$x_2' = x_2, \tag{3.12}$$

$$x_3' = x_3, \tag{3.13}$$

where $\gamma = \frac{1}{\sqrt{1-v^2}}$ as before. Hence, $\psi_1(t, x) = \psi_0(t', x')$ is a solution of the semilinear wave equation. We see that given any standing wave $u(x)e^{-i\omega t}$, we can form a solitary wave solution that travels at a velocity v. If $u(x)$ solves the reduced static equation, then we can perform a Lorentz transformation in a similar manner as above to obtain solitary wave solutions of the form

$$\psi_v(t, x_1, x_2, x_3) = u(\gamma(x_1 - vt), x_2, x_3)e^{k \cdot x - \omega t}, \tag{3.14}$$

where

$$\omega = \gamma\omega_0, k = \gamma\omega_0 v. \tag{3.15}$$

This method furnishes solitary wave solutions whenever a solution to the reduced static equation is known. These solutions are also the critical points in $H^1(\mathbb{R}^3)$ of the *reduced action* functional

$$J(u) = \frac{1}{2}\int |\nabla u|^2 dx + \int \left\{ W(u) - \frac{1}{2}\omega_0^2 u^2 \right\} dx. \tag{3.16}$$

As noted in Benci and Fortunato (2007), the existence of nontrivial critical points is guaranteed due to a theorem of Berestycki and Lions. We will examine a more complicated case later, so we omit the details of the proof here.

3.2 A Perturbation of Maxwell's Equations

One interesting application of semilinear perturbations used to produce solitary wave solutions is found in Benci and Fortunato (2004). In this paper, the authors present an alternative formulation of Maxwell's equations based on a semilinear perturbation of the Lagrangian. They begin by considering the perturbed action functional

$$S = \frac{1}{2}\int\int \left| \frac{\partial \mathbf{A}}{\partial t} + \nabla\phi \right|^2 - |\nabla \times \mathbf{A}|^2 + W(|\mathbf{A}|^2 - \phi^2)dxdt. \tag{3.17}$$

Here, **H** denotes the magnetic field and **A** is the magnetic potential related to that field. Similarly, **E** denotes the electric field and ϕ is the corresponding electric potential. Note that **A** is a vector potential and ϕ is a scalar potential. The argument of W is chosen so that the equations remain invariant under the Poincaré group, so it maintains the fundamental symmetries desired in a relativistic theory.

Using this Lagrangian, we find the Euler–Lagrange equations

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{A}}{\partial t}+\nabla\phi\right)+\nabla\times(\nabla\times\mathbf{A}) = W'\left(|\mathbf{A}|^2-\phi^2\right)\mathbf{A} \tag{3.18}$$

$$-\nabla\cdot\left(\frac{\partial \mathbf{A}}{\partial t}+\nabla\phi\right) = W'\left(|\mathbf{A}|^2-\phi^2\right)\phi. \tag{3.19}$$

Making the substitutions

$$\mathbf{H} = \nabla\times\mathbf{A} \tag{3.20}$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}-\nabla\phi \tag{3.21}$$

$$\rho = W'(|\mathbf{A}|^2-\phi^2)\phi \tag{3.22}$$

$$\mathbf{J} = W'(|\mathbf{A}|^2-\phi^2)\mathbf{A}, \tag{3.23}$$

we arrive at Maxwell's equations in the presence of matter:

$$\nabla\times\mathbf{H}-\frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}(\mathbf{A},\phi) \tag{3.24}$$

$$\nabla\cdot\mathbf{E} = \rho(\mathbf{A},\phi) \tag{3.25}$$

$$\nabla\times\mathbf{E}+\frac{\partial \mathbf{H}}{\partial t} = 0 \tag{3.26}$$

$$\nabla\cdot\mathbf{H} = 0. \tag{3.27}$$

Using Noether's theorem, which states that any differentiable symmetry of the action has a corresponding conservation law, one can derive expressions for several first integrals of the motion. For example, time invariance of the Lagrangian for the semilinear Maxwell's equations yields an expression for the energy,

$$\mathcal{E} = \frac{1}{2}\int\left(\left|\frac{\partial \mathbf{A}}{\partial t}\right|^2-|\nabla\phi|^2+|\nabla\times\mathbf{A}|^2-W(|\mathbf{A}|^2-\phi^2)\right)dx. \tag{3.28}$$

Invariance under spatial translations yields an expression for the momentum,

$$\mathbf{P} = \int\sum_{i=1}^{3}\left(\frac{\partial A_i}{\partial t}+\frac{\partial\phi}{\partial x_i}\right)\nabla A_i dx. \tag{3.29}$$

Using the expression for the charge density, we can also derive the charge:

$$C = \int W'(|\mathbf{A}|^2-\phi^2)\phi dx. \tag{3.30}$$

The expression for the energy can be recast in a more familiar form by multiplying Gauss's law by ϕ, integrating in x, and adding the result to the above expression for the energy. This gives

$$\mathcal{E} = \frac{1}{2}\int \left(|\mathbf{E}|^2 + |\mathbf{H}|^2\right) dx - \int \left(\rho\phi + \frac{1}{2}W(\sigma)\right) dx, \tag{3.31}$$

where

$$\sigma = |\mathbf{A}|^2 - \phi^2.$$

The left-hand term is the traditional energy of the electromagnetic field, and the right-hand term can be interpreted as the energy of the matter associated with bond energy or nuclear fields. The authors go on to show existence of solitary waves by first finding static solutions in simple cases (either $\mathbf{A} = 0$ or $\phi = 0$) and applying Lorentz transformations to these solutions. However, there are a great deal of technical assumptions and additional background needed in order to carry out the proof of existence. Thus, we instead turn to a slightly different model for the interaction of matter with electromagnetic waves based on the coupling of a nonlinear Klein–Gordon equation with Maxwell's equations. Before investigating the existence of solutions to said model, we introduce some useful preliminary theorems.

Chapter 4

Preliminaries

4.1 Analysis Results

In this section, we present some results in real analysis. The first is a result from Castro (1978) concerning the reduction of functionals satisfying certain geometric conditions:

Lemma 4.1. *Let H be a real, separable Hilbert space. Suppose there exist X and Y, which are closed subspaces of H, such that $H = X \bigoplus Y$ and for some $m > 0$*

$$\langle \nabla J(x + y_1) - \nabla J(x + y_2), y_1 - y_2 \rangle \geq m||y_1 - y_2||^2 \tag{4.1}$$

for every $x \in X, y_1 \in Y, y_2 \in Y$.

Then, there exists a continuous function $\phi : X \to Y$ satisfying:

(i) $J(x + \phi(x)) = \ min \{J(x + y), y \in Y\}$,

(ii) The functional $\widetilde{J} : x \to \mathbb{R}, x \to J(x + \phi(x))$ is of class C^1,

(iii) x is a critical point of $\widetilde{J}$ if and only if $x + \phi(x)$ is a critical point of J.

This lemma will be useful in the hunt for critical points, as it provides conditions for reducing the possible critical points to a subspace of the original Hilbert space. A second lemma is a result in Ambrosetti and Rabinowitz (1973) concerning the exsitence of critical points for even functionals which need not be bounded from above or below.

Lemma 4.2. *Let E be an infinite dimensional Banach space over $\mathbb{R}$. Let $B_r = \{u \in E; ||u|| < r\}$ and $S_r = \partial B_r$. Let $J \in C^1(E, \mathbb{R})$. Suppose J satisfies $J(0) = 0$ and* *(i)* *There exists a* $\rho > 0$ *such that* $I > 0$ *in* $B_\rho - \{0\}$ *and* $J \geq \alpha > 0$ *on* S_ρ, *then* $u = 0$ *is a local minimum for* J,

(ii) *If* $(u_m) \subset E$ *with* $0 < J(u_m), J(u_m)$ *bounded above, and* $J'(u_m) \to 0$, *then* (u_m) *possesses a convergent subsequence,*

(iii) $J(u) = J(-u)$ *for all* $u \in E$,

(iv) *For any finite dimensional* $\widetilde{E} \subset E, \widetilde{E} \cap \hat{A}_c$ *is bounded, where* $\hat{A}_c = \{u \in E; J(u) \geq c\}$.

For each $m \in \mathbb{N}$, *let*

$$c_m = \sup_{h \in \Gamma^*} \inf_{u \in S \cup E_{m-1}^{\perp}} J(h(u)), \tag{4.2}$$

where $\Gamma^* = \{h \in C(E, E); h(0) = 0, h(B_1) \subset \hat{A}_0, h$ *is odd* $\}$.

Then,

$$0 < \alpha \leq c_m \leq b_m < \infty, c_m \leq c_{m+1}, c_m \text{ is a critical value of } J. \tag{4.3}$$

This is a variant of the celebrated Mountain Pass Theorem, applied to the particular case of even functionals satisfying certain geometric constraints. As *m* is arbitrary, this lemma actually furnishes an infinite number of critical points.

Chapter 5

Electrostatic KGM System

5.1 Physical Background

Now, we follow the approach of Benci and Fortunato (2002) in our investigation of the nonlinear Klein–Gordon–Maxwell (KGM) system, a set of elliptic equations derived from a coupling of a nonlinear Klein–Gordon equation and Maxwell's equation in empty space. The nonlinear Klein–Gordon (NKG) equation can be interpreted as a perturbed relativistic matter field. In the linear case, we see that substituting a plane wave ansatz $e^{i\{kx-\omega t\}}$ into the Klein–Gordon equation yields

$$\frac{\partial\psi}{\partial t^2} - \Delta\psi + m^2\psi = 0 \tag{5.1}$$

$$k^2 e^{i\{kx-\omega t\}} - \omega^2 e^{i\{kx-\omega t\}} + m^2 e^{i\{kx-\omega t\}} = 0 \tag{5.2}$$

$$\omega^2 = 1 + \frac{m^2}{k^2}. \tag{5.3}$$

Hence, the frequencies and wavenumbers of different components are dependent on a dispersion relation. In order to produce solitary wave solutions, we need to balance the effects of dispersion, which tends to spread out a wave, with a nonlinear focusing term. One common nonlinear Klein–Gordon equation being studied is

$$\frac{\partial\psi}{\partial t^2} - \Delta\psi + m^2\psi - |\psi|^{p-2}\psi = 0, \tag{5.4}$$

where $\psi = \psi(x,t), m > 0, p > 2, \psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$. Following the paradigm of transforming a wave equation into an elliptic equation, substituting the ansatz

$$\psi(x,t) = u(x)e^{i\omega t}, \qquad u : \mathbb{R}^3 \to \mathbb{R}, \omega \in \mathbb{R}, \tag{5.5}$$

into the NKG equation gives

$$-\Delta u + (m^2 - \omega^2)u - |u|^{p-2}u = 0, \tag{5.6}$$

which is the Euler–Lagrange equation relative to the functional

$$f(u) = \frac{1}{2}\int [|\nabla u|^2 + (m^2 - \omega^2)u^2]dx - \frac{1}{p}\int |u|^p dx. \tag{5.7}$$

The critical points are obtained via a Mountain Pass theorem described in Ambrosetti and Rabinowitz (1973), and solitary wave solutions are found by applying a Lorentz transformation.

In order to couple the nonlinear Klein–Gordon equation with Maxwell's equations, we substitute the derivatives $\frac{\partial}{\partial t}, \nabla$ with the *gauge covariant* derivatives

$$\frac{\partial}{\partial t} + ie\phi, \qquad \nabla - ie\mathbf{A}; \tag{5.8}$$

this stems from wanting to retain invariances under transformations with local symmetry; the constant e denotes electric charge. Substituting these expressions into the nonlinear Klein–Gordon Lagrangian and setting

$$\psi(x,t) = u(x,t)e^{iS(x,t)}, \qquad u, S \in \mathbb{R} \tag{5.9}$$

yields

$$\mathcal{L}_{NKG} = \frac{1}{2}\{u_t^2 - |\nabla u|^2 - (|\nabla u - e\mathbf{A}|^2 - (S_t + e\phi)^2 + m^2)u^2\} + \frac{1}{p}|u|^p. \tag{5.10}$$

After adding the Lagrangian density of the electromagnetic field,

$$\mathcal{L}_{EM} = \frac{1}{2}\left[|\mathbf{A}_t + \nabla\phi|^2 - |\nabla \times \mathbf{A}|^2\right], \tag{5.11}$$

the total action is given by

$$S = \int\int \mathcal{L}_{NKG} + \mathcal{L}_{EM}. \tag{5.12}$$

We then take variational derivatives with respect to u, S, ϕ, and $\mathbf{A}$, yielding the following system of equations:

$$\Box u + [|\nabla S - e\mathbf{A}|^2 - (S_t + e\phi)^2 + m^2]u = |u|^{p-2}u, \tag{5.13}$$

$$\frac{\partial}{\partial t}\left[S_t + e\phi)u^2\right] = \nabla \cdot [(\nabla S - e\mathbf{A})u^2], \tag{5.14}$$

$$\nabla \cdot (\mathbf{A}_t + \nabla\phi) = e(S_t + e\phi)u^2, \tag{5.15}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{\partial}{\partial t}(\mathbf{A}_t + \nabla\phi) = e(\nabla S - e\mathbf{A})u^2. \tag{5.16}$$

By setting

$$u = u(x), \qquad S = \omega t, \qquad \mathbf{A} = 0, \qquad \phi = \phi(x), \tag{5.17}$$

we reduce the system to a set of two equations

$$-\Delta u + [m^2 - (\omega + \phi)^2]u = |u|^{p-2}u, \tag{5.18}$$

$$-\Delta \phi + (\omega + \phi)u^2 = 0. \tag{5.19}$$

Note that we have set $e = 1$ for simplicity. This system describes the interaction of an electrostatic field with a relativistic matter field, and is in terms of the wave function amplitude $u(x)$ and electric potential $\phi(x)$. We work with the functional $F : H^1 \times D^{1,2} \to \mathbb{R}$ defined by

$$F(u, \phi) = \frac{1}{2} \int |\nabla u|^2 - |\nabla \phi|^2 + [m^2 - (\omega + \phi)^2 u^2] dx - \frac{1}{p} \int |u|^p dx. \tag{5.20}$$

It is clear that taking the variational derivatives with respect to u and ϕ yield the system above.

5.2 Proof of Existence

In order to prove existence of solutions, we follow the ideas of Benci and Fortunato (2002) and D'Aprile and Mugnai (2004), but use a different method to obtain the reduced functional. We show the result of Benci and Fortunato (2002), namely the result

Theorem 5.1. *If* $|\omega| < |m|$ *and* $6 > p > 4$, *then the nonlinear Klein–Gordon–Maxwell system has infinitely many radially symmetric solutions* $(u, \phi), u \in H^1, \phi \in D^{1,2}$.

Proof. Let us divide the proof into four main parts:

1. Recast in terms of reduced action functional $J(u)$.
2. Show critical points $J\big|_{H_r^1}$ are critical points of J.
3. Show $J\big|_{H_r^1}$ satisfies Palais–Smale condition.
4. Show $J\big|_{H_r^1}$ satisfies geometric hypotheses of Mountain Pass Theorem.

With these conditions satisfied, we can use Equation 4.2 to furnish infinitely many critical points. We begin by using Equation 4.1 for the functional reduction. There are a few differences in the type of functional described in Castro (1978) and the functional $F(u,\phi)$. For instance, F is functional of two variables, and it is concave rather than convex. This only changes a few inequalities and turns minima into maxima, so we will still be able to use the theorem in a slightly altered form. We simply need to show that

$$\langle \nabla F(u,\phi_1) - \nabla F(u,\phi_2), (0, \phi_1 - \phi_2)\rangle \leq -m||(0,\phi_1-\phi_2)||^2. \tag{5.21}$$

To aid calculation, we first compute

$$\begin{aligned}
\nabla F(u,\phi_1) &= \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{\mathbb{R}^3} -|\nabla(\phi_1+\epsilon v)|^2 - 2\omega(\phi+\epsilon v)u^2 && (5.22)\\
&\quad -(\phi_1+\epsilon v)^2u^2 + |\nabla\phi_1|^2 + 2\omega\phi_1 u^2 + \phi_1 u^2 dx\\
&= \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{\mathbb{R}^3} -|\nabla(\phi_1+\epsilon v)|^2 + |\nabla\phi_1|^2 && (5.23)\\
&\quad - 2\epsilon v u^2 - 2\phi_1\epsilon v u^2 - \epsilon^2 u^2 v^2 dx\\
&= \int_{\mathbb{R}^3} -vu^2 - \phi_1 v u^2 - \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{\mathbb{R}^3} 2\epsilon\langle\nabla\phi_1,\nabla v\rangle && (5.24)\\
&\quad + \epsilon^2\langle\nabla v,\nabla v\rangle dx\\
&= \int_{\mathbb{R}^3} -vu^2 - \phi_1 v u^2 - \langle\nabla\phi_1,\nabla v\rangle dx. && (5.25)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\langle \nabla F(u,\phi_1) - \nabla J(u,\phi_2), (0,\phi_1-\phi_2)\rangle &= \int_{\mathbb{R}^3} -\langle\nabla\phi_1,\nabla v\rangle + \langle\nabla\phi_2,\nabla v\rangle dx && (5.26)\\
&= -\int_{\mathbb{R}^3} |\nabla(\phi_1-\phi_2)|^2 dx && (5.27)\\
&= -||\phi_1-\phi_2||_{D^{1,2}}. && (5.28)
\end{aligned}$$

Hence, due to Equation 4.1, we see that there exists a continuous function $\Phi(u)$ such that (u,ϕ) is a critical point of $F(u,\phi)$ if and only if u is a critical point of $J(u) = F(u,\Phi(u))$ and $\phi = \Phi(u)$.

For $u \in H^1$ that solves Equation 5.19, we multiply by $\Phi(u)$ and integrate, yielding

$$-\int \omega u^2\Phi(u)dx = \int |\nabla\Phi(u)|^2 dx + \int u^2\Phi(u)^2 dx. \tag{5.29}$$

This allows us to rewrite the functional $J(u)$ as an even functional,

$$J(u) = \frac{1}{2}\int (|\nabla u|^2 + |\nabla\Phi(u)|^2 + u^2\Phi(u)^2 + (m^2 - \omega^2)u^2)dx - \frac{1}{p}\int |u|^p dx \tag{5.30}$$

which has the symmetry properties necessary to employ Equation 4.2. However, because J is invariant under translations, there is lack of compactness. This means that for any nontrivial solution u, the sequence $u_n(x) = v(x + z_n)$ with $|u_n| \to \infty$ does not satisfy the Palais–Smale compactness condition. To overcome the lack of compactness, we restrict the functional to the subspace of radial functions,

$$H_r^1 = \{u \in H^1(\mathbb{R}^3) : u = u(r), r = |x|\}. \tag{5.31}$$

Also, since this subspace is compactly embedded in L_r^p, the restricted functional $J\Big|_{H_r^1}$ does not exhibit strong indefiniteness.

Next, we need to show that any critical point $u \in H_r^1$ of $J\Big|_{H_r^1}$ is also a critical point of J. Because this subspace is the set of fixed points for the orthogonal group (group of translations and reflections), we simply notice that J is invariant under orthogonal transformations, so that a critical point of the restricted functional is a critical point of J.

Showing that $J\Big|_{H_r^1}$ satisfies the Palais–Smale condition is slightly more involved; the full proof can be found in Benci and Fortunato (2002). They first establish weak convergence using arguments in the dual space H^{-1}, and then use Sobolev embedding theorems to achieve strong convergence in the norm.

The final portion of the proof is satisfying the geometric hypotheses of the Mountain Pass Theorem, Equation 4.2. It is clear that $J(0) = 0$. They then use the continuous embedding of H_r^1 in L^p, which to establish that there exists $\rho > 0$ such that

$$\inf_{||u||_{H^1}=\rho} J(u) > 0, \tag{5.32}$$

and use the equivalence of norms in finite dimensions to show that $J(u) \to -\infty$ as $||u|| \to \infty$ for u in finite dimensional subspaces. Since the functional is even and all the other hypotheses are satisfied, Equation 4.2 can be invoked, meaning that there at least m distinct pairs of critical points of $J\Big|_{H_r^1}$, where m is the dimension of the subspace. However, as m is arbitrary, this means that there are an infinite number of critical points; that is, an infinite number of radial solutions. □

5.3 Extensions and Current Research

The nonlinear Klein–Gordon–Maxwell system is still being actively studied, with papers being published as recently as 2011. We give a brief outline of some of the extensions that have been investigated. The result shown in the previous section was based on Benci and Fortunato (2002), in which the authors used a reduction method to show the existence of infinitely many radially symmetric solutions via a version of the Mountain Pass Theorem in Ambrosetti and Rabinowitz (1973).

Following this approach, several authors have made contributions to the study of this system. Cassani (2004) considered the problem with a different nonlinearity $f(u) = \mu|u|^{p-2}u + |u|^{2^*-2}u$, where $\mu \geq 0, p \in [4,6), 2^* = \frac{2n}{n-2} = 6$ (this last term is the critical exponent associated with the Sobolev embedding theorem). Using a suitable Pohozaev identity, he proved that the system only admits the trivial solution for $\mu = 0$, but that nontrivial solutions exist for

(i) $p \in (4,6), |m| > |\omega| > 0, \mu > 0$,

(ii) $p = 4, |m| > |\omega| > 0, \mu > 0$ sufficiently large.

In D'Aprile and Mugnai (2004), the authors show that the unique solution ϕ_u of the second equation of the system satisfies a stronger L^∞ estimate, allowing them to prove the existence of infinitely many radially symmetric solutions for $f(u) - |u|^{p-2}u$ with the conditions

(i) $p \in [4,6), m > \omega > 0$,

(ii) $p \in (2,4), m\sqrt{\frac{p-2}{2}} > \omega > 0$.

Recently, Wang (2011) has generalized several of these results further. For the nonlinearity $f(u) = \mu|u|^{p-2}u + |u|^{2^*-2}u, \mu > 0$, he showed existence of nontrivial solutions for

(i) $p \in (4,6), m > \omega > 0, \mu > 0$,

(ii) $p \in (3,4], m > \omega > 0, \mu > 0$ sufficiently large,

(iii) $p \in (2,3], m\sqrt{(p-2)(4-p)} > \omega > 0$ and $\mu > 0$ sufficiently large.

He also proved that the system admits a ground-state (minimizing) solution for $f(u) = |u|^{p-2}u, 2 < p < 2^* = 6$ under the conditions

(i) $p \in (4,6), m > \omega > 0,$

(ii) $p \in (2,4], m > \sqrt{g(p)}\omega > 0,$

where $g(p) = 1 + \frac{(4-p)^2}{4(p-2)}$.

These approaches only consider even, p-power type nonlinearities; it would be very interesting to find solutions for odd perturbations, but it is made more difficult to find critical points with the lack of symmetry.

5.4 Other Approaches

One approach is to use spherical symmetry to reduce the system to a set of coupled ODEs. In spherical coordinates, the Laplacian of a function f is

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{N-1}{r}\frac{\partial f}{\partial r},$$

where N is the dimension of the system. Thus, the system becomes

$$(r^{N-1}u')' = f'(u)r^{N-1} - [m^2 - (e\phi - \omega)^2 u]r^N - 1 \tag{5.33}$$

$$(r^{N-1}\phi')' = r^{N-1}e(e\phi - \omega)u^2. \tag{5.34}$$

Note that the sign of ω is altered; this is allowed since if (u, ϕ) is a solution corresponding to ω, then $(u, -\phi)$ is a solution corresponding to -ω. After integrating, we obtain

$$u(r) = a + \int_o^r s^{1-N}(\int_0^s f'(u) - [m^2 - (e\phi - \omega)^2 u]t^{N-1}dt)ds \tag{5.35}$$

$$\phi(r) = b + \int_0^r s^{1-N}(\int_0^s t^{N-1}e(e\phi - \omega)u^2 dt)ds. \tag{5.36}$$

where $a = u(0), b = \phi(0)$. Let $X = C[0,\epsilon] \times C[0,\epsilon]$, where $\epsilon > 0$ and $C[a,b]$ denotes the set of continuous functions on the interval $[a,b]$. We claim that the above system admits a unique solution; in particular, the solution will be a continuously differentiable function defined on $[0,\epsilon] \times [0,\epsilon]$. We now prove the following theorem:

Theorem 5.2. *If $f' : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, there exists $\epsilon > 0$ and a unique continuously differentiable function $(u(r), \phi(r)) : [0, \epsilon] \times [0, \epsilon] \to \mathbb{R}$ that solves the initial value problem*

$$u'(r) = g(\phi, u, r), u(0) = a \tag{5.37}$$

$$\phi'(r) = h(\phi, u, r), \phi(0) = b, \tag{5.38}$$

where $a, b \in \mathbb{R}$ and

$$g(\phi, u, r) = r^{1-N}(\int_0^r f'(u) - [m^2 - (e\phi - \omega)^2 u]t^{N-1}dt), \tag{5.39}$$

$$h(\phi, u, r) = r^{1-N}(\int_0^r t^{N-1}e(e\phi - \omega)u^2 dt) \tag{5.40}$$

Proof. Define $F(u, \phi) : X \to X$ to be the integral operator

$$F(u, \phi) = (a + \int_0^r g(u, \phi, s)ds, b + \int_0^r h(u, \phi, s)ds), \tag{5.41}$$

for all $s \in [0, \epsilon]$. By showing that F is a contraction mapping, then by the contraction mapping principle we will have guaranteed the existence and uniqueness of fixed points of the map; that is, that there exists a unique solution for specified initial conditions. Thus, we must show that

$$||F(u_1, \phi_1) - F(u_2, \phi_2)|| \leq C||(u_1 - u_2, \phi_1 - \phi_2)||, \tag{5.42}$$

where the norm is the sup norm for continuous functions. By working in a region within ϵ of the initial conditions, we can easily bound the left-hand side. From the triangle inequality, we have

$$\begin{aligned} ||F(u_1, \phi_1) - F(u_2, \phi_2)|| \leq &| \int_0^r g(\phi_1, u_1, s) - g(\phi_2, u_2, s)ds| \\ &+ | \int_0^r h(\phi_1, u_1, s) - h(\phi_2, u_2, s)ds|. \end{aligned}$$

We bound the first term as follows

$$
\begin{aligned}
|\int_0^r g(\phi_1, u_1, s) - g(\phi_2, u_2, s)ds| &\leq \int_0^s |f'(u_1) - [m^2 - (e\phi_1 - \omega)^2]u_1 \\
&\quad - f'(u_2) + [m^2 - (e\phi_2 - \omega)^2]u_2|ds \\
&\leq \int_0^s |f'(u_1) - f'(u_2) - m^2|u_1 - u_2| \\
&\quad + eu_1(e\phi_1 + e\phi_2 - \omega)(\phi_1 - \phi_2) \\
&\quad + (e\phi_2 - \omega)^2|u_1 - u_2|ds \\
&\leq \epsilon[|f'(u_1) - f'(u_2)| - m^2||u_1 - u_2|| \\
&\quad + eu_1(e||\phi_1 + \phi_2|| - \omega)||\phi_1 - \phi_2|| \\
&\quad + ||e\phi_2 - \omega||^2||u_1 - u_2||] \\
&\leq \epsilon[(M + (||e\phi_2 - \omega||^2 - m^2))||u_1 - u_2|| \\
&\quad + e||u_1||(e||\phi_1 + \phi_2|| - \omega)||\phi_1 - \phi_2||],
\end{aligned}
$$

where the last inequality follows if we assume that f' is locally Lipschitz continuous. Similarly, for the second term, we have

$$
\begin{aligned}
|\int_0^r g(\phi_1, u_1, s) - g(\phi_2, u_2, s)ds| &\leq |e| \int_0^s |e\phi_1 - \omega|u_1^2 - (e\phi_2 - \omega)u_1^2 \\
&\quad + (e\phi_2 - \omega)u_1^2 - (e\phi_2 - \omega)u_2^2|ds \\
&\leq |e| \int_0^s |u_1^2((e\phi_1 - \omega) - (e\phi_2 - \omega)) \\
&\quad + (u_1^2 - u_2^2)(e\phi_2 - \omega)|ds \\
&\leq |e|\epsilon[||u_1||^2 e||\phi_1 - \phi_2|| \\
&\quad + ||u_1 + u_2||||u_1 - u_2||||e\phi_2 - \omega||)] \\
&\leq \epsilon[(||u_1||^2 e^2)||\phi_1 - \phi_2|| \\
&\quad + (|e|||u_1 + u_2||||e\phi_2 - \omega||)||u_1 - u_2||].
\end{aligned}
$$

Everything is now in terms of $||u_1 - u_2||$ and $||\phi_1 - \phi_2||$, so we simply choose ϵ such that the constant term in Equation 5.42 is 1, verifying that F is a contraction map. Hence, the system admits a unique solution. □

Based on physical considerations, we want the potential ϕ to vanish at infinity. However, when conducting Mathematica simulations via shooting methods to get a sense of the behavior, it proved to be very difficult to find a value of b that yields a potential vanishing at infinity. We had taken u to have compact support, but this may not necessarily be a physical solution.

We did not proceed with this approach, but it seems that it could be enlightening to analyze the system in the framework of ODEs and dynamical systems theory. While the solutions obtained from variational methods are more generalized, there could still be interesting behavior worth studying, and developing different numerical schemes could help with the investigation of this behavior.

Chapter 6

Conclusion

We began by investigating the physical significance of semilinear wave equations, especially relativistic equations. In particular, we discussed how these equations naturally arise from variational principles and the types of symmetries and invariances they possess. We introduced the concept of solitary waves, which are fascinating coherent structures arising from a balance of linear and nonlinear effects. Also, we presented a physical model whose solitary wave solutions can be interpreted as matter particles with space extension and finite energy, giving an alternative to thinking of particles as singularities of a field.

Future work on the Klein–Gordon–Maxwell system and other related wave equations will likely focus on the growth conditions of the nonlinearity, as many recent papers have been devoted to relaxing the restrictions of the p-power nonlinearity. Other possible directions for future study would be to consider more complicated cases besides magnetostatics and electrostatics, different forms of perturbations, and detailed numerical studies and stability criteria of the system expressed as ODEs.

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